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ON THE LOCAL WELL-POSEDNESS OF THE BENJAMIN-ONO AND MODIFIED BENJAMIN-ONO EQUATIONS

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Abstract. We prove that the Benjamin-Ono equation is locally well-posed in $H^s(\mathbf{R})$ for $s > 9/8$ and that for arbitrary initial data, the modified (cubic nonlinearity) Benjamin-Ono equation is locally well-posed in $H^s(\mathbf{R})$ for $s \geq 1$.

1. Introduction

We consider the initial value problems for the Benjamin-Ono (BO) equation

$$\begin{cases} \partial_t u + H\partial_x^2 u + u\,\partial_x u = 0 \qquad (x,t) \in \mathbf{R}^2 \\ u(x,0) = u_0(x) \end{cases} \tag{1.1}$$

and a modified Benjamin-Ono (mBO) equation (with cubic nonlinearity)

$$\begin{cases} \partial_t u + H\partial_x^2 u + u^2\,\partial_x u = 0 \qquad (x,t) \in \mathbf{R}^2 \\ u(x,0) = u_0(x) \end{cases} \tag{1.2}$$

where H is the Hilbert transform.

The Benjamin-Ono equation models behavior of long internal waves in deep stratified fluids ([B], [O]). Both equations satisfy the conservation laws

$$\begin{aligned} I_1(u) &= \int_{-\infty}^{\infty} u(x,t)\,dx \\ I_2(u) &= \int_{-\infty}^{\infty} u^2(x,t)\,dx \end{aligned} \tag{1.3}$$

and the Benjamin-Ono equation possesses an infinite number of conservation laws, including

$$\begin{aligned} I_3(u) &= \int_{-\infty}^{\infty} \left(|D_x^{\frac{1}{2}} u(x,t)|^2 + \frac{1}{3} u^3(x,t) \right) dx \\ I_4(u) &= \int_{-\infty}^{\infty} \left((\partial_x u)^2(x,t) + \frac{3}{4}(u^2 D_x u)(x,t) + \frac{1}{8} u^4(x,t) \right) dx. \end{aligned} \tag{1.4}$$

The Benjamin-Ono equation has global weak solutions in $L^2(\mathbf{R})$, $H^{\frac{1}{2}}(\mathbf{R})$, and $H^1(\mathbf{R})$ ([GV], [To], [Sa]), and it has been known for some time that it is globally

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well-posed in $H^s(\mathbf{R})$ for $s \geq 3/2$ (see [P] and the references therein). Recently, H. Koch and N. Tzvetkov [KoTz] proved local well-posedness for $s > 5/4$ by a substantially simpler argument. Our first result is the following improvement:

Theorem 1.1. *Let $s > 9/8$. For any $u_0 \in H^s(\mathbf{R})$, there exists $T \gtrsim \|u_0\|_{H^s}^{-4}$ and a unique solution u of the Benjamin-Ono equation (1.1) satisfying*

$$u \in C([0,T] : H^s(\mathbf{R})) \qquad \text{and} \quad \partial_x u \in L^1([0,T] : L^\infty(\mathbf{R})).$$

Moreover, for any $R > 0$, the map $u_0 \mapsto u(t)$ is continuous from the ball $\{u_0 \in H^s(\mathbf{R}) : \|u_0\|_{H^s} < R\}$ to $C([0,T] : H^s(\mathbf{R}))$.

Observe that a desirable goal is to extend local well-posedness to $s \geq 1$, since global well-posedness would then hold in $H^1(\mathbf{R})$ due to the conservation law (1.4). Just as this paper was completed, T. Tao ([Ta]) announced a proof of this global well-posedness in $H^1(\mathbf{R})$, by performing an appropriate gauge transformation that eliminates the derivative (on high-frequency components) in the nonlinear term. Nevertheless, we expect that the simplicity of our argument for the range $s > 9/8$, and its wide scope of applicability, should be of independent interest. For instance, a very similar proof to the one given here yields, for the "dispersion-generalized" Benjamin-Ono equation

$$\begin{cases} \partial_t u + \partial_x D_x^{1+a} u + u\,\partial_x u = 0 & (x,t) \in \mathbf{R}^2, \quad 0 \leq a < 1 \\ u(x,0) = u_0(x) \end{cases}$$

local well-posedness in $H^s(\mathbf{R})$ for $s > \frac{9}{8} - \frac{3a}{8}$. This improves the best previously known result given in [KPV1], where local well-posedness was proved for $s > \frac{3}{2} - \frac{3a}{4}$. Since these equations are not completely integrable (for $0 < a < 1$), it is unclear whether Tao's gauge transformation applies to them.

For the modified Benjamin-Ono equation (1.2), it has been known that it is locally well-posed for $s > 3/2$ ([I]) and, for small initial data, for $s > 1$ ([KPV3]). Also very recently, the latter result (i.e. for small data) was extended to $s > 1/2$ by L. Molinet and F. Ribaud ([MR]). We show that for arbitrary initial data, mBO is locally well-posed in $H^s(\mathbf{R})$ for $s \geq 1$.

Theorem 1.2. *Let $s \geq 1$. For any $u_0 \in H^s(\mathbf{R})$, there exists $T \gtrsim \min(1, \|u_0\|_{H^1}^{-25})$ and a unique solution u of the modified Benjamin-Ono equation (1.2) satisfying*

$$u \in C([0,T] : H^s(\mathbf{R})) \qquad \text{and} \quad \partial_x u \in L^4([0,T] : L^\infty(\mathbf{R})).$$

Moreover, for any $R > 0$, the map $u_0 \mapsto u(t)$ is continuous from the ball $\{u_0 \in H^s(\mathbf{R}) : \|u_0\|_{H^s} < R\}$ to $C([0,T] : H^s(\mathbf{R}))$.

Our method is to refine the energy method and smoothing effect approach (such as in [KPV1], taking advantage of the Christ-Kiselev lemma 2.5 below). It is worth pointing out that it is not possible to use the contraction principle to prove local well-posedness in $H^s(\mathbf{R})$ for the Benjamin-Ono equation ([MSaTz]). On the other hand, the results of Kenig-Ponce-Vega and Molinet-Ribaud cited above for modified Benjamin-Ono were proved by contraction methods. These cannot apply for $s < 1/2$, since mBO is not C^3 well-posed in this range ([MR]).

The following notation will be used throughout this article: $D^s = (-\Delta)^{s/2}$ and $J^s = (I-\Delta)^{s/2}$ denote the Riesz and Bessel potentials of order $-s$, respectively. We write

$$\begin{aligned}\|f\|_{L^p_x L^q_t} &= \big\|\|f(x,\cdot)\|_{L^q(\mathbf{R})}\big\|_{L^p(\mathbf{R})}\\ \|f\|_{L^p_x L^q_T} &= \big\|\|f(x,\cdot)\|_{L^q([0,T])}\big\|_{L^p(\mathbf{R})}\end{aligned}$$

with similar definitions for $L^q_t L^p_x$ and $L^q_T L^p_x$. Finally, we say $A \lesssim B$ if there exists a constant $c>0$ such that $A \le cB$ (it will be clear from context what parameters c may depend on).

