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RELATIONS IN THE QUANTUM COHOMOLOGY RING OF G/B

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Abstract. The ideal of relations in the quantum cohomology of the flag manifold G/B has been determined by B. Kim in [K]. We are going to point out a limited number of properties that, if they are satisfied by an $\mathbb{R}[q_1, \dots, q_l]$-linear product $\circ$ on $H^*(G/B)\otimes\mathbb{R}[q_1, \dots, q_l]$, then the ring $(H^*(G/B)\otimes\mathbb{R}[q_1, \dots, q_l], \circ)$ is isomorphic to Kim's ring.

1. Introduction

Let us consider the complex flag manifold G/B, where G is a connected, simply connected, simple, complex Lie group and $B \subset G$ a Borel subgroup. Let T be a maximal torus of a compact real form of G, $\mathfrak{t}$ its Lie algebra and $\Phi \subset \mathfrak{t}^*$ the corresponding set of roots. Consider an arbitrary W-invariant inner product $\langle\ ,\ \rangle$ on $\mathfrak{t}$. The Weyl group W can be realized as the subgroup of the orthogonal group of $(\mathfrak{t}, \langle\ ,\ \rangle)$ which is generated by the reflections about the hyperplanes $\ker\alpha$, $\alpha \in \Phi^+$. To any root α corresponds the coroot

$$\alpha^\vee := \frac{2\alpha}{\langle\alpha, \alpha\rangle}$$

which is an element of $\mathfrak{t}$, by using the identification of $\mathfrak{t}$ and $\mathfrak{t}^*$ induced by $\langle\ ,\ \rangle$. If $\{\alpha_1, \dots, \alpha_l\}$ is a system of simple roots then $\{\alpha_1^\vee, \dots, \alpha_l^\vee\}$ is a system of simple coroots. Consider $\{\lambda_1, \dots, \lambda_l\} \subset \mathfrak{t}^*$ the corresponding system of fundamental weights, which are defined by $\lambda_i(\alpha_j^\vee) = \delta_{ij}$.

Let us recall the presentation of the cohomology[1] ring of G/B, as obtained by Borel in [B]. First of all, one can assign to any weight $\lambda \in \mathfrak{t}^*$ a group homomorphism $T \to S^1$; the latter can be extended canonically to a group homomorphism $B \to \mathbb{C}^*$ and gives rise in this way to the complex line bundle $L_\lambda = G \times_B \mathbb{C}$ over G/B. One shows that the ring homomorphism $S(\mathfrak{t}^*) \to H^*(G/B)$ induced by $\lambda_i \mapsto c_1(L_{\lambda_i})$, $1 \le i \le l$, is surjective; moreover it induces the ring isomorphism

$$H^*(G/B) \simeq \mathbb{R}[\{\lambda_i\}]/I_W,$$

where I_W is the ideal of $S(\mathfrak{t}^*) = \mathbb{R}[\lambda_1, \dots, \lambda_l] = \mathbb{R}[\{\lambda_i\}]$ generated by the W-invariant polynomials of strictly positive degree. We identify $H^*(G/B)$ with

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[1]Only cohomology with *real* coefficients will be considered throughout this paper.

Borel's presentation and denote them both by $\mathcal{H}$. So

$$\mathcal{H} = H^*(G/B) = \mathbb{R}[\{\lambda_i\}]/I_W,$$

where $c_1(L_{\lambda_i})$ is identified with the coset $[\lambda_i]$ of λ_i, $1 \le i \le l$. There are two more things we would like to recall here:

- by a result of Chevalley [C], there exist l homogeneous, functionally independent polynomials $u_1, \dots, u_l \in S(\mathfrak{t}^*)$, which generate I_W;
- on $\mathcal{H}$ there exists a natural inner product $(\ ,\ )$, induced by the Poincaré pairing.

Let us consider now the Hamiltonian system of Toda lattice type, which consists of the standard symplectic manifold $(\mathbb{R}^{2l}, \sum_{i=1}^l dr_i \wedge ds_i)$ with the Hamiltonian function

$$E(\{r_i\},\{s_i\}) = \sum_{i,j=1}^{l} \langle \alpha_i^\vee, \alpha_j^\vee \rangle r_i r_j + \sum_{i=1}^{l} e^{2s_i}. \tag{1}$$

The following result, proved by Goodman and Wallach in [G-W], gives details concerning the integrals of motion of this system (note that the latter is completely integrable):

Theorem 1.1. (see [G-W]) *There exist l functionally independent functions*

$$E = \tilde{F}_1, \tilde{F}_2, \dots, \tilde{F}_l : \mathbb{R}^{2l} \to \mathbb{R}$$

each of them uniquely determined by:

(i) $\tilde{F}_k(\{r_i\},\{s_i\}) = F_k(\{e^{2s_i}\},\{r_i\})$, *where F_k is a polynomial in variables $e^{2s_1}, \dots, e^{2s_l}$, $r_1, \dots, r_l$, homogeneous with respect to $e^{s_1}, \dots, e^{s_l}, r_1, \dots, r_l$;*
(ii) $\{\tilde{F}_k, E\} = 0$, *where $\{\ ,\ \}$ denotes the Poisson bracket of functions on $\mathbb{R}^{2l}$;*
(iii) $F_k(0, \dots, 0, \lambda_1, \dots, \lambda_l) = u_k(\lambda_1, \dots, \lambda_l)$ *as elements of $S(\mathfrak{t}^*)$.*

Consider now the formal multiplicative variables $q_1, \dots, q_l$ which are assigned degree 4 (note that the coset of λ_j in $\mathbb{R}[\{\lambda_i\}]/I_W$, which is the same as $c_1(L_{\lambda_j})$ in $H^*(G/B)$, has degree 2). Occasionally, q_i will stand for e^{t_i}, $1 \le i \le l$, where $t_1, \dots, t_l$ are real numbers, so that the differential operators $\frac{\partial}{\partial t_i}$ on $\mathcal{H} \otimes \mathbb{R}[\{q_i\}]$ will be well defined.

