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A VANISHING THEOREM ON MANIFOLDS OF POSITIVE SPECTRUM

Yiu-Ming Lo

1. Introduction

In recent work of Witten-Yau, Cai-Galloway, and X. Wang, they investigated the relation of the homology group $H_{n-1}(M, Z) = 0$ under the assumption that M is a conformally compact manifold of dimension ≥ 3. The great interest relies on the close relation between this class of manifolds and the AdS/CFT correspondence which links the conformal field theory and supergravity together.

A manifold M is conformally compact if its complete metric is of the form

$$ds^2 = \rho^{-2} ds_0^2,$$

where ds_0^2 is some background metric defined on the manifold with boundary $\widetilde{M} = M \cup \partial M$ and ρ is a defining function satisfying

$$\rho = 0 \quad \text{on} \quad \partial M$$

and

$$\mathrm{d}\rho \neq 0 \quad \text{on} \quad \partial M.$$

In his thesis [19] and [20], X. Wang proved the following theorem:

Theorem 1 (Wang). *Let M^n be a conformally compact manifold of dimension $n \geq 3$ with Ricci curvature bounded from below by*

$$Ric_M \geq -(n-1).$$

Let $\lambda_1(M)$ denote the lower bound of the spectrum of the Laplacian on M. If

$$\lambda_1(M) \geq n-2,$$

then either

(1) $H^1(L^2(M)) = 0$; *or*

(2) $M = \mathbf{R} \times N$ *with the warped product metric* $ds^2 = dt^2 + \cosh^2 t \; ds_N^2$, *where N is a compact manifold with Ricci curvature bounded from below by*

$$Ric_N \geq -(n-2).$$

In particular, M either has only one end or it must be a warped product given as above.

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Due to Mazzeo's theorem on conformally compact manifolds, we are able to identify the L^2-cohomology group $H^1(L^2(M))$ with the relative cohomology group $H^1(M, \partial M)$ and hence we obtain the vanishing of $H^1(M, \partial M)$.

Later this type of theorems were generalized by Leung-Wan and Li-Wang, respectively in different directions. In [7], Leung and Wan extended X. Wang's arguments to harmonic maps and generalized his result to a wider class of manifolds, namely, the class of asymptotically hyperbolic conformally compact manifold of order C^1. In fact, they showed that:

Theorem 2 (Leung-Wan)**.** *Suppose that (M^n, g), $n \geq 3$ is an asymptotically hyperbolic conformally compact manifold of order C^1 such that $Ric_M \geq -(n-1)g$ and $\lambda_1(M) \geq n-2$. Suppose that $f : M \to N$ is a smooth harmonic map of finite total energy from M into a complete non-positively curved manifold N. If $\lambda_1(M) > n-2$, then f is a constant map. If $\lambda_1(M) = n-2$, then either f is a constant map, or $M = \mathbf{R} \times \Sigma$ with the warped product metric $g = dt^2 + \cosh^2(t) ds_\Sigma^2$, where (Σ, ds_Σ^2) is a compact manifold with $Ric_\Sigma \geq -(n-2)$.*

As an application, they showed that the homotopy classes in $[(M, \partial M), (N, *)]$ are trivial, or M splits as a warped product of the real line and some compact manifold.

In another direction, Li and Wang [12] generalized the theorem to a class of manifolds with positive spectrum.

Theorem 3 (Li-Wang)**.** *Let M be a complete Riemannian manifold of dimension $n \geq 3$. Suppose $\lambda_1(M) > 0$ and*

$$Ric_M \geq -\frac{(n-1)\lambda_1(M)}{n-2}.$$

Then either

1. *M has only one end with infinite volume; or*
2. *$M = \mathbf{R} \times N$ with the warped product metric $ds^2 + \cosh^2\left(\sqrt{\frac{\lambda_1(M)}{n-2}}t\right) ds_N^2$, where N is a compact manifold with Ricci curvature bounded from below by*

$$Ric_N \geq -\lambda_1(M).$$

By combining the idea of Li-Wang and Leung-Wan, we are able to prove:

Theorem 4 (Main Theorem)**.** *Let M be a complete Riemannian manifold of dimension $n \geq 3$ and N be a manifold of non-positive sectional curvature. Suppose $\lambda_1(M) > 0$ and*

$$Ric_M \geq -\frac{(n-1)\lambda_1(M)}{n-2}$$

Either M splits as a warped product of the real line and a compact manifold, or any smooth map $h : M \to N$ which is constant outside a compact set is homotopic to a constant map.