2. Linear estimates and local smoothing

In this section, we provide the linear estimates and local smoothing properties for solutions to BO and mBO. Consider the corresponding linear IVP

$$\begin{cases}\partial_t v + H\partial_x^2 v = 0 & (x,t)\in\mathbf{R}^2\\ v(x,0)=v_0(x)\end{cases} \tag{2.1}$$

and

$$\begin{cases}\partial_t w + H\partial_x^2 w = f & (x,t)\in\mathbf{R}^2\\ w(x,0)=0\end{cases} \tag{2.2}$$

whose solutions are given by

$$\begin{aligned}v(x,t) &= S(t)v_0(x)\\ w(x,t) &= \int_0^t S(t-t')f(\cdot,t')\,dt'\end{aligned}$$

where $S(t)u_0(x) = c\int_{-\infty}^{\infty} e^{ix\xi}\, e^{it\xi|\xi|}\hat{u}_0(\xi)\,d\xi$.

We first state the standard Strichartz estimate and sharp Kato smoothing effect for the unitary group $\{S(t)\}_{t\in\mathbf{R}}$ (see e.g. [KPV1]).

Lemma 2.1. *For $v_0\in L^2$,*

$$\|S(t)v_0\|_{L^q_t L^p_x} \le c\|v_0\|_{L^2} \quad \textit{with } (q,p) = \left(\frac{4}{\theta},\frac{2}{1-\theta}\right) \tag{2.3}$$

for any $\theta\in[0,1]$, and

$$\|D_x^{1/2}S(t)v_0\|_{L^\infty_x L^2_t} = c\|v_0\|_{L^2}. \tag{2.4}$$

The following version of the local smoothing effect for solutions to BO was established in [KPV1]; a careful examination of the proof[1] shows that the result extends (as stated below) to solutions of mBO as well.

[1] Here it is necessary to use the energy estimate for mBO given in Lemma 2.7.

Lemma 2.2. *Fix $\delta > 0$, and let I be an interval of unit length. If $u \in C([0,T] : H^{s+1}(\mathbf{R}))$ is a solution to BO, then*

$$\left(\int_0^T\!\!\int_I |D_x^{s-\frac{1}{2}}\partial_x u(x,t)|^2\,dx\,dt\right)^{1/2} \le c\|u_0\|_{H^s}\left(1+T+T\|u_0\|_{H^{\frac{1}{2}+\delta}}+\|\partial_x u\|_{L^1_T L^\infty_x}\right) \cdot \exp(c\|\partial_x u\|_{L^1_T L^\infty_x}) \tag{2.5}$$

for $s \ge \frac{1}{2}$. Similarly, if u is a solution to mBO, then

$$\left(\int_0^T\!\!\int_I |D_x^{s-\frac{1}{2}}\partial_x u(x,t)|^2\,dx\,dt\right)^{1/2} \le c\|u_0\|_{H^s}\left(1+T+(1+T)\|u_0\|_{H^{\frac{1}{2}+\delta}}+\|\partial_x u\|_{L^1_T L^\infty_x}\right) \cdot \exp(c\|u\|^2_{L^2_T L^\infty_x})\exp(c\|\partial_x u\|^2_{L^2_T L^\infty_x}). \tag{2.6}$$

Next we recall the maximal function estimate proved in [KPV1].

Lemma 2.3. *Assume $v_0 \in H^{\frac{1}{2}+\delta}$ for some fixed $\delta > 0$. Then*

$$\|S(t)v_0\|_{L^2_x L^\infty_T} \le \left(\sum_{j=-\infty}^{\infty}\|S(t)v_0\|^2_{L^\infty([j,j+1)\times[0,T])}\right)^{1/2} \le c(1+T)\|v_0\|_{H^{\frac{1}{2}+\delta}}. \tag{2.7}$$

Using duality arguments and complex interpolation, we combine the Kato-type smoothing and maximal function estimates to obtain additional linear estimates needed in the proof of Theorem 1.2.

Lemma 2.4.
(a) *For $T \in [0,1]$,*

$$\left\|\partial_x \int_0^t S(t-t')f(\cdot,t')\,dt'\right\|_{L^4_T L^\infty_x} \le c\|D_x^{1/2}f\|_{L^1_x L^2_T}. \tag{2.8}$$

(b) *For $\delta, \theta, T \in [0,1]$ and $\varepsilon > 0$,*

$$\|D_x^\theta S(t)v_0\|_{L_x^{2/(1-\theta)}L_T^{2/\theta}} \le c\|D_x^{\frac{1}{2}+\varepsilon}v_0\|_{L^2} + c\|v_0\|_{L^2} \tag{2.9}$$

and

$$\|D_x^\theta \int_0^t S(t-t')f(\cdot,t')\,dt'\|_{L_x^{2/(1-\theta)}L_T^{2/\theta}} \le cT^{\frac{\delta(1-\theta)}{2}}\|D_x^{\frac{\delta(1-\theta)}{2}+\varepsilon}f\|_{L^p_x L^2_T} + cT^{1/2}\|f\|_{L^2_x L^2_T} \tag{2.10}$$

where $p = p(\delta,\theta) = \frac{2}{2-\delta(1-\theta)}$.

A useful lemma of Christ and Kiselev ([ChKi]) allows one to deduce the inequalities (2.8) and (2.10) from the corresponding "nonretarded" ones. The version of this lemma that we use is the one presented and proved in [MR], [SmSo].

Lemma 2.5. *Let* $\mathcal{T}f(t) = \int_{-\infty}^{\infty} K(t,t')f(t')\,dt'$ *be a linear operator, where* $K : \mathcal{S}(\mathbf{R}^2) \to C(\mathbf{R}^3)$. *Assume that* $\|\mathcal{T}f\|_{L_x^{p_1} L_t^{q_1}} \le c\|f\|_{L_x^{p_2} L_t^{q_2}}$ *for some* $p_1, p_2, q_1, q_2 \in [1,\infty]$ *with*

$$\min(p_1, q_1) > \max(p_2, q_2). \tag{2.11}$$

Then

$$\left\| \int_0^t K(t,t')f(t')\,dt' \right\|_{L_x^{p_1} L_t^{q_1}} \le c\|f\|_{L_x^{p_2} L_t^{q_2}}.$$

A similar result[2] *holds for the spaces* $L_t^{q_1} L_x^{p_1}$ *and/or* $L_t^{q_2} L_x^{p_2}$ *instead of* $L_x^{p_1} L_t^{q_1}$, $L_x^{p_2} L_t^{q_2}$. *Moreover, the case* $q_1 = \infty$; $p_2, q_2 < \infty$ *is allowed even if (2.11) fails; more precisely, if* $T \le 1$ *and* $\|\mathcal{T}f\|_{L_x^{p_1} L_T^{\infty}} \le c\|f\|_{L_x^{p_2} L_T^{q_2}}$ *with* $p_2 \ne \infty$, $q_2 \ne \infty$, *then*

$$\left\| \int_0^t K(t,t')f(t')\,dt' \right\|_{L_x^{p_1} L_T^{\infty}} \le c\|f\|_{L_x^{p_2} L_T^{q_2}}.$$

Proof of Lemma 2.4.
(a) A standard TT^* argument using Lemma 2.1 yields the linear estimate

$$\left\| \partial_x \int_{-\infty}^{\infty} S(t-t')f(\cdot,t')\,dt' \right\|_{L_t^4 L_x^{\infty}} \le c\|D_x^{1/2} f\|_{L_x^1 L_t^2}.$$