Our goal is to prove the following result:

Theorem 1.2. *Let $\circ$ be an $\mathbb{R}[\{q_i\}]$-linear product on $\mathcal{H} \otimes \mathbb{R}[\{q_i\}]$ with the following properties:*

(i) $\circ$ *preserves the graduation induced by* $\deg[\lambda_i] = 2$ *and* $\deg q_i = 4$*;*
(ii) $\circ$ *is a deformation of the usual product, in the sense that if we formally replace all q_i by 0, we obtain the usual product on $\mathcal{H}$;*
(iii) $\circ$ *is commutative;*
(iv) $\circ$ *is associative;*
(v) $\sum_{i,j=1}^l \langle \alpha_i^\vee, \alpha_j^\vee \rangle [\lambda_i] \circ [\lambda_j] = \sum_{i=1}^l \langle \alpha_i^\vee, \alpha_i^\vee \rangle q_i$;
(vi) $\frac{\partial}{\partial t_i}([\lambda_j] \circ a) = \frac{\partial}{\partial t_j}([\lambda_i] \circ a)$, *for any $a \in \mathcal{H}$, $1 \le i, j \le l$.*

Then the ring $(\mathcal{H}\otimes\mathbb{R}[\{q_i\}],\circ)$ *is generated by* $[\lambda_1],\ldots,[\lambda_l],q_1,\ldots,q_l$, *subject to the relations*

$$F_k(\{-\langle\alpha_i^\vee,\alpha_i^\vee\rangle q_i\},\{[\lambda_i]\circ\})=0, \tag{2}$$

$1\le k\le l$, *where the polynomials* F_k *are given by Theorem 1.1.*

Our proof is purely algebraic, but the following geometric ideas stay behind it: We assign to $\circ$ the 1-form ω on $H^2(G/B)$ with values in End$\mathcal{H}$, given by

$$\omega_t(X)(Y)=X\circ Y,$$

where $t=t_1[\lambda_1]+\ldots+t_l[\lambda_l]\in H^2(G/B)$ and $X,Y\in\mathcal{H}$ (the convention $q_i=e^{t_i}$ is in force). Consider the Dubrovin type connection $\nabla^h=d+\frac{1}{h}\omega$ (cf. [D]) on the vector bundle $\mathcal{H}\times H^2(G/B)\to H^2(G/B)$. Conditions (iv) and (vi) say that ∇^h is a flat connection, for all $h\neq 0$. Let (,) denote the Poincaré pairing on $\mathcal{H}$. We are able to construct parallel sections $s: H^2(G/B)\to\mathcal{H}$ of the connection dual to ∇^h, i.e. the one corresponding to ω^T, where

$$(\omega_t^T(X)(Y),Z)=(Y,\omega_t(X)(Z)),$$

$X,Y,Z\in\mathcal{H}$. More precisely, we find certain "formal" solutions s of the system

$$h\frac{\partial s}{\partial t_i}=\omega_t^T([\lambda_i])(s),$$

$1\le i\le l$ (for the details, see section 4). The main difficulty is to show that the integrals of motion of the quantum Toda lattice are *quantum differential operators* for $\circ$, i.e. they vanish all functions $(s,1): H^2(G/B)\to\mathbb{R}$, where s is a parallel section as before: by results of Givental [G] (see also [C-K, section 10.3]), such differential operators induce relations, and it is not difficult to see that those relations are just (2). Now from condition (v) we can deduce that the degree 2 integral of motion — call it H — *is* a quantum differential operator. Because H commutes with any other integral of motion, the latter is also a quantum differential operator (this idea has also been used by B. Kim in [K]).

Remarks. 1. We only have to show that the relations (2) hold in $(\mathcal{H}\otimes\mathbb{R}[\{q_i\}],\circ)$: by a general result of Siebert and Tian [S-T], they generate the whole ideal of relations.

2. Properties of the three-point Gromov-Witten invariants $\langle\ |\ |\ \rangle_d$ (see for instance Fulton-Pandharipande [F-P]) show that the hypotheses of Theorem 1.2 are satisfied by the quantum product $\star$ on the (small) quantum cohomology ring of G/B: Condition (v) follows immediately from the equation

$$[\lambda_i]\star[\lambda_j]=[\lambda_i][\lambda_j]+\delta_{ij}q_j, \tag{3}$$

and the fact that the degree two homogeneous generator of I_W (see the mention to Chevalley's result from above) is

$$u_1=\sum_{i,j=1}^{l}\langle\alpha_i^\vee,\alpha_j^\vee\rangle\lambda_i\lambda_j.$$

As about (3), it can be proved in an elementary way (see [K] or [M1]). Condition (vi) is a direct consequence of the definition

$$[\lambda_i]\star a = \sum_{d=(d_1,\dots,d_l)\geq 0} ([\lambda_i]\star a)_d q_1^{d_1}\dots q_l^{d_l} \text{ with } (([\lambda_i]\star a)_d, b) = \langle[\lambda_i]|a|b\rangle_d \text{ for all } b \in H^*(G/B)$$

and the "divisor property"

$$\langle[\lambda_i]|a|b\rangle_d = d_i\langle a|b\rangle_d.$$

We recover in this way Kim's result on $QH^*(G/B)$ (see [K]). The main achievement of our paper is that it shows that Kim's presentation of $QH^*(G/B)$ can be deduced in an elementary way, by using very few of the properties of the quantum product $\star$. For instance, the Frobenius property

$$(a \star b, c) = (a, b \star c), \qquad a, b, c \in \mathcal{H}$$

is not needed in our proof.