In particular, if M has only one end, then for any compact set K the homotopy class $[(M, M-K), (N, \text{a point})]$ is trivial.

In fact, Li and Wang [12] showed that for a subclass of bounded harmonic functions with finite Dirichlet integral constructed in [10] there exists some a such that

$$\int_{E(R+1)\setminus E(R)} (f-a)^2 \leq C\exp(-2\sqrt{\lambda_1(E)}R). \tag{1.1}$$

By following their argument, we can show that the similar energy estimate holds for certain class of harmonic maps.

The main key of the proof is based on the fact that a power of the energy density of a harmonic map is in $L^2(M)$. Then the Bochner formula forces the energy density must be zero on a manifold of one end.

We remark that there is certain subtlety of our formation of our theorem. For a harmonic map f which is homotopic to a map constant outside a compact set, apriorily, it might not be a constant map, although the unique continuation theorem implies that f is constant if it is constant on some open set.

Throughout the whole paper, we denote $E(R) = E \cap B_p(R)$ and $\partial E(R) = E \cap \partial B_p(R)$, where E is an end of M. We also denote the bottom of the L^2 spectrum of the Laplacian on E satisfying Dirichlet boundary conditions on ∂E by $\lambda_1(E)$. Thus for any compactly supported smooth function ϕ on E

$$\lambda_1(E)\int_E \phi^2 \leq \int_E |\nabla\phi|^2.$$

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2. Vanishing Theorem for Harmonic Maps

First let us recall a fundamental theorem on the existence of harmonic maps with finite total energy and then we will construct a class of harmonic maps.

For a smooth map $h : M \to N$, the energy density of h is defined to be

$$e(h) := tr_M(h^* ds_N^2),$$

where tr_M is the trace with respect to the metric ds_M^2. The total energy of h is

$$E(h) := \int_M e(h)\ dv_M.$$

Theorem 5 (Schoen-Yau). *Let M be a complete Riemannian manifold with*

$$Ric_M \geq -(n-1)k^2,$$

and N be a complete manifold of non-positive sectional curvature. Let $h : M \to N$ be a smooth map of finite total energy. Then there exists a harmonic map $f : M \to N$ such that f is homotopic to h on compact sets of M and f has finite energy.

Let us give an outline of the proof for our purpose. Let h be a map from M to N which is constant outside a compact set of M. Let Ω_i be a sequence of compact manifolds with boundary such that $M = \cup_i \Omega_i$. Then by Hamilton's theorem of Dirichlet problem for harmonic maps, we can find harmonic maps $f_i : \Omega_i \to N$ which are homotopic to $h|_{\Omega_i}$. We also have

$$E(f_i) \leq E(h|_{\Omega_i}).$$

Then it is standard to show that there exists a subsequence of f_i which converges uniformly on compact sets in M.

We denote by $\mathcal{K}$ the class of harmonic maps which can be constructed as above for some $h : M \to N$ which is constant outside a compact set of M.

Since the usual distance function is not globally smooth, we have to go through the universal coverings of M and N and define the homotopic distance function. For the sake of completeness, we reproduce the construction here. Let f and g be homotopic maps from M to N and let $\widetilde{M}$ and $\widetilde{N}$ be the universal covers of M and N respectively. Then $\pi_1(M,*)$ and $\pi_1(N,*)$ act as groups of isometries on $\widetilde{M}$ and $\widetilde{N}$ respectively so that $M = \widetilde{M}/\pi_1(M,*)$ and $N = \widetilde{N}/\pi_1(N,*)$. Let $\widetilde{r}$ be the distance function on $\widetilde{N}$. Since $\widetilde{N}$ has non-positive curvature, $\widetilde{r}$ is smooth on $\widetilde{N} \times \widetilde{N} \backslash$diagonal. Now $\pi_1(N,*)$ acts on $\widetilde{N} \times \widetilde{N}$ as a group of isometries by

$$\alpha(x,y) = (\alpha(x), \alpha(y)) \text{ for } \alpha \in \pi_1(\mathrm{N},*).$$