Indeed, the $L_t^4 L_x^{\infty}$ Strichartz estimate and smoothing effect (2.4) imply that

$$\begin{aligned}
&\left\| S(t) \int_{-\infty}^{\infty} H D_x^{1/2} S(-t')(D_x^{1/2} f)(t')\,dt' \right\|_{L_t^4 L_x^{\infty}} \\
\lesssim\; &\left\| \int_{-\infty}^{\infty} D_x^{1/2} S(-t')(D_x^{1/2} f)(t')\,dt' \right\|_{L_x^2} \\
\lesssim\; &\sup_{\|g\|_{L^2} \le 1} \left| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} D_x^{1/2} S(-t')(D_x^{1/2} f)(x,t') g(x)\,dt'\,dx \right| \\
\lesssim\; &\sup_{\|g\|_{L^2} \le 1} \left| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} D_x^{1/2} f(x,t') D_x^{1/2} S(t') g(x)\,dx\,dt' \right| \\
\lesssim\; &\sup_{\|g\|_{L^2} \le 1} \|D_x^{1/2} f\|_{L_x^1 L_t^2} \, \|D_x^{1/2} S(t) g\|_{L_x^{\infty} L_t^2} \le \|D_x^{1/2} f\|_{L_x^1 L_t^2}.
\end{aligned}$$

Therefore, by Lemma 2.5,

$$\left\| \partial_x \int_0^t S(t-t')f(\cdot,t')\,dt' \right\|_{L_t^4 L_x^{\infty}} \le c\|D_x^{1/2} f\|_{L_x^1 L_t^2} \tag{2.12}$$

which in turn implies (2.8).

[2]The $L_t^{q_1} L_x^{p_1}$, $L_x^{p_2} L_t^{q_2}$ (resp. $L_x^{p_1} L_t^{q_1}$, $L_t^{q_2} L_x^{p_2}$; or $L_t^{q_1} L_x^{p_1}$, $L_t^{q_2} L_x^{p_2}$) version of the Christ-Kiselev lemma holds with the condition $q_1 > \max(p_2, q_2)$ (resp. $\min(p_1, q_1) > q_2$; or $q_1 > q_2$) instead of (2.11).

(b) The proof is in part similar[3] to the one in [KPV3] that showed

$$\|D_x^{\frac{3\theta-1}{4}} S(t)v_0\|_{L_x^{4/(1-\theta)} L_t^{2/\theta}} \le c\|v_0\|_{L^2}$$

and

$$\|D_x^{\frac{5\theta}{4}} \int_0^t S(t-t')f(\cdot,t')\,dt'\|_{L_x^{4/(1-\theta)} L_t^{2/\theta}} \le cT^{\frac{1-\theta}{2}} \|D_x^{1/4} f\|_{L_x^{2/(1+\theta)} L_T^2}. \tag{2.13}$$

To adapt the argument to the operator $D_x^{1/2}$ (instead of $D_x^{1/4}$), one interpolates the smoothing effect

$$\|D_x^{1+i\alpha} S(t)v_0\|_{L_x^\infty L_t^2} \le c\|v_0\|_{H^{\frac{1}{2}}}$$

with

$$\|D_x^{i\alpha} S(t)v_0\|_{L_x^2 L_T^\infty} \le c\|v_0\|_{H^{\frac{1}{2}+\varepsilon}}$$

to obtain (2.9). For (2.10), we decompose f into small and large frequencies (using smooth cutoffs) so that $f = f_1 + f_2$ with $\operatorname{supp} \hat{f}_1 \subset \{|\xi| < 2\}$ and $\operatorname{supp} \hat{f}_2 \subset \{|\xi| > 1\}$. Recall that for any fixed $\kappa > 0$, there exist finite measures μ, ν such that $|\xi|^\kappa = (1+|\xi|^2)^{\kappa/2}\hat{\mu}(\xi)$ and $(1+|\xi|^2)^{\kappa/2} = (1+|\xi|^\kappa)\hat{\nu}(\xi)$ (see for example [St], pp. 133–134). Therefore, combining the inequalities $\|D_x^{\frac{1}{2}} J^{i\alpha} S(t)u_0\|_{L_x^\infty L_T^2} \le c\|u_0\|_{L^2}$ and $\|S(t)u_0\|_{L_x^2 L_T^\infty} \le c\|J^{\frac{1}{2}+\varepsilon}u_0\|_{L^2}$ yields (by a TT^* argument as above)

$$\left\| J_x^{i\alpha-\varepsilon} \int_{-\infty}^{\infty} S(t-t')f_2(\cdot,t')\,dt' \right\|_{L_x^2 L_T^\infty} \lesssim \|f_2\|_{L_x^1 L_T^2}$$

and hence

$$\left\| J_x^{i\alpha-\varepsilon} \int_0^t S(t-t')f_2(\cdot,t')\,dt' \right\|_{L_x^2 L_T^\infty} \lesssim \|f_2\|_{L_x^1 L_T^2} \tag{2.14}$$

by Lemma 2.5. On the other hand, by Lemma 2.3,

$$\left\| J_x^{-\frac{1}{2}-\varepsilon+i\alpha} \int_0^t S(t-t')f_2(\cdot,t')\,dt' \right\|_{L_x^2 L_T^\infty} \lesssim T^{1/2}\|f_2\|_{L_x^2 L_T^2}. \tag{2.15}$$

Interpolating (2.14) and (2.15) gives

$$\left\| J_x^{-\frac{\delta}{2}-\varepsilon+i\alpha} \int_0^t S(t-t')f_2(\cdot,t')\,dt' \right\|_{L_x^2 L_T^\infty} \lesssim T^{\frac{\delta}{2}}\|f_2\|_{L_x^{2/(2-\delta)} L_T^2} \tag{2.16}$$

for $\delta \in [0,1]$.

[3]The direct analogue of (2.13) is

$$\|D_x^{\frac{3\theta}{2}} \int_0^t S(t-t')f(\cdot,t')\,dt'\|_{L_x^{2/(1-\theta)} L_T^{2/\theta}} \le cT^{\frac{1-\theta}{2}} \|D_x^{\frac{1}{2}+\varepsilon} f\|_{L_x^{2/(1+\theta)} L_T^2} + cT^{1/2}\|f\|_{L_x^2 L_T^2}.$$

but the extra ε-derivative in the $\|D_x^{\frac{1}{2}+\varepsilon} f\|_{L_x^{2/(1+\theta)} L_T^2}$ term causes difficulties when this estimate is applied with $f = u^2\partial_x u$ (see (4.2) below).