3. In [M2] we constructed the "combinatorial" quantum cohomology ring and then we used Theorem 1.2 in order to prove that its isomorphism type is the one expected by the theorem of Kim.

4. The main result of [M3] is an extension of Theorem 1.2: we were able to obtain a similar connection between the small quantum cohomology of the *infinite dimensional* generalized flag manifold and the integrals of motion of the *periodic* Toda lattice.

2. Toda lattices according to Goodman and Wallach

The goal of this section is to present two results of Goodman and Wallach[G-W], which will be essential ingredients for the proof of Theorem 1.2. Let us consider the $(ax+b)$-algebra corresponding to the coroot system of G. By definition, this is the Lie algebra

$$(\mathfrak{b} = \mathfrak{t}^* \oplus \mathfrak{u}, [\ ,\ ]),$$

where $\mathfrak{u}$ has a basis $X_1, \dots, X_l$ such that:

$$[\lambda_i, \lambda_j] = 0, \quad [\lambda_i, X_j] = \delta_{ij}X_j, \quad [X_i, X_j] = 0, \tag{4}$$

$1 \leq i, j \leq l$. The set $S(\mathfrak{b})$ of polynomial functions on $\mathfrak{b}^*$ is a Poisson algebra and by (4) we have

$$\{\lambda_i, \lambda_j\} = 0, \quad \{\lambda_i, X_j\} = \delta_{ij}X_j, \quad \{X_i, X_j\} = 0,$$

$1 \leq i, j \leq l$.

On the other hand, one can easily see that the Poisson bracket of functions on the standard symplectic manifold $(\mathbb{R}^{2l}, \sum_{i=1}^l dr_i \wedge ds_i)$ satisfies

$$\{r_i, r_j\} = 0, \quad \{r_i, e^{s_j}\} = \delta_{ij}e^{s_j}, \quad \{e^{s_i}, e^{s_j}\} = 0,$$

$1 \leq i, j \leq l$. We deduce that the Poisson subalgebra $\mathbb{R}[e^{s_1}, \dots, e^{s_l}, r_1, \dots, r_l]$ of $C^\infty(\mathbb{R}^{2l})$ is isomorphic to $S(\mathfrak{b})$ via

$$X_i \mapsto e^{s_i}, \quad \lambda_i \mapsto r_i, \tag{5}$$

$1 \leq i \leq l$. In this way, integrals of motion of the Hamiltonian system determined by (1) can be obtained from elements of the space $S(\mathfrak{b})^{\{,\}}$, which is the $\{,\}$-commutator in $S(\mathfrak{b})$ of the polynomial

$$\sum_{i,j=1}^{l} \langle \alpha_i^\vee, \alpha_j^\vee \rangle \lambda_i \lambda_j + \sum_{i=1}^{l} X_i^2. \tag{6}$$

Let us consider now the universal enveloping algebra

$$U(\mathfrak{b}) = T(\mathfrak{b})/\langle x \otimes y - y \otimes x - [x,y], x, y \in \mathfrak{b} \rangle$$

with the canonical filtration $\{0\} = U_0(\mathfrak{b}) \subset U_1(\mathfrak{b}) \subset \ldots$ (see e.g. [H, section 17.3]). We say that an element f of $U(\mathfrak{b})$ has *degree* m if m is the smallest positive integer with the property that $f \in U_m(\mathfrak{b})$. There exists a vector space isomorphism

$$\phi : S(\mathfrak{b}) \to U(\mathfrak{b})$$

induced by the symmetrization map followed by the canonical projection (see [H, Corollary E, section 17.3]). Since $\mathfrak{t}^*$ and $\mathfrak{u}$ are abelian, the element of $S(\mathfrak{b})$ described by (6) is mapped by ϕ to

$$\Omega := \sum_{i,j=1}^{l} \langle \alpha_i^\vee, \alpha_j^\vee \rangle \lambda_i \lambda_j + \sum_{i=1}^{l} X_i^2,$$

the right hand side being regarded this time as an element of $U(\mathfrak{b})$.

The complete integrability of the Toda lattice follows from the following two theorems of Goodman and Wallach:

Theorem 2.1. (see [G-W]) *The Poisson bracket commutator $S(\mathfrak{b})^{\{,\}}$ is mapped by ϕ isomorphically onto the space $U(\mathfrak{b})^{[,]}$ of all $f \in U(\mathfrak{b})$ with the property that $[f, \Omega] = 0$.*

Theorem 2.2. (see [G-W]) *The map $\mu : U(\mathfrak{b}) \to U(\mathfrak{t}^*) = S(\mathfrak{t}^*)$ induced by the natural Lie algebra homomorphism $\mathfrak{b} \to \mathfrak{t}^*$ establishes an algebra isomorphism between $U(\mathfrak{b})^{[,]}$ and the ring $S(\mathfrak{t}^*)^W$ of W-invariant polynomials. Hence there exist $\Omega = \Omega_1, \Omega_2, \ldots, \Omega_l \in U(\mathfrak{b})$, each of them uniquely determined by*

(i) $[\Omega_k, \Omega] = 0$,
(ii) $\mu(\Omega_k) = u_k$ *and* $\deg \Omega_k = \deg u_k$.

Moreover, Ω_k is contained in the subring of $U(\mathfrak{b})$ which is spanned by the elements of the form $X^{2I}\lambda^J$.