Thus $\widetilde{r}$ induces a function $r : \widetilde{N} \times \widetilde{N}/\pi_1(N,*) \to \mathbf{R}$. Let $F : M \times [0,1] \to N$ be a homotopy of f with g so that $F(p,0) = f(p)$ and $F(p,1) = g(p)$ for all $p \in M$. We now choose a lifting $\widetilde{F} : \widetilde{M} \times [0,1] \to \widetilde{N}$, and let

$$\widetilde{f}(p) = F(p,0) \text{ and } \widetilde{g}(p) = F(p,1)$$

for all $p \in M$. This defines lifting $\widetilde{f}, \widetilde{g}$ of f, g. Thus if $\gamma \in \pi_1(M,*)$, there exists $\alpha \in \pi_1(N,*)$ with

$$\widetilde{f}(\gamma(p)) = \alpha\widetilde{f}(p) \text{ and } \widetilde{g}(\gamma(p)) = \alpha\widetilde{g}(p) \text{ for all } p \in \widetilde{M}. \tag{2.1}$$

We define a map $\widetilde{h} : \widetilde{M} \to \widetilde{N} \times \widetilde{N}$ by $\widetilde{h}(p) = (\widetilde{f}(p), \widetilde{g}(p))$ and it induces a map

$$h : M \to \widetilde{N} \times \widetilde{N}/\pi_1(N,*).$$

We now define $\rho(f,g) : M \to \mathbf{R}$ by

$$\rho(f,g) = r^2 \circ h.$$

Then $\rho(f,g)$ is smooth on M. We call $\rho(f,g)$ to be the homotopic distance between f and g. Moreover, if f is harmonic and g is constant, then the hessian comparison theorem implies $\rho(f,g)$ is subharmonic and

$$\Delta\rho(f,g) \geq 2e(f). \tag{2.2}$$

Lemma 2.1. *Let M be a complete Riemannian manifold. Suppose E is an end of M such that $\lambda_1(E) > 0$. Then for any smooth harmonic map $f \in \mathcal{K}$, we have the energy decay estimate*

$$\int_{E(R+1)\setminus E(R)} e(f) \leq C \exp(-2\sqrt{\lambda_1(E)}R)$$

for some constant $C > 0$ depending on f, $\lambda_1(E)$ and n.

Proof of Lemma 2.1. Let $f \in \mathcal{K}$. Since the initial map of f is constant outside a compact set, without loss of generality, we may assume it is constant on the end E. We will denote the homotopic distance between f and its initial map by $\rho(f)$.

It now follows from (2.2) that $\rho(f)$ is a subharmonic function on E and

$$\Delta\rho \geq 2e(f) \geq 0. \tag{2.3}$$

Thus letting ϕ be a non-negative cut-off function on M and multiplying (2.3) by ϕ^2 and applying integration by parts gives

$$\begin{aligned}\int_M \phi^2 e(f) &\leq \frac{1}{2}\int_M \phi^2 \Delta(\rho(f)) \\ &= -\frac{1}{2}\int_M \langle \nabla\phi^2, \nabla(\rho(f))\rangle \\ &\leq \int_M \phi|\nabla\phi||\nabla\rho|\sqrt{e(f)} \\ &\leq \frac{1}{2}\int_M \phi^2 e(f) + \frac{1}{2}\int_M |\nabla\phi|^2|\nabla\rho|^2.\end{aligned}$$

By using $|\nabla \mathrm{dist}|^2 = 1$, we obtain

$$\int_M \phi^2 e(f) \leq C\int_M |\nabla\phi|^2\rho(f) \tag{2.4}$$

for some absolute constant $C > 0$. So it suffices to establish the decay estimate for $\rho(f)$.