Now we also know the following analogue of the smoothing effect (2.4):

$$\left\| J_x^{1+i\alpha} \int_0^t S(t-t') f_2(\cdot, t')\, dt' \right\|_{L_x^\infty L_T^2} \lesssim \|f_2\|_{L_x^1 L_T^2} \tag{2.17}$$

(see (2.9) in [KPV3], where (2.17) is proved with $D_x^{1+i\alpha}$ instead of $J_x^{1+i\alpha}$), so interpolating (2.16) and (2.17) yields the desired result (2.10) for f_2 (with just the first term on the right-hand side). For f_1, it is easiest to estimate the left-hand side of (2.10) directly, using the result (2.9) already established. Thus

$$\begin{aligned}
\left\| D_x^\theta \int_0^t S(t-t') f_1(\cdot, t')\, dt' \right\|_{L_x^{2/(1-\theta)} L_T^{2/\theta}} &\le \int_0^T \left\| D_x^\theta S(t) S(-t') f_1(\cdot, t') \right\|_{L_x^{2/(1-\theta)} L_T^{2/\theta}} dt' \\
&\lesssim \int_0^T (\|D_x^{\frac{1}{2}+\varepsilon} f_1\|_{L_x^2} + \|f_1\|_{L_x^2})\, dt \\
&\lesssim T^{1/2} \|f_1\|_{L_x^2 L_T^2} \\
&\lesssim T^{1/2} \|f\|_{L_x^2 L_T^2}.
\end{aligned}$$

□

Remark: By (2.7) and the proof of the last statement in Lemma 2.5, we obtain the following analogue of (2.16): for any $\delta < 1$, there exists $p \in (1,2)$ such that

(2.18)
$$\left\| \int_0^t S(t-t') f(\cdot, t')\, dt' \right\|_{\ell_j^2(L^\infty([j,j+1)\times[0,T]))} \lesssim T^{\frac{\delta}{2}} \|D_x^{\frac{1}{2}} f\|_{L_x^p L_T^2} + T^{1/2} \|f\|_{L_x^2 L_T^2}.$$

To analyze the products that arise from the nonlinear term of the BO and mBO equations, we require the following Leibniz rules for fractional derivatives. For detailed proofs of these facts, see [KPV2].

Lemma 2.6.
(a) *Let* $\alpha = \alpha_1 + \alpha_2 \in (0,1)$ *with* $\alpha_i \in (0, \alpha)$, $p \in [1, \infty)$, *and* $p_1, p_2 \in (1, \infty)$ *such that* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. *Then*

$$\|D^\alpha (fg) - f D^\alpha g - g D^\alpha f\|_{L^p} \le c \|D^{\alpha_1} f\|_{L^{p_1}} \|D^{\alpha_2} g\|_{L^{p_2}}.$$

Moreover, if $p > 1$, *then the case* $\alpha_2 = 0$, $1 < p_2 \le \infty$ *is also allowed.*
(b) *Let* $\alpha = \alpha_1 + \alpha_2 \in (0,1)$ *with* $\alpha_i \in [0, \alpha]$, *and let* $p, p_1, p_2, q, q_1, q_2 \in (1, \infty)$ *such that* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ *and* $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. *Then*

$$\|D_x^\alpha (fg) - f D_x^\alpha g - g D_x^\alpha f\|_{L_x^p L_t^q} \le c \|D_x^{\alpha_1} f\|_{L_x^{p_1} L_t^{q_1}} \|D_x^{\alpha_2} g\|_{L_x^{p_2} L_t^{q_2}}.$$

Moreover, the following additional cases are allowed: $(\alpha_1, q_1) = (0, \infty)$; $(p, q) = (1, 2)$; *and* $q = 1$, *provided that* $\alpha_i \in (0, \alpha)$.

We remark that all of these results remain valid with $\tilde{D}_x = H D_x$ instead of D_x.

Next we turn to the energy estimates satisfied by solutions to the Benjamin-Ono equations (1.1) and (1.2). Note that their L^2 norms are preserved by the second conservation law (1.3).

Lemma 2.7.
(a) *Let* $s \geq 0$ *and* $u \in C([0,T] : H^{s+2}(\mathbf{R}))$ *be a solution to the IVP (1.1) for the Benjamin-Ono equation. Then*

$$\sup_{t\in[0,T]} \|u(t)\|_{H^s} \leq c\|u_0\|_{H^s} \exp\left(c\int_0^T \|\partial_x u(t)\|_{L^\infty}\,dt\right) \leq c\|u_0\|_{H^s} \exp(cT^{1/2}\|\partial_x u\|_{L^2_T L^\infty_x}). \tag{2.19}$$

(b) *Let* $s \geq 0$ *and* $u \in C([0,T] : H^{s+2}(\mathbf{R}))$ *be a solution to the IVP (1.2) for the modified Benjamin-Ono equation. Then*

$$\begin{aligned}\sup_{t\in[0,T]} \|u(t)\|_{H^s} &\leq c\|u_0\|_{H^s} \exp\left(c\int_0^T \|u(t)\|^2_{L^\infty}\,dt\right)\exp\left(c\int_0^T \|\partial_x u(t)\|^2_{L^\infty}\,dt\right)\\ &\leq c\|u_0\|_{H^s} \exp\left(cT^{1/2}\|u\|^2_{L^4_T L^\infty_x}\right)\exp\left(cT^{1/2}\|\partial_x u\|^2_{L^4_T L^\infty_x}\right).\end{aligned} \tag{2.20}$$

Proof. Part (a) of the lemma is contained in [KPV1], [P]; we indicate the changes needed in the cubic nonlinearity case (b). Differentiating the equation (1.2), we have

$$\partial_t D^s_x u + H\partial^2_x D^s_x u + \left(D^s_x(u^2\partial_x u) - u^2 D^s_x\partial_x u - (D^s_x u^2)\partial_x u\right) + u^2\partial_x D^s_x u + (D^s_x u^2)\partial_x u = 0.$$

Multiplying by $D^s_x u$, integrating by parts, and applying Lemma 2.6 yields

$$\begin{aligned}\frac{d}{dt}\|D^s_x u(t)\|^2_{L^2} &\lesssim \|D^s_x(u^2\partial_x u) - u^2 D^s_x\partial_x u - (D^s_x u^2)\partial_x u\|_{L^2_x}\,\|D^s_x u\|_{L^2_x} \,+\\ &\quad + \|\partial_x(u^2)\|_{L^\infty_x}\,\|D^s_x u\|^2_{L^2_x} \,+\, \|\partial_x u\|_{L^\infty_x}\,\|D^s_x(u^2)\|_{L^2_x}\,\|D^s_x u\|_{L^2_x}\\ &\lesssim \|D^s_x(u^2)\|_{L^2_x}\,\|\partial_x u\|_{L^\infty_x}\,\|D^s_x u\|_{L^2_x} + \|\partial_x(u^2)\|_{L^\infty_x}\,\|D^s_x u\|^2_{L^2_x}\\ &\lesssim \|u\|_{L^\infty_x}\,\|\partial_x u\|_{L^\infty_x}\,\|D^s_x u\|^2_{L^2_x}.\end{aligned}$$

By Gronwall's inequality,

$$\begin{aligned}\|D^s_x u\|^2_{L^\infty_T L^2_x} &\leq c\|D^s_x u_0\|^2_{L_2} \exp\left(c\int_0^T \|u\|_{L^\infty_x}\,\|\partial_x u\|_{L^\infty_x}\,dt\right)\\ &\leq c\|D^s_x u_0\|^2_{L^2} \exp\left(c\int_0^T \left(\|u\|^2_{L^\infty_x} + \|\partial_x u\|^2_{L^\infty_x}\right)dt\right).\end{aligned}$$

□

Finally, we give the key linear estimate used in the proof of Theorem 1.1 that reformulates and generalizes the one given by Koch and Tzvetkov ([KoTz]) in their demonstration of local well-posedness of BO for $s > 5/4$.