Remark. The integrals of motion of the Toda lattice mentioned in Theorem 1.1 are obtained from $\phi^{-1}(\Omega_k)$, $1 \leq k \leq l$ by the transformations (5): by the last statement of Theorem 2.2, the result are polynomial expressions in variables $e^{2s_1}, \ldots, e^{2s_l}, r_1, \ldots, r_l$ and these are what we denoted $F_k(e^{2s_1}, \ldots, e^{2s_l}, r_1, \ldots, r_l)$.

3. Relations in $(\mathcal{H}\otimes\mathbb{R}[q_i],\circ)$

Consider the representation ρ of $\mathfrak{b}$ on $C^\infty(\mathbb{R}^l)$ given by:

$$\rho(\lambda_i)=2\frac{\partial}{\partial t_i},\quad \rho(X_i)=\frac{2\sqrt{-1}}{h}\sqrt{\langle\alpha_i^\vee,\alpha_i^\vee\rangle}e^{\frac{t_i}{2}}\cdot,$$

$1\le i\le l$, where h is a nonzero real parameter. The differential operators

$$D_k=h^{\deg\Omega_k}\rho(\Omega_k),$$

$1\le k\le l$, will be the crucial objects of the proof of Theorem 1.2.

Since F_k is homogeneous in variables e^{s_i},r_i, it follows that Ω_k — being obtained from F_k after applying ϕ, up to the replacements (5) — has a presentation as a homogeneous, symmetric polynomial in the variables X_i,λ_i. We use the commutation relations (4) in order to express Ω_k as a linear combination of elements of the form $X^{2I}\lambda^J$ (see Theorem 2.2). The polynomial expression we obtain in this way appears as

$$\Omega_k=F_k(\{X_i^2\},\{\lambda_i\})+f_k(\{X_i^2\},\{\lambda_i\})$$

where

$$\deg f_k<\deg F_k.$$

Consequently D_k appears as a polynomial expression

$$D_k(e^{t_1},\dots,e^{t_l},h\frac{\partial}{\partial t_1},\dots,h\frac{\partial}{\partial t_l},h),$$

the last "variable", h, being due to the possible occurrence of f_k.

Amongst all D_k, $1\le k\le l$, the operator $D_1=h^2\rho(\Omega)$ plays a privileged role, and we write

$$H:=\frac{1}{4}D_1=h^2\sum_{i,j=1}^{l}\langle\alpha_i^\vee,\alpha_j^\vee\rangle\frac{\partial^2}{\partial t_i\partial t_j}-\sum_{j=1}^{l}\langle\alpha_j^\vee,\alpha_j^\vee\rangle e^{t_j}. \tag{7}$$

Below we will see that the polynomial

$$D_k(\{Q_i\},\{\Lambda_i\},h)\in\mathbb{R}[Q_1,\dots,Q_l,\Lambda_1,\dots,\Lambda_l,h]$$

obtained from D_k by the replacements $e^{t_i}\mapsto Q_i$, $h\frac{\partial}{\partial t_i}\mapsto\Lambda_i$, $1\le i\le l$, satisfies the hypotheses of the following theorem.

Theorem 3.1. *Let $\circ$ be a product on $\mathcal{H}\otimes\mathbb{R}[\{q_i\}]$ with the properties (i)-(vi) from Theorem 1.2. Suppose that $D=D(\{Q_i\},\{\Lambda_i\},h)\in\mathbb{R}[\{Q_i\},\{\Lambda_i\},h]$ satisfies*

(a) $[D(\{e^{t_i}\},\{h\frac{\partial}{\partial t_i}\},h),H(\{e^{t_i}\},\{h\frac{\partial}{\partial t_i}\},h)]=0$,
(b) $D(0,\dots,0,\Lambda_1,\dots,\Lambda_l,h)$ *does not depend on h,*
(c) $D(0,\dots,0,\lambda_1,\dots,\lambda_l,0)\in S(\mathfrak{t}^*)^W$.

Then the relation $D(\{q_i\},\{[\lambda_i]\circ\},0)=0$ holds in the ring $(\mathcal{H}\otimes\mathbb{R}[\{q_i\}],\circ)$.

The proof of this theorem will be done in the next section. Now we will show how can be used Theorem 3.1 in order to prove the main result of the paper.

Proof of Theorem 1.2. By Remark 1 in the introduction, we only have to show that the relations (2) hold for all $1 \leq k \leq l$. To this end we note that $D = D_k$ satisfies the hypotheses of Theorem 3.1: (a) follows from the fact that ρ is a Lie algebra representation, and (b) and (c) from Theorem 2.2 (ii). We obtain the relation $D_k(\{q_i\}, \{[\lambda_i]\circ\}, 0) = 0$, which is just (2). $\square$

4. Proof of Theorem 3.1

Let us begin by picking a basis of $\mathcal{H}$ which consists of homogeneous elements (e.g. the Schubert basis): this will allow us to identify $\mathcal{H}$ with $\mathbb{R}^n$, where $n = \dim \mathcal{H}$, and the endomorphism $[\lambda_i]\circ$ of $\mathcal{H}$ with an element B_i of the space $M_n(\mathbb{R}[e^{t_j}])$ of $n \times n$ matrices whose coefficients are polynomials in $e^{t_1}, \dots, e^{t_l}$. Let $\circ$ be a product which satisfies the hypotheses of Theorem 1.2.