Let f_i be a sequence of harmonic maps which converge uniformly to f on compact sets and $\rho_i : M \to \mathbf{R}$ be the corresponding homotopic distance between f_i and the initial map. Here we extend f_i to a constant map outside Ω_i. Thus each ρ_i converges uniformly to $\rho(f)$ on compact sets and satisfies

$$\Delta\rho_i \geq 2e(f_i) \geq 0 \quad \text{on} \quad \Omega_i, \qquad \rho_i = 0 \quad \text{on} \quad \partial\Omega_i. \tag{2.5}$$

By scaling the metric, we may assume the $\lambda_1(E) = 1$ and we want to prove

$$\int_{E(R+1)\setminus E(R)} \rho(f) \leq C\exp(-2R).$$

First, we show that for any $0 < \delta < 1$,

$$\int_E \exp(2\delta r)\rho(f) \leq \frac{C}{(1-\delta)^2},$$

where $r(x)$ is the geodesic distance to the fixed point p in M. In particular, $\rho(f)$ is in $L^2(E)$.

To do this, let ϕ be the non-negative cut-off function

$$\phi(x) = \begin{cases} \frac{r(x)-R_0}{R_0} & \text{on} \quad E(2R_0)\backslash E(R_0), \\ 1 & \text{on} \quad E\backslash E(2R_0), \end{cases}$$

and R_i a sequence divergent to infinity. By using integration by parts and (2.5) and Cauchy-Schwarz inequality, we have for any $\epsilon > 0$,

(2.6)

$$\begin{aligned} &\int_{E(R_i)} |\nabla(\phi \exp(\delta r)\sqrt{\rho_i})|^2 \\ &\quad = \int_{E(R_i)} |\nabla(\phi \exp(\delta r))|^2 \rho_i + 2\int_{E(R_i)} \phi \exp(\delta r)\sqrt{\rho_i}\langle \nabla(\phi \exp(\delta r)), \nabla\sqrt{\rho_i}\rangle \\ &\qquad + \int_{E(R_i)} (\phi \exp(\delta r)^2 |\nabla \sqrt{\rho_i}|^2 \\ &\quad = \int_{E(R_i)} |\nabla(\phi \exp(\delta r))|^2 \rho_i - \frac{1}{2}\int_{E(R_i)} \phi^2 \exp(2\delta r)\Delta\rho_i \\ &\qquad + \int_{E(R)} \phi^2 \exp(2\delta r)|\nabla\sqrt{\rho_i}|^2 \\ &\quad \leq \int_{E(R_i)} |\nabla(\phi \exp(\delta r))|^2 \rho_i - \int_{E(R_i)} \phi^2 \exp(2\delta r) e(f_i) \\ &\qquad + \int_{E(R)} \phi^2 \exp(2\delta r) e(f_i) \\ &\quad = \int_{E(R_i)} |\nabla(\phi \exp(\delta r))|^2 \rho_i \\ &\quad \leq (1+\epsilon)\delta^2 \int_{E(R_i)} \phi^2 \exp(2\delta r)\rho_i + \left(1 + \frac{1}{\epsilon}\right)\frac{1}{R_0}\int_{E(2R_0)\backslash E(R_0)} \exp(2\delta r)\rho_i. \end{aligned}$$

By using the fact that $\lambda_1(E) = 1$ and choosing $\epsilon = \frac{1-\delta}{\delta}$, we obtain

$$(1-\delta)^2 \int_{E(R_i)} \exp(2\delta r)\rho_i \leq \frac{1}{R_0^2}\int_{E(2R_0)\backslash E(R_0)} \exp(2\delta r)\rho_i.$$

Since ρ_i converges to $\rho(f)$ compactly uniformly, by letting $i \to \infty$, we obtain

$$(1-\delta)^2 \int_{E} \exp(2\delta r)\rho(f) \leq \frac{1}{R_0^2}\int_{E(2R_0)\backslash E(R_0)} \exp(2\delta r)\rho(f).$$