Proposition 2.8. *Let $\alpha \in [0,1]$ and $T \in (0,1]$. Assume $w \in C([0,T] : H^3(\mathbf{R}))$ is a solution*[4] *to the linear equation*

$$\partial_t w + H\partial_x^2 w = F.$$

For any $\varepsilon > 0$,

$$\|\partial_x w\|_{L^2_T L^\infty_x} \le c\|D_x^{1+\frac{\alpha}{4}+\varepsilon} w\|_{L^\infty_T L^2_x} + c\|D_x^{1-\frac{3\alpha}{4}+\varepsilon} F\|_{L^2_T L^2_x} + c\|w\|_{L^\infty_T L^2_x} + c\|F\|_{L^2_T L^2_x}.$$

Remarks:

(1) We take $\alpha = 1/2$ in the proof of Theorem 1.1, which is the optimal choice of parameter in our argument. Indeed, given a linear estimate of the form

$$\|\partial_x w\|_{L^2_T L^\infty_x} \lesssim \|D_x^a w\|_{L^\infty_T L^2_x} + \|D_x^b F\|_{L^2_T L^2_x} + \|w\|_{L^\infty_T L^2_x} + \|F\|_{L^2_T L^2_x} \tag{2.21}$$

we want to apply the smoothing effect (2.5) and "absorb" as many derivatives as possible on F; this approach requires that $a = b + \frac{1}{2}$. Thus local well-posedness of BO holds for $s \ge a$ whenever (2.21) holds such a pair of exponents (a, b). However, we have been kindly informed by L. Vega that such an interpolation-type estimate fails for any $a < \frac{9}{8}$.

(2) Koch and Tzvetkov ([KoTz]) consider instead the linearized BO equation $\partial_t w + H\partial_x^2 w + V\partial_x w = F$. The strength of their estimate corresponds to the case $\alpha = 1$ in the version considered here, which explains our improvement of their result by an $\frac{1}{8}$-derivative.

Proof of Proposition 2.8. Let $g = \sum_\lambda g_\lambda$ denote a Littlewood-Paley decomposition of a function g (in the frequency variable dual to x), where g_λ has frequency $\sim \lambda > 1$ and the sum is taken over all dyadic integers. More precisely, choose $\eta \in C_0^\infty(\frac{1}{2} < |\xi| < 2)$ and $\chi \in C_0^\infty(|\xi| < 2)$ such that $1 = \sum_{k=1}^\infty \eta(2^{-k}\xi) + \chi(\xi)$, and for $\lambda = 2^k$, define $g_\lambda = Q_k(g)$ where $\widehat{Q_0 g}(\xi) = \chi(\xi)\hat{g}(\xi)$ and $\widehat{Q_k g}(\xi) = \eta(2^{-k}\xi)\hat{g}(\xi)$ for $k \ge 1$. Recall that for $1 < p < \infty$, $\|g\|_{L^p} \approx \left\|\left(\sum_k |Q_k g|^2\right)^{1/2}\right\|_{L^p} = \left\|\left(\sum_\lambda |g_\lambda|^2\right)^{1/2}\right\|_{L^p}$. Fix $\varepsilon > 0$. For $p > 1/\varepsilon > 2$, we have $\|g\|_{L^\infty} \lesssim \|J^\varepsilon g\|_{L^p} \approx \left\|\left(\sum_\lambda |J^\varepsilon g_\lambda|^2\right)^{1/2}\right\|_{L^p} = \left\|\sum_\lambda |J^\varepsilon g_\lambda|^2\right\|_{L^{p/2}}^{1/2} \lesssim \left(\sum_\lambda \|J^\varepsilon g_\lambda\|_{L^p}^2\right)^{1/2}$, so $\|g\|_{L^2_T L^\infty_x} \lesssim \left(\sum_\lambda \|J^\varepsilon g_\lambda\|_{L^2_T L^p_x}^2\right)^{1/2}$.

Clearly then it suffices to show that for $p > 2$,

$$\|\partial_x w_\lambda\|_{L^2_T L^p_x} \lesssim \|D_x^{1+\frac{\alpha}{4}-\frac{\alpha}{2p}} w_\lambda\|_{L^\infty_T L^2_x} + \|D_x^{1-\frac{3\alpha}{4}-\frac{\alpha}{2p}} F_\lambda\|_{L^2_T L^2_x}$$

for any frequency $\lambda = 2^k$ with $k \ge 1$. (The case $k = 0$ is easily handled using Lemma 2.1 and Hölder's inequality, yielding the last two terms in (2.21).) Fix such $\lambda \ge 2$, and observe that $\partial_t w_\lambda + H\partial_x^2 w_\lambda = F_\lambda$.

Consider a partition $[0,1] = \cup I_j$, $I_j = [a_j, b_j]$ of the unit interval into subintervals of length $|I_j| \sim \lambda^{-\alpha}$, so that $T = b_j$ for some j, and note that there are

[4]Here we are establishing an *a priori* estimate for solutions w having at least two derivatives in L^2_x, with bounds that are independent of the H^3_x-norms of w and F.

$O(\lambda^\alpha)$ of them. (For the last subinterval and the one with endpoint T, we require only that the length be at least $\lambda^{-\alpha}$ and at most $2\lambda^{-\alpha}$; the other subintervals can be taken to have length $\lambda^{-\alpha}$.)

For $p > 2$, set $q = 4 + \frac{8}{p-2} = \frac{4p}{p-2}$. Applying Hölder's inequality, Lemma 2.1, and Jensen's inequality, we obtain (summing over indices j such that $b_j \leq T$)

$$\begin{aligned}
\|\partial_x w_\lambda\|_{L^2_T L^p_x} &= \left(\sum_j \|\partial_x w_\lambda\|^2_{L^2_{I_j} L^p_x} \right)^{1/2} \\
&\leq \lambda^{-\alpha(\frac{1}{2}-\frac{1}{q})} \left(\sum_j \|\partial_x w_\lambda\|^2_{L^q_{I_j} L^p_x} \right)^{1/2} \\
&\lesssim \lambda^{-\frac{\alpha}{2}+\frac{\alpha}{q}} \sum_j \left(\|S(t-a_j)\partial_x w_\lambda(a_j)\|^2_{L^q_{I_j} L^p_x} + \left\| \int_{a_j}^t S(t-t')\partial_x F_\lambda(t')dt' \right\|^2_{L^q_{I_j} L^p_x} \right)^{1/2} \\
&\lesssim \lambda^{-\frac{\alpha}{2}+\frac{\alpha}{q}} \left(\sum_j \|\partial_x w_\lambda(a_j)\|^2_{L^2_x} + \sum_j \left(\int_{I_j} \|\partial_x F_\lambda\|_{L^2_x} dt \right)^2 \right)^{1/2} \\
&\lesssim \lambda^{-\frac{\alpha}{2}+\frac{\alpha}{q}} \left(\sum_j \|\partial_x w_\lambda\|^2_{L^\infty_T L^2_x} \right)^{1/2} + \lambda^{-\frac{\alpha}{2}+\frac{\alpha}{q}} \left(\sum_j \lambda^{-2\alpha}\lambda^{\alpha} \int_{I_j} \|\partial_x F_\lambda\|^2_{L^2_x} dt \right)^{1/2} \\
&\lesssim \lambda^{-\frac{\alpha}{2}+\frac{\alpha}{q}} \cdot \lambda^{\frac{\alpha}{2}} \|\partial_x w_\lambda\|_{L^\infty_T L^2_x} + \lambda^{-\alpha+\frac{\alpha}{q}} \left(\int_0^T \|\partial_x F_\lambda\|^2_{L^2_x} dt \right)^{1/2} \\
&\lesssim \|D_x^{1+\frac{\alpha}{q}} w_\lambda\|_{L^\infty_T L^2_x} + \|D_x^{1-\alpha+\frac{\alpha}{q}} F_\lambda\|_{L^2_T L^2_x}.
\end{aligned}$$

Since $\frac{\alpha}{q} = \frac{\alpha(p-2)}{4p} = \frac{\alpha}{4} - \frac{\alpha}{2p}$ and $1 - \alpha + \frac{\alpha}{q} = 1 - \frac{3\alpha}{4} - \frac{\alpha}{2p}$, we are done. (Note that when $p > 1/\varepsilon$ and $\alpha \leq 1$, we have $\varepsilon - \frac{\alpha}{2p} > \frac{1}{p} - \frac{1}{2p} = \frac{1}{2p} > 0$.)