Lemma 4.1. *Fix $i \in \{1, \dots, l\}$ and take $a \in \mathcal{H}$. Write*

$$[\lambda_i] \circ a = \sum_{d=(d_1,\dots,d_l)\geq 0} ([\lambda_i] \circ a)_d q^d \tag{8}$$

with $([\lambda_i] \circ a)_d \in \mathcal{H}$. If $d = (d_1, \dots, d_l) \neq 0$ such that $([\lambda_i] \circ a)_d \neq 0$, then $d_i \neq 0$. In other words, any non-zero term in the right hand side of (8) which is different from $([\lambda_i] \circ a)_0 = [\lambda_i]a$ must be a multiple of q_i.

Proof. Condition (vi) from Theorem 1.2 reads

$$\frac{\partial}{\partial t_i} B_j = \frac{\partial}{\partial t_j} B_i.$$

Hence there exists $M \in M_n(\mathbb{R}[e^{t_j}])$ such that

$$B_i = B_i' + \frac{\partial}{\partial t_i} M,$$

where B_i' is constant, for any $1 \leq i \leq l$. It remains to notice that the derivative with respect to t_i of a monomial in $e^{t_1}, \dots, e^{t_l}$ contains only nonzero powers of e^{t_i}, or else it is 0. $\square$

As pointed out in the introduction, $\mathcal{H}$ has a natural inner product $(\ ,\ )$, namely the Poincaré pairing. Denote by $([\lambda_i]\circ)^T$ the endomorphism of $\mathcal{H}$ which is transposed to $[\lambda_i]\circ$ with respect to this product, i.e.

$$([\lambda_i] \circ a, b) = (a, ([\lambda_i]\circ)^T b), \quad a, b \in \mathcal{H}.$$

Also denote by A_i the matrix of $([\lambda_i]\circ)^T$ with respect to the basis of $\mathcal{H}$ which is the dual with respect to $(\ ,\ )$ of our original basis: of course A_i coincides with the transposed of the matrix of $[\lambda_i]\circ$ with respect to the original basis. Now we want the ordering of the original basis of $\mathcal{H}$ to be decreasing with respect to the degrees of its elements. From condition (i) from Theorem 1.2 and Lemma 4.1 it follows that for any $i \in \{1, \dots, l\}$, the matrix A_i can be decomposed as

$$A_i = A_i' + A_i''(e^{t_j})$$

where A_i' is strictly lower triangular and its coefficients do not depend on t and A_i'' is strictly upper triangular, its coefficients being linear combinations of

$$e^{td} := e^{t_1 d_1} \dots e^{t_l d_l},$$

where $d_1, \dots, d_l$ are nonnegative integer numbers with

$$d_i > 0.$$

Consider the PDE system:

$$h \frac{\partial}{\partial t_i} s = ([\lambda_i] \circ)^T (s), \tag{9}$$

$1 \le i \le l$, where the map $s = s(t_1, \dots, t_l)$ takes values in $\mathcal{H}$ and h is a nonzero real parameter. Some algebraic formalism is needed in order to provide solutions to (9). Let $\mathcal{R}$ be an arbitrary commutative, associative real algebra with unit. For $V = \mathcal{R}^n$ or $V = M_n(\mathcal{R})$ we denote by

$$V[t_i][[e^{t_i}]] := V \otimes \mathcal{R}[t_1, \dots, t_l][[e^{t_1}, \dots, e^{t_l}]]$$

the space of formal series

$$f = \sum_{d=(d_1,\dots,d_l)\ge 0} f_d e^{td}$$

where f_d is a polynomial in variables $t_1, \dots, t_l$ with coefficients in V. The operator $\frac{\partial}{\partial t_i}$ acts in a natural way on $V[t_i][[e^{t_i}]]$ via

$$\frac{\partial}{\partial t_i}(f_d e^{td}) = (\frac{\partial f_d}{\partial t_i} + d_i f_d) e^{td}.$$

We use the same formula

$$(\sum_{d\ge 0} f_d e^{td})(\sum_{d \ge 0} g_d e^{td}) = \sum_{d\ge 0}(\sum_{d_1+d_2=d} f_{d_1} g_{d_2}) e^{td} \tag{10}$$

in order to define both:

- an action of $M_n(\mathcal{R})[t_i][[e^{t_i}]]$ on $\mathcal{R}^n[t_i][[e^{t_i}]]$ (take $f_d \in M_n(\mathcal{R})[t_i]$, $g_d \in \mathcal{R}^n[t_i]$);
- a multiplication on $M_n(\mathcal{R})[t_i][[e^{t_i}]]$ (take $f_d, g_d \in M_n(\mathcal{R})[t_i]$).

Alternatively, we can use the ring structure of $\mathcal{R}[t_i][[e^{t_i}]]$ induced by the same formula (10) (take $f_d, g_d \in \mathcal{R}[t_i]$), the identifications

$$M_n(\mathcal{R})[t_i][[e^{t_i}]] = M_n(\mathcal{R}[t_i][[e^{t_i}]]),\ \mathcal{R}^n[t_i][[e^{t_i}]] = (\mathcal{R}[t_i][[e^{t_i}]])^n$$

and the usual matrix multiplication rules.