Thus we have for some $C > 0$

$$\int_{E} \exp(2\delta r)\rho(f) \leq \frac{C}{(1-\delta)^2}. \tag{2.7}$$

Once we get this estimate (2.7), we can improve the estimate as in [12]. First, setting $\delta = 1$ in (2.6) and $\lambda_1(E) = 1$ implies

$$\begin{aligned}\int_E \psi^2 \exp(2r)\rho_i &\leq \int_E |\nabla(\psi \exp(r))|^2 \rho_i \\ &\leq \int_E |\nabla\psi|^2 \exp(2r)\rho_i + 2\int_E \psi \exp(2r)\langle\nabla\psi, \nabla r\rangle\rho_i \\ &\quad + \int_E \psi^2 \exp(2r)\rho_i,\end{aligned}$$

which gives

$$-2\int_E \psi \exp(2r)\langle\nabla\psi, \nabla r\rangle\rho_i \leq \int_E |\nabla\psi|^2 \exp(2r)\rho_i.$$

Since ψ is a compactly supported function, by letting i go to infinity, we obtain

$$-2\int_E \psi \exp(2r)\langle\nabla\psi, \nabla r\rangle\rho \leq \int_E |\nabla\psi|^2 \exp(2r)\rho.$$

Then we choose our cut-off function ψ. For $R_0 < R_1 < R$, let ψ be

$$\psi(x) = \begin{cases} \frac{r(x)-R_0}{R_1-R_0} & \text{on} \quad E(R_1)\backslash E(R_0) \\ \frac{R-r(x)}{R-R_1} & \text{on} \quad E(R)\backslash E(R_1) \end{cases}$$

Then we obtain, for any fixed $0 < t < R - R_1$,

$$\begin{aligned}&\frac{2t}{(R-R_1)^2}\int_{E(R-t)\backslash E(R_1)} \exp(2r)\rho(f) \\ &\qquad \leq \Big(\frac{2}{R_1-R_0} + \frac{1}{(R_1-R_0)^2}\Big)\int_{E(R_1)\backslash E(R_0)} \exp(2r)\rho(f) \\ &\qquad\quad + \frac{1}{(R-R_1)^2}\int_{E(R)\backslash E(R_1)} \exp(2r)\rho(f).\end{aligned} \tag{2.8}$$

Based on (2.8), after an iterative argument (see [12] for details), we can show that for any positive integer k and $R \geq 1$

$$\int_{E(R)\backslash E(R_0+1)} \exp(2r)\rho(f) \leq CR^2 + 2^{-k}\int_{E(R+k)\backslash E(R_0+1)} \exp(2r)\rho(f)$$

But using (2.7) and choosing δ sufficiently small, we see that the second term goes to 0 as $k \to \infty$. Thus we have

$$\int_{E(R)} \exp(2r)\rho(f) \leq CR^2 \quad \text{for all} \quad R \leq R_0. \tag{2.9}$$

By applying the same iterative argument, we can further improve the estimate. First, we obtain, for all $R \leq R_0$

$$\int_{E(R)} \exp(2r)\rho(f) \leq CR,$$

and repeat the iterative argument again to get

$$\int_{E(R+2)\setminus E(R)} \exp(2r)\rho(f) \leq C,$$

for some constant $C > 0$ independent of R, which implies

$$\int_{E(R+1)\setminus E(R)} \rho(f) \leq C\exp(-2R). \tag{2.10}$$

Now for $R_0 < R_0 + 1 < R < R + 1$, we can choose ϕ in (2.4) to be

$$\phi(x) = \begin{cases} r(x) - R_0 & \text{on} \quad E(R_0+1)\setminus E(R_0) \\ 1 & \text{on} \quad E(R)\setminus E(R_0+1) \\ R - r(x) & \text{on} \quad E(R+1)\setminus E(R). \end{cases}$$

Then the lemma follows immediately from (2.4) and (2.10). □

Now we are going to prove a vanishing theorem of harmonic maps.

Lemma 2.2. *Let M be a complete Riemannian manifold of dimension $n \geq 3$. Suppose $\lambda_1(M) > 0$ and*

$$Ric_M \geq -\frac{(n-1)\lambda_1(M)}{n-2}.$$

Suppose that $f : M \to N$ is a smooth harmonic map from M into a complete manifold N of non-positive sectional curvature. If $f \in \mathcal{K}$ and M has only one infinite volume end, f must be a constant map.

Proof of Lemma 2.2. Let $f \in \mathcal{K}$ be a harmonic map constructed as above and R and K be the Riemannian curvature tensor of M and N, respectively.