□

3. Proof of Theorem 1.1

Fix $s > 9/8$, and set $\varepsilon = s - 9/8 > 0$. Without loss of generality, we may assume that

$$\Lambda := \|u_0\|_{L^2} + \|D_x^{9/8+\varepsilon} u_0\|_{L^2} \leq \delta$$

for $\delta > 0$ small enough (to be specified later). Indeed, if $u(x,t)$ is a solution to (1.1), then $u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t)$ is also a solution with initial data $\lambda u_0(\lambda x)$. Since $\|\lambda u_0(\lambda x)\|_{L^2} = \lambda^{\frac{1}{2}}\|u_0\|_{L^2}$ and $\|\lambda D^s(u_0(\lambda x))\|_{L^2} = \lambda^{\frac{1}{2}+s}\|D^s u_0\|_{L^2}$, we can always reduce to the case of small initial data by rescaling. Moreover, observe that if there exists a solution $u \in C([0,1] : H^s(\mathbf{R}))$ whenever $\|u_0\|_{H^s} \lesssim \delta$, then for arbitrary data there exists a solution for time $T \gtrsim \|u_0\|_{H^s}^{-4}$. In the following, we always assume that $T \leq 1$, and we frequently apply the Cauchy-Schwarz inequality in the t-integrations without further comment.

In view of the energy estimate (2.19), the key point is to establish *a priori* control of $\|\partial_x u\|_{L^2_T L^\infty_x}$. Once this is done, the proofs of existence, uniqueness, and continuous dependence on initial data can be completed as in [P], [I], [KPV1], [KoTz], etc.

Applying Proposition 2.8 (with $\alpha = 1/2$, $w = u$, and $F = -u\partial_x u$), the energy estimate (2.19), and the L^2 conservation law, we have

$$\begin{aligned}\|\partial_x u\|_{L^2_T L^\infty_x} &\lesssim \|D_x^{9/8+\varepsilon} u\|_{L^\infty_T L^2_x} + \|D_x^{5/8+\varepsilon}(u\partial_x u)\|_{L^2_T L^2_x} \\ &\quad + \|u\|_{L^\infty_T L^2_x} + \|u\|_{L^\infty_T L^2_x} \|\partial_x u\|_{L^2_T L^\infty_x} \\ &\lesssim \Lambda + \Lambda \exp(c\|\partial_x u\|_{L^2_T L^\infty_x}) + \|D_x^{5/8+\varepsilon}(u\partial_x u)\|_{L^2_T L^2_x}.\end{aligned}$$

By Lemma 2.6(a) and (2.19) again,

$$\begin{aligned}\|D_x^{5/8+\varepsilon}(u\partial_x u)\|_{L^2_T L^2_x} &\lesssim \|u D_x^{5/8+\varepsilon}\partial_x u\|_{L^2_T L^2_x} \\ &\quad + \left(\int_0^T \|D_x^{5/8+\varepsilon} u\|^2_{L^2_x} \|\partial_x u\|^2_{L^\infty_x}\, dt\right)^{1/2} \\ &\lesssim \|u D_x^{5/8+\varepsilon}\partial_x u\|_{L^2_T L^2_x} \\ &\quad + \Lambda \|\partial_x u\|_{L^2_T L^\infty_x} \exp(c\|\partial_x u\|_{L^2_T L^\infty_x}) \\ &\lesssim \|u D_x^{5/8+\varepsilon}\partial_x u\|_{L^2_T L^2_x} + \Lambda \exp(c\|\partial_x u\|_{L^2_T L^\infty_x}).\end{aligned}$$

Now the smoothing effect (2.5) provides a gain of half a derivative. Indeed,

$$\begin{aligned}\|u D_x^{5/8+\varepsilon}\partial_x u\|_{L^2_T L^2_x} &= \left(\sum_j \|u D_x^{\frac{1}{2}+\frac{1}{8}+\varepsilon}\partial_x u\|^2_{L^2([j,j+1)\times[0,T])}\right)^{1/2} \\ &\lesssim \left(\sum_j \|u\|^2_{L^\infty([j,j+1)\times[0,T])}\right)^{1/2} \\ &\quad \cdot\Lambda(1 + \Lambda + \|\partial_x u\|_{L^2_T L^\infty_x}) \exp(c\|\partial_x u\|_{L^2_T L^\infty_x}) \\ &\lesssim \left(\sum_j \|u\|^2_{L^\infty([j,j+1)\times[0,T])}\right)^{1/2} \\ &\quad \cdot\Lambda(\Lambda + 1) \exp(c\|\partial_x u\|_{L^2_T L^\infty_x}).\end{aligned}$$

Using the integral equation

$$u(t) = S(t)u_0 + \int_0^t S(t-t')u\partial_x u(t')\, dt'$$

as well as Lemma 2.3 and the L^2 conservation law, we find that for fixed (small) $\eta > 0$,

$$\begin{aligned}\left(\sum_j \|u\|^2_{L^\infty([j,j+1)\times[0,T])}\right)^{1/2} &\lesssim \Lambda + \|u\partial_x u\|_{L^1_T L^2_x} + \|D_x^{\frac{1}{2}+\eta}(u\partial_x u)\|_{L^1_T L^2_x} \\ &\lesssim \Lambda + \|u\|_{L^\infty_T L^2_x}\|\partial_x u\|_{L^1_T L^\infty_x} \\ &\quad + \|D_x^{\frac{1}{2}+\eta}(u\partial_x u)\|_{L^1_T L^2_x} \\ &\lesssim \Lambda + \Lambda\|\partial_x u\|_{L^2_T L^\infty_x} + \|D_x^{\frac{1}{2}+\eta}(u\partial_x u)\|_{L^2_T L^2_x}.\end{aligned}$$

Repeating the previous calculation (note that $\frac{1}{2} + \eta < \frac{5}{8} + \varepsilon$) leads to

$$\begin{aligned}\|D_x^{\frac{1}{2}+\eta}(u\partial_x u)\|_{L^2_T L^2_x} &\lesssim \Lambda \exp(c\|\partial_x u\|_{L^2_T L^\infty_x}) \\ &\quad + \left(\sum_j \|u\|^2_{L^\infty([j,j+1)\times[0,T])}\right)^{1/2} \\ &\quad \times\Lambda(\Lambda+1)\exp(c\|\partial_x u\|_{L^2_T L^\infty_x}).\end{aligned}$$

Set

$$\phi(T) = \|\partial_x u\|_{L^2_T L^\infty_x} + \left(\sum_j \|u\|^2_{L^\infty([j,j+1)\times[0,T])}\right)^{1/2}$$

which is a continuous, nondecreasing function of T. Combining all of the preceding, we know that

$$\phi(T) \lesssim \Lambda\exp(c\phi(T)) + \Lambda(\Lambda+1)\phi(T)\exp(c\phi(T)) + \Lambda + \Lambda\phi(T)$$

so

$$\phi(T) \le C\Lambda + C\Lambda\exp(C\phi(T)) \tag{3.1}$$

provided that $\Lambda \le \delta \le 1$.