Our aim is to find solutions s of the system (9) in the space $\mathcal{H}[t_i][[e^{t_i}]] = \mathbb{R}^n[t_i][[e^{t_i}]]$, where $\mathcal{H}$ has been identified with $\mathbb{R}^n$ via the basis which is the dual with respect to $(\ ,\ )$ of the original basis (see above). The following result will help us to this end:

Proposition 4.2. *Let $\mathcal{A}_1, \ldots, \mathcal{A}_l \in M_n(\mathcal{R}[e^{t_i}])$ be matrices which satisfy:*

(a) *$\mathcal{A}_i$ commutes with $\mathcal{A}_j$ for any two i, j;*

(b) *$\frac{\partial}{\partial t_i}\mathcal{A}_j = \frac{\partial}{\partial t_j}\mathcal{A}_i$ for any two i, j;*

(c) *for any $i \in \{1, \ldots, l\}$ we can decompose $\mathcal{A}_i$ as*

$$\mathcal{A}_i' + \mathcal{A}_i''(e^{t_j})$$

where $\mathcal{A}_i'$ is strictly lower triangular and its coefficients do not depend on t, and $\mathcal{A}_i''$ is strictly upper triangular, its coefficients being linear combinations of $e^{td} := e^{t_1 d_1} \ldots e^{t_l d_l}$, where $d_1, \ldots, d_l$ are nonnegative integer numbers with

$$d_i > 0.$$

Consider the PDE system

$$\frac{\partial g}{\partial t_i} = \mathcal{A}_i g, \tag{11}$$

$1 \le i \le l$, where $g = \sum_{d \ge 0} g_d e^{td} \in \mathcal{R}^n[t_i][[e^{t_i}]]$. The system has a unique solution g with g_0^0 (the constant term of the polynomial $g_0 \in \mathcal{R}^n[t_i]$) prescribed.

The following elementary lemma will be needed in the proof:

Lemma 4.3. *Let $\mathcal{A} \in M_n(\mathcal{R})$ be a matrix and $g \in \mathcal{R}^n[t]$ a polynomial. Consider the differential equation:*

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \mathcal{A}f + g,$$

where f is in $\mathcal{R}^n[t]$.

(i) If $\mathcal{A}$ is invertible, then we have a unique solution f.

(ii) If $\mathcal{A}$ is nilpotent, then the equation has a unique solution f with the constant term $f_0 \in \mathcal{R}^n$ prescribed.

Proof. Put $g = \sum_{k=0}^{p} g_k t^k$ and look for f as $\sum_{j=0}^{m} f_j t^j$, where $g_k, f_j \in \mathcal{R}^n$. The proof is straightforward. □

Proof of Proposition 4.2. We will prove this result by induction on $l \ge 1$. First take $l = 1$ and solve the equation

$$\frac{\mathrm{d}g}{\mathrm{d}t} = \mathcal{A}_1 g,$$

where $g = \sum_{k \ge 0} g_k e^{tk}$, $g_k \in \mathcal{R}^n[t]$. Decompose $\mathcal{A}_1$ as $\sum_{k \ge 0} (\mathcal{A}_1)^k e^{tk}$, where $(\mathcal{A}_1)^k \in M_n(\mathcal{R})$. Identify the coefficients of e^{tk} and then determine the polynomials $g_0, g_1, g_2, \ldots$ recursively by Lemma 4.3 (notice that the matrix $(\mathcal{A}_1)^0 = \mathcal{A}_1'$ is strictly lower triangular).

The induction step from $l-1$ to l now follows. The idea is to put $\mathcal{S} = \mathcal{R}[t_l][[e^{t_l}]]$ and note that we have

$$\mathcal{R}^n[t_1, \ldots, t_l][[e^{t_1}, \ldots, e^{t_l}]] = \mathcal{S}^n[t_1, \ldots, t_{l-1}][[e^{t_1}, \ldots, e^{t_{l-1}}]]. \tag{12}$$

In other words, the g we are looking for can be written as:

$$g = \sum_{d=(d_1,\dots,d_l)\geq 0} g_d e^{t_1d_1+\dots+t_ld_l} = \sum_{r=(r_1,\dots,r_{l-1})\geq 0} h_r e^{t_1r_1+\dots+t_{l-1}r_{l-1}},$$

where $g_d \in \mathcal{R}^n[t_1,\dots,t_l]$ and $h_r \in \mathcal{R}^n[t_l][[e^{t_l}]][t_1,\dots,t_{l-1}] = \mathcal{S}^n[t_1,\dots,t_{l-1}]$. The identification given by (12) maps $\mathcal{A}_i g$ to $\mathcal{A}_i h$, where the latter $\mathcal{A}_i$ is regarded as an element of $M_n(\mathcal{S}[e^{t_1},\dots,e^{t_{l-1}}])$, $1 \leq i \leq l$.

Our aim is to solve the system

$$\frac{\partial h}{\partial t_i} = \mathcal{A}_i h, \ 1 \leq i \leq l-1,$$

where $h \in \mathcal{S}^n[t_1,\dots,t_{l-1}][[e^{t_1},\dots,e^{t_{l-1}}]]$. The elements

$$\mathcal{A}_1,\dots,\mathcal{A}_{l-1} \text{ of } M_n(\mathcal{S}[e^{t_1},\dots,e^{t_{l-1}}])$$

satisfy the conditions (a), (b) and (c) (with $l-1$ instead of l). By the induction hypothesis, we know that the solution of the latter PDE is uniquely determined by the degree zero term h_0^0 of the polynomial $h_0 \in \mathcal{S}^n[t_1,\dots,t_{l-1}]$. We require that $h_0^0 \in \mathcal{S}^n = \mathcal{R}^n[t_l][[e^{t_l}]]$ is the solution of the equation

$$\frac{\partial h_0^0}{\partial t_l} = \mathcal{A}_l^0 h_0^0 \tag{13}$$

where $\mathcal{A}_l^0 \in M_n(\mathcal{R}[e^{t_l}])$ is the first term of the decomposition

$$\mathcal{A}_l = \sum_{r\geq 0} \mathcal{A}_l^r e^{t_1r_1+\dots+t_{l-1}r_{l-1}}.$$

In order to be more precise, we write

$$h_0^0 = \sum_{k\geq 0} f_k(t_l) e^{kt_l}$$

where $f_k(t_l) \in \mathcal{R}^n[t_l]$, $k \geq 0$, and then we identify the coefficients of e^{t_lk} in both sides of (13). One obtains the following sequence of differential equations:

$$\frac{\mathrm{d}f_k}{\mathrm{d}t_l} + kf_k = (\mathcal{A}_l^0 h_0^0)_k = \sum_{u+v=k} (\mathcal{A}_l^0)^u f_v \tag{14}$$

where $(\mathcal{A}_l^0 h_0^0)_k$ symbolizes the coefficient of e^{kt_l} in $\mathcal{A}_l^0 h_0^0$ and $(\mathcal{A}_l^0)^u \in M_n(\mathcal{R})$ is the coefficient of e^{t_lu} in $\mathcal{A}_l^0 \in M_n(\mathcal{R}[e^{t_l}])$.