For a smooth harmonic map from M to N, we have

$$|\nabla^2 f|^2 \geq \Big(1 + \frac{1}{n-1}\Big)|\nabla\sqrt{e(f)}|^2 \tag{2.11}$$

(See also [7], [16]). Then we apply the curvature assumption and (2.11) to the Bochner formula for harmonic maps

$$\frac{1}{2}\Delta e(f) = |\nabla^2 f|^2 + R_{ij} f_{\alpha i} f_{\alpha j} - K_{\alpha\beta\gamma\delta} f_{\alpha i} f_{\beta j} f_{\gamma i} f_{\delta j}$$

and obtain

$$\Delta h \geq -\frac{(n-1)\lambda_1(M)}{(n-2)} h + \frac{|\nabla h|^2}{(n-1)h},$$

where $h = \sqrt{e(f)}$. Setting $g = h^{\frac{n-2}{n-1}} = e(f)^{\frac{n-2}{2(n-1)}}$, this differential inequality can be rewritten as

$$\Delta g \geq -\lambda_1(M) g. \tag{2.12}$$

Using the decay estimate and the differential inequality (2.12) and argue as in [12], we can derive an L^2 estimate of g

$$\int_{B_p(2R)\backslash B_p(R)} g^2 \leq CR.$$

In fact, the Cauchy-Schwarz inequality and Lemma 2.1 imply

$$\begin{aligned}
&\int_{B_p(2R)\backslash B_p(R)} g^2 \\
&\leq \Big(\int_{B_p(2R)\backslash B_p(R)} \exp(2\sqrt{\lambda_1(M)}r)e(f)\Big)^{\frac{n-2}{n-1}} \\
&\quad\times \Big(\int_{B_p(2R)\backslash B_p(R)} \exp(-2(n-2)\sqrt{\lambda_1(M)}r)\Big)^{\frac{1}{n-1}} \\
&\leq C\Big(\int_{B_p(2R)\backslash B_p(R)} \exp(-2(n-2)\sqrt{\lambda_1(M)}r)\Big)^{\frac{1}{n-1}}.
\end{aligned}$$

Then an application of the volume comparison theorem shows that the second term in the last inequality can be bounded by R.

Now let ϕ be a non-negatively cut-off function on M. Since

$$\int_M |\nabla(\phi g)|^2 = \int_M |\nabla\phi|^2 g^2 + \frac{1}{2}\int_M \langle\nabla\phi^2, \nabla g^2\rangle + \int_M \phi^2|\nabla g|^2,$$

by using $\lambda_1(M) = 1$ and integration by parts, we have

$$\begin{aligned}
\lambda_1(M)\int_M \phi^2 g^2 &\leq \int_M |\nabla(\phi g)|^2 \\
&= \int_M |\nabla\phi|^2 g^2 + \frac{1}{2}\int_M \langle\nabla\phi^2, \nabla g^2\rangle + \int_M \phi^2|\nabla g|^2 \\
&= \int_M |\nabla\phi|^2 g^2 - \frac{1}{2}\int_M \phi^2\Delta g^2 + \int_M \phi^2|\nabla g|^2 \\
&= \int_M |\nabla\phi|^2 g^2 - \int_M \phi^2 g\Delta g
\end{aligned}$$

which implies

$$\int_M \phi^2 g(\Delta g + \lambda_1(M)g) \leq \int_M |\nabla\phi|^2 g^2. \tag{2.13}$$

For $R > R_0$, let us choose ϕ such that

$$\phi = \begin{cases} 1 & \text{on} \quad B_p(R) \\ 0 & \text{on} \quad M\backslash B_p(2R) \end{cases}$$

and

$$|\nabla\phi| \leq C/R \quad \text{on} \quad B_p(2R)\backslash B_p(R)$$

for some constant $C > 0$. Then the right hand (2.13) can be estimated by

$$\int_M |\nabla\phi|^2 g^2 \leq \frac{C}{R^2}\int_{B_p(2R)\setminus B_p(R)} g^2.$$

By the L^2 estimate of g, this tends to 0 as $R \to \infty$. We can conclude from (2.13) that g either must be identically 0 or it must satisfy

$$\Delta g = -\lambda_1(M) g.$$

This equality forces all inequalities to be equalities. In particular, we have

$$K_{\alpha\beta\gamma\delta} f_{\alpha i} f_{\beta j} f_{\gamma i} f_{\delta j} = 0,$$

and

$$|\nabla^2 f|^2 = \Big(1 + \frac{1}{n-1}\Big)|\nabla|\nabla f|^2|. \tag{2.14}$$

From the first equality, we conclude that f must be a constant map provided that the image of f has strictly negative curvature.