Note that

$$\begin{aligned}\phi(0) &= \left(\sum_j \left(\sup_{x\in I_j} |u(x,0)|\right)^2\right)^{1/2} = \left(\sum_j \|u_0\|^2_{L^\infty(I_j)}\right)^{1/2} \\ &\lesssim (\|u_0\|^2_{L^2} + \|D^s u_0\|^2_{L^2})^{1/2} \\ &\le C_0\Lambda\end{aligned}$$

and take $C > C_0$ above.

We claim that there exists $\delta > 0$ and a constant $M > 0$ such that if $\Lambda \le \delta$, then $\phi(1) \le M$. To see this, let

$$\Phi(y,\eta) = y - C\eta - C\eta\exp(Cy).$$

Since $\Phi(0,0) = 0$ and $\frac{\partial\Phi}{\partial y}(0,0) = 1$, the implicit function theorem guarantees that for $|\eta| \le \delta$ sufficiently small, there exists a smooth function $A(\eta)$ such that $A(0) = 0$ and $\Phi(A(\eta),\eta) = 0$. Clearly $A(\eta) > 0$ for $\eta > 0$ since $\Phi(y,\eta) < 0$ for $y \le 0$. Moreover, since $\frac{\partial\Phi}{\partial y}(0,0) = 1$,$\Phi(\cdot,\eta)$ is increasing near $A(\eta)$ provided that δ is chosen small enough.

Assume $\Lambda \le \delta$, and set $M = A(\Lambda)$. Then

$$\phi(0) \le C_0\Lambda < C\Lambda \le C\Lambda + C\Lambda \exp(C\,A(\Lambda)) = M.$$

Suppose $\phi(T) > M$ for some $T \in (0,1)$, and let $T_0 = \inf\{T \in (0,1) : \phi(T) > M\}$. Then $T_0 > 0$, $\phi(T_0) = M$, and there exists a decreasing sequence $\{T_n\}$ converging to T_0 such that $\phi(T_n) > M$. Now $\Phi(\phi(T),\Lambda) \le 0$ for all $T \in [0,1]$ by (3.1). On the other hand, $\Phi(\cdot,\Lambda)$ is increasing near M, so $\Phi(\phi(T_n),\Lambda) > \Phi(\phi(T_0),\Lambda) = \Phi(M,\Lambda) = \Phi(A(\Lambda),\Lambda) = 0$ for n sufficiently large. This is a contradiction, and we conclude that $\phi(T) \le M$ for all $T \in (0,1)$. Hence $\phi(1) \le M$ as claimed.

4. Proof of Theorem 1.2

Our main goal is to obtain *a priori* estimates for $\|\partial_x u\|_{L^4_T L^\infty_x}$ and $\|u\|_{L^4_T L^\infty_x}$ (see (2.20)), where u is a solution to mBO and thus satisfies the integral equation

$$u(t) = S(t)u_0 + \int_0^t S(t-t')u^2\partial_x u(t')\,dt'.$$

(In the following, we always take $T \le 1$.) We consider the (worse) term $\|\partial_x u\|_{L^4_T L^\infty_x}$ first. By Lemma 2.1 and Lemma 2.4, we have

$$\begin{aligned} \|\partial_x u\|_{L^4_T L^\infty_x} &\lesssim \|u_0\|_{H^1} + \left\|\partial_x \int_0^t S(t-t')u^2\partial_x u(t')\,dt'\right\|_{L^4_T L^\infty_x} \\ &\lesssim \|u_0\|_{H^1} + \|D_x^{1/2}(u^2\partial_x u)\|_{L^1_x L^2_T}. \end{aligned} \tag{4.1}$$

Set

$$\Lambda(T) = \sup_{p\in[1,2]} \|D_x^{1/2}(u^2\partial_x u)\|_{L^p_x L^2_T} + \|u^2\partial_x u\|_{L^2_x L^2_T}.$$

For $p \in [1,2]$, $p_1 \in [\frac{6}{5},3]$, $p_2 \in [2,\infty]$ with $\frac{1}{p_1} + \frac{1}{6} = \frac{1}{p}$ and $\frac{1}{p_2} + \frac{1}{3} = \frac{1}{p_1}$, we apply Lemma 2.6 and Hölder's inequality to obtain

$$\begin{aligned} \|D_x^{\frac12}(u^2\partial_x u)\|_{L^p_x L^2_T} &\le \|D_x^{\frac12}(u^2\partial_x u) - u^2 D_x^{\frac12}\partial_x u - D_x^{\frac12}(u^2)\partial_x u\|_{L^p_x L^2_T} + \\ &\quad + \|u^2 D_x^{\frac12}\partial_x u\|_{L^p_x L^2_T} + \|D_x^{\frac12}(u^2)\partial_x u\|_{L^p_x L^2_T} \\ &\lesssim \|u^2 D_x^{\frac12}\partial_x u\|_{L^p_x L^2_T} + \|D_x^{\frac12} u^2\|_{L^{p_1}_x L^6_T}\,\|\partial_x u\|_{L^6_x L^3_T} \\ &\lesssim \|u^2 D_x^{\frac12}\partial_x u\|_{L^p_x L^2_T} + \|u\|_{L^{p_2}_x L^\infty_T}\,\|D_x^{\frac12} u\|_{L^3_x L^6_T}\,\|\partial_x u\|_{L^6_x L^3_T}. \end{aligned} \tag{4.2}$$

To handle the factors in the second term, we first differentiate the integral equation for u and apply the linear estimates in Lemma 2.4 (with $\theta = 2/3$) to see that for any (positive) $\delta < 1$, there exists $p = p(\delta) \in (1,2)$ such that

$$\begin{aligned}
\|\partial_x u\|_{L^6_x L^3_T} &\lesssim \|D_x^{\frac{1}{3}+\frac{1}{2}+\varepsilon} u_0\|_{L^2} + \|D_x^{\frac{1}{3}} u_0\|_{L^2} + T^{\delta/6}\|D_x^{1/2}(u^2\partial_x u)\|_{L^p_x L^2_T} \\
&\quad + T^{1/2}\|D_x^{1/3}(u^2\partial_x u)\|_{L^2_x L^2_T} \\
&\lesssim \|u_0\|_{H^1} + T^{\delta/6}\|D_x^{1/2}(u^2\partial_x u)\|_{L^p_x L^2_T} + T^{1/2}\|D_x^{1/2}(u^2\partial_x u)\|_{L^2_x L^2_T} \\
&\quad + T^{1/2}\|u^2\partial_x u\|_{L^2_x L^2_T} \\
&\lesssim \|u_0\|_{H^1} + T^{\delta/6}\Lambda(T).
\end{aligned}$$

Similarly (taking $\theta = 1/3$), for any $\delta < 1$, there exists $p \in [1,2]$ such that

$$\begin{aligned}
\|D_x^{\frac{1}{2}} u\|_{L^3_x L^6_T} &\lesssim \|u_0\|_{H^1} + T^{\delta/3}\|D_x^{\frac{1}{6}+\frac{\delta}{3}+\varepsilon}(u^2\partial_x u)\|_{L^p_x L^2_T} \\
&\quad + T^{1/2}\|D_x^{\frac{1}{6}}(u^2\partial_x u)\|_{L^2_x L^2_T} \\
&\lesssim \|u_0\|_{H^1} + T^{\delta/3}\|D_x^{1/2}(u^2\partial_x u)\|_{L^p_x L^2_T} \\
&\quad + T^{1/2}\|D_x^{1/2}(u^2\partial_x u)\|_{L^2_x L^2_T} + T^{1/2}\|u^2\partial_x u\|_{L^2_x L^2_T} \\
&\lesssim \|u_0\|_{H^1} + T^{\delta/3}\Lambda(T).
\end{aligned}$$

The case $\theta = 0$ and the energy estimate (2.20) lead to

$$\begin{aligned}
\|u\|_{L^{p_2}_x L^\infty_T} &\lesssim \|u\|^{1-\frac{2}{p_2}}_{L^\infty_x L^\infty_T} \|u\|^{\frac{2}{p_2}}_{L^2_x L^\infty_T} \\
&\lesssim \Big(\sup_{0<t<T} \|u(t)\|^{1-\frac{2}{p_2}}_{H^{\frac{1}{2}+\varepsilon}}\Big) \|u\|^{\frac{2}{p_2}}_{L^2_x L^\infty_T} \\
&\lesssim \Big(\|u_0\|_{H^1} + T^{\delta/2}\Lambda(T)\Big)^{2/p_2} \|u_0\|^{1-\frac{2}{p_2}}_{H^1} \exp(cT^{1/2}\|u\|^2_{L^4_T L^\infty_x}) \\
&\quad \cdot \exp(cT^{1/2}\|\partial_x u\|^2_{L^4_T L^\infty_x}) \\
&\lesssim (1 + \|u_0\|_{H^1} + T^{\delta/2}\Lambda(T))\, \|u_0\|^{1-\frac{2}{p_2}}_{H^1} \exp(cT^{1/2}\|u\|^2_{L^4_T L^\infty_x}) \\
&\quad \cdot \exp(cT^{1/2}\|\partial_x u\|^2_{L^4_T L^\infty_x}).
\end{aligned}$$

We turn next to the main term in (4.2), using the smoothing effect (2.6) and the maximal function estimate (2.7) to absorb half a derivative. In detail, since

$p \in [1,2]$ (so that $2/p \geq 1$), we have

$$\begin{aligned}
\|u^2 D_x^{\frac{1}{2}} \partial_x u\|_{L^p_x L^2_T} &= \left(\int_{-\infty}^{\infty} \left(\int_0^T |u^2 D_x^{\frac{1}{2}} \partial_x u|^2 \, dt \right)^{p/2} dx \right)^{1/p} \\
&= \left(\sum_{j=-\infty}^{\infty} \int_j^{j+1} \left(\int_0^T |u^2 D_x^{\frac{1}{2}} \partial_x u|^2 \, dt \right)^{p/2} dx \right)^{1/p} \\
&\leq \left(\sum_{j=-\infty}^{\infty} \int_j^{j+1} \left(\sup_{0<t<T} |u| \right)^{2p} \left(\int_0^T |D_x^{\frac{1}{2}} \partial_x u|^2 \, dt \right)^{p/2} dx \right)^{1/p} \\
&\leq \left(\sum_{j=-\infty}^{\infty} \left(\int_j^{j+1} \left(\sup_{0<t<T} |u| \right)^{\frac{4p}{2-p}} dx \right)^{\frac{2-p}{2}} \left(\int_j^{j+1} \int_0^T |D_x^{\frac{1}{2}} \partial_x u|^2 \, dt \, dx \right)^{p/2} \right)^{1/p} \\
&\lesssim \left(\sum_{j=-\infty}^{\infty} \|u\|^{2p}_{L^\infty([j,j+1)\times[0,T])} \right)^{1/p} (1 + \|u_0\|_{H^1} + T^{3/4} \|\partial_x u\|_{L^4_T L^\infty_x}) \, \|u_0\|_{H^1} \\
&\qquad \cdot \exp(cT^{1/2} \|u\|^2_{L^4_T L^\infty_x}) \exp(cT^{1/2} \|\partial_x u\|^2_{L^4_T L^\infty_x}).
\end{aligned}$$

Now $\|(a_j)\|_{\ell^{2p}} \leq \|(a_j)\|_{\ell^2}$, so by (2.18),

$$\begin{aligned}
&\left(\sum_{j=-\infty}^{\infty} \|u\|^{2p}_{L^\infty([j,j+1)\times[0,T])} \right)^{1/p} \\
&\quad \lesssim \sum_{j=-\infty}^{\infty} \|u\|^2_{L^\infty([j,j+1)\times[0,T])} \\
&\quad \lesssim \sum_{j=-\infty}^{\infty} \|S(t)u_0\|^2_{L^\infty([j,j+1)\times[0,T])} \\
&\qquad + \sum_{j=-\infty}^{\infty} \left\| \int_0^t S(t-t') u^2 \partial_x u(t') \, dt' \right\|^2_{L^\infty([j,j+1)\times[0,T])} \\
&\quad \lesssim (\|u_0\|_{H^1} + T^{\delta/2} \|D_x^{1/2}(u^2 \partial_x u)\|_{L^{p(\delta)}_x L^2_T} + T^{1/2} \|u^2 \partial_x u\|_{L^2_x L^2_T})^2 \\
&\quad \lesssim (\|u_0\|_{H^1} + T^{\delta/2} \Lambda(T))^2.
\end{aligned}$$

Observe that

$$\begin{aligned}
\|u^2 \partial_x u\|_{L^2_x L^2_T} &\lesssim \|u\|_{L^\infty_T L^\infty_x} \|u\|_{L^\infty_T L^2_x} \|\partial_x u\|_{L^2_T L^\infty_x} \\
&\lesssim T^{1/4} \|u_0\|_{L^2} \|\partial_x u\|_{L^4_T L^\infty_x} \|u_0\|_{H^1} \exp(cT^{1/2} \|u\|^2_{L^4_T L^\infty_x}) \\
&\qquad \cdot \exp(cT^{1/2} \|\partial_x u\|^2_{L^4_T L^\infty_x})
\end{aligned}$$

so we can control $\Lambda(T)$.

Finally, we need to estimate $\|u\|_{L^4_T L^\infty_x}$, which is easy. From the integral equation and the Strichartz estimate (2.3), we deduce that

$$\begin{aligned}\|u\|_{L^4_T L^\infty_x} &\lesssim \|u_0\|_{L^2} + \|u^2\partial_x u\|_{L^1_T L^2_x} \\ &\lesssim \|u_0\|_{L^2} + T^{3/4}\|u_0\|_{L^2}\,\|\partial_x u\|_{L^4_T L^\infty_x}\,\|u_0\|_{H^1}\exp(cT^