We solve the sequence (14) of differential equations by using Lemma 4.3. First we write (14) as:

$$\frac{\mathrm{d}f_k}{\mathrm{d}t_l} + kf_k = (\mathcal{A}_l^0)^0 f_k + b,$$

where $b \in \mathcal{R}^n[t_l]$ depends only on $f_0,\dots,f_{k-1}$. The matrix $(\mathcal{A}_l^0)^0 \in M_n(\mathcal{R})$ is obviously $\mathcal{A}_l'$ (see condition (c)), hence it is strictly lower triangular. A simple recursive procedure provides solutions: specifying only $f_0^0 = g_0^0$ determines first f_0 and then $f_1, f_2, \dots$.

The only thing that remains to be proved is that the g we just constructed satisfies:

$$\frac{\partial g}{\partial t_l} = \mathcal{A}_l g. \tag{15}$$

To this end, we notice first that

$$\frac{\partial}{\partial t_i}(\frac{\partial g}{\partial t_l} - \mathcal{A}_l g) = \mathcal{A}_i(\frac{\partial g}{\partial t_l} - \mathcal{A}_l g),$$

for all $1 \leq i \leq l-1$. Also $\frac{\partial g}{\partial t_l} - \mathcal{A}_l g$ can be written as

$$\frac{\partial g}{\partial t_l} - \mathcal{A}_l g = \sum_{r=(r_1,\dots,r_{l-1})\geq 0} q_r e^{t_1 r_1 + \dots + t_{l-1} r_{l-1}},$$

with $q_r \in \mathcal{R}^n[t_l][[e^{t_l}]][t_1, \dots, t_{l-1}]$. The degree zero term q_0^0 of q_0 is obviously $\frac{\partial h_0^0}{\partial t_l} - \mathcal{A}_l^0 h_0^0$. From the choice of h_0^0 it follows that $q_0^0 = 0$. By the induction hypothesis, q_0^0 determines $\frac{\partial g}{\partial t_l} - \mathcal{A}_l g$ uniquely, hence the latter is zero. $\square$

We apply Proposition 4.2 for $\mathcal{R} = \mathbb{R}$ and $\mathcal{A}_i = \frac{1}{h} A_i$ and deduce:

Corollary 4.4. *For any $a \in \mathcal{H}$ there exists*[2] *$s_a \in \mathcal{H}[t_i][[e^{t_i}]]$ which is a solution of the system (9) and satisfies the condition $(s_a)_0^0 = a$.*

There exists a $\mathbb{R}[t_i][[e^{t_i}]]$-bilinear extension of the product $\circ$ to $\mathcal{H}[t_i][[e^{t_i}]]$. Similarly, the intersection pairing $(\cdot,\cdot)$ can be extended to a $\mathbb{R}[t_i][[e^{t_i}]]$-bilinear map

$$\mathcal{H}[t_i][[e^{t_i}]] \times \mathcal{H}[t_i][[e^{t_i}]] \to \mathbb{R}[t_i][[e^{t_i}]].$$

The differential operator $D(\{e^{t_i}\}, \{h\frac{\partial}{\partial t_i}\}, h)$ acts on $\mathbb{R}[t_i][[e^{t_i}]]$ and $\mathcal{H}[t_i][[e^{t_i}]]$ in an obvious way. This action plays an important role, as we can see in the following lemma:

Lemma 4.5. *(i) Suppose that the differential operator $D(\{e^{t_i}\}, \{h\frac{\partial}{\partial t_i}\}, h)$ satisfies*

$$D.(s_a, 1) = 0 \textit{ for all } a \in \mathcal{H} \textit{ and all } h \neq 0,$$

where s_a is given by Corollary 4.4. Then we have the relation $D(\{q_i\}, \{[\lambda_i]\circ\}, 0) = 0$.

(ii) The following equation holds:

$$H.(s_a, 1) = 0,$$

for all $a \in \mathcal{H}$, where the differential operator H is given by (7).

Proof. (i) For any $a \in \mathcal{H}$ and any $f \in \mathcal{H} \otimes \mathbb{R}[e^{t_i}]$ we have that

$$\begin{aligned} h\frac{\partial}{\partial t_i}(s_a, f) &= (h\frac{\partial}{\partial t_i} s_a, f) + (s_a, h\frac{\partial}{\partial t_i} f) \\ &= (([\lambda_i]\circ)^T s_a, f) + (s_a, h\frac{\partial}{\partial t_i} f) = (s_a, ([\lambda_i]\circ + h\frac{\partial}{\partial t_i}) f). \end{aligned}$$

[2]Proposition 4.2 also says that such an s_a is unique, but we do not need that.