Otherwise, the image of f is flat. Moreover, by tracing back the proof of (2.14), we have

$$|\nabla|\nabla f||^2 = \Big|\frac{\sum_\alpha |\nabla f^\alpha|\nabla|\nabla f^\alpha|}{\sqrt{\sum_\alpha |\nabla f^\alpha|^2}}\Big|^2 = \sum_\alpha |\nabla|\nabla f^\alpha||^2, \tag{2.15}$$

where f^α's are components of the harmonic map f with respect to the normal coordinates on N. From the equality of triangle inequality and Cauchy-Schwarz inequality, we have the vector $\nabla|\nabla f^1|, \dots, \nabla|\nabla f^m|$ are nonnegative multiples of a nonzero one provided that they are not all zero and

$$|\nabla f^\alpha| = c\,|\nabla|\nabla f^\alpha||$$

for some c. We also have

$$|\nabla^2 f^\alpha|^2 = \Big(1 + \frac{1}{n-1}\Big)|\nabla|\nabla f^\alpha|^2| \quad \text{for each } \alpha. \tag{2.16}$$

Now, from the argument of X. Wang (see [20], [21]), we conclude that for each α, ∇f^α is a scalar multiple of $\nabla|\nabla f^\alpha|$. Therefore, the image under *df* is of rank 1 provided $\nabla f^1, \dots, \nabla f^m$ are not all zero. It implies that the image of f is contained by a geodesic in N.

Thus $f : M \to f(M)$ is a harmonic map of rank 1 and we have 2 cases to consider: either $f(M)$ is contained in $\mathbf{R}$ or $\mathbf{S}^1$. If $f(M)$ is contained in $\mathbf{R}$, then f can be considered as a harmonic function. By the construction of f and $\lambda_1(M) > 0$, we have

$$\int_M f^2 \leq \lambda_1(M)\int_M |\nabla f|^2 = \lambda_1(M)\int_M e(f) < \infty,$$

provided that f is a limit of compactly supported functions. Thus f is a L^2 harmonic function, and hence must be constant. Otherwise, f must not be constant and hence the argument of Li-Wang implies that M splits into a warped product with 2 infinite volume ends. So we have a contradiction.

If $f(M)$ is contained by $\mathbf{S}^1$, then f can be identified as a harmonic 1-form on M with integral period. Thus this case is reduced to the case considered by X. Wang. If f is nontrivial, then X. Wang's argument again shows that M splits into a warped product with 2 infinite volume ends, which contradicts to the assumption. Thus f must be constant. □

In view of Lemma 2.2, we have

Theorem 2.3. *Let M be a complete Riemannian manifold of dimension $n \geq 3$ and N be a manifold of non-positive sectional curvature. Suppose $\lambda_1(M) > 0$ and*

$$Ric_M \geq -\frac{(n-1)\lambda_1(M)}{n-2}.$$

Let h be any smooth map from M to N which is constant outside a compact set. Then either M splits into $\mathbf{R} \times N$ with the warped product metric $ds^2 + \cosh^2\Big(\sqrt{\frac{\lambda_1(M)}{n-2}}t\Big)ds_N^2$, where N is a compact manifold with Ricci curvature bounded from below by $Ric_N \geq -\lambda_1(M)$, or h is homotopic to a constant map.

Corollary 2.4. *Under the same assumption on M and N, if M has only one end, then any map h constant outside a compact set is homotopic to a constant map. In particular, for any compact set K the homotopy class $[(M, M - K), (N, \text{a point})]$ is trivial.*

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Department of Mathematics, University of California, Irvine, CA 92697-3875
E-mail address: ylo@math.uci.edu