We deduce that

$$(16)\quad D(e^{t_i}, h\frac{\partial}{\partial t_i}, h).(s_a, f)$$
$$= (s_a, D(e^{t_1}, \dots, e^{t_l}, [\lambda_1] \circ +h\frac{\partial}{\partial t_1}, \dots, [\lambda_l] \circ +h\frac{\partial}{\partial t_l}, h).f).$$

Replacing f by 1 and denoting

$$\mathcal{D} = D(e^{t_1}, \dots, e^{t_l}, [\lambda_1] \circ +h\frac{\partial}{\partial t_1}, \dots, [\lambda_l] \circ +h\frac{\partial}{\partial t_l}, h).1,$$

we obtain

$$(17)\qquad (\mathcal{D}, s_a) = 0$$

for all $a \in \mathcal{H}$.

For the rest of the proof, "degree" will refer to the variables $e^{t_1}, \dots, e^{t_l}$. Note that $\mathcal{D}$ is an element of $\mathcal{H} \otimes \mathbb{R}[e^{t_i}]$. Decompose it as

$$\mathcal{D} = \mathcal{D}_0 + \mathcal{D}_1 + \ldots + \mathcal{D}_m,$$

where $\mathcal{D}_k \in \mathcal{H} \otimes \mathbb{R}[e^{t_i}]$ denotes the sum of all monomials of degree k, $0 \leq k \leq m$. Recall that the degree zero term of s_a is the polynomial $(s_a)_0 \in \mathcal{H} \otimes \mathbb{R}[t_1, \dots, t_l]$, with $(s_a)_0^0 = a$. The degree zero term of $(\mathcal{D}, s_a)$ is $(\mathcal{D}_0, (s_a)_0)$. From the vanishing of the latter we obtain that

$$(\mathcal{D}_0, (s_a)_0^0) = (\mathcal{D}_0, a) = 0,$$

for all $a \in \mathcal{H}$, hence $\mathcal{D}_0 = 0$.

Also the sum of the terms of degree 1 in $(\mathcal{D}, s_a)$ is zero. Since $\mathcal{D}_0 = 0$, this implies that

$$(\mathcal{D}_1, (s_a)_0) = 0.$$

As before, we deduce that $\mathcal{D}_1 = 0$. We continue this process and show inductively that $\mathcal{D}_k = 0$, for all $0 \leq k \leq m$, hence

$$\mathcal{D} = 0.$$

Now we let h approach zero and deduce the desired relation:

$$D(q_i, [\lambda_i] \circ, 0) = 0.$$

(ii) When computing $H.(s_a, 1)$ we only need the fact that

$$h^2 \frac{\partial^2}{\partial t_i \partial t_j}(s_a, 1) = (s_a, [\lambda_i] \circ [\lambda_j]),$$

which can be deduced immediately from (16). This implies that

$$H.(s_a, 1) = (s_a, \sum_{i,j=1}^{l} \langle \alpha_i^\vee, \alpha_j^\vee \rangle [\lambda_i] \circ [\lambda_j] - \sum_{i=1}^{l} \langle \alpha_i^\vee, \alpha_i^\vee \rangle e^{t_i}) = 0,$$

where we have used condition (v) from Theorem 1.2. □

Another important step will be made by the following lemma:

Lemma 4.6. (Kim's lemma, see [K]) *Let*[3] $g = g_0 + \sum_{d>0} g_d e^{td} \in \mathbb{R}[t_i][[e^{t_i}]]$ *be a formal series with the properties* $g_0 = 0$ *and* $H.g = 0$. *Then* $g = 0$.

Proof. Suppose $g \neq 0$. Fix $d \in \mathbb{Z}^l$, $d_i \geq 0$, with $g_d \neq 0$ and $|d| := \sum_{i=1}^l d_i > 0$ minimal. From $H.g = 0$ it follows that

$$\sum_{i,j=1}^{l} \langle \alpha_i^\vee, \alpha_j^\vee \rangle \frac{\partial^2}{\partial t_i \partial t_j}(g_d e^{td}) = 0. \tag{18}$$

On the other hand, we have

$$\sum_{i,j=1}^{l} \langle \alpha_i^\vee, \alpha_j^\vee \rangle \frac{\partial^2}{\partial t_i \partial t_j}(e^{td}) = \sum_{i,j=1}^{l} \langle \alpha_i^\vee, \alpha_j^\vee \rangle d_i d_j e^{td} = || \sum_{j=1}^{l} d_j \alpha_j^\vee ||^2 e^{td} > 0.$$

Hence (18) is impossible. □

And now we are in a position to prove Theorem 3.1:
Proof of Theorem 3.1. By Lemma 4.5, it is sufficient to show that

$$g := D.(s_a, 1) = \sum_{d \geq 0} g_d e^{td}$$

equals zero. Taking into account (16), we have that

$$\begin{aligned} g &= (s_a, D(e^{t_1}, \dots, e^{t_l}, [\lambda_1] \circ +h\frac{\partial}{\partial t_1}, \dots, [\lambda_l] \circ +h\frac{\partial}{\partial t_l}, h).1) \\ &= (s_a, D(0, \dots, 0, [\lambda_1]\circ, \dots, [\lambda_l]\circ, h) + R), \end{aligned}$$

where $R \equiv 0 \bmod \{e^{t_i}\}$. Hence the polynomial g_0 must be the same as

$$(s_a, D(0, \dots, 0, [\lambda_1], \dots, [\lambda_l], h))_0$$

(in our notation, the subscript 0 indicates the constant term with respect to $\{e^{t_i}\}$). By conditions (b) and (c), the latter expression is zero, hence $g_0 = 0$. It remains to notice that $H.g = 0$ (which follows from $H.(s_a, 1) = 0$ and $[D, H] = 0$) and apply Lemma 4.6. □

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[3]Here $d > 0$ means $d \geq 0$ and $d \neq 0$.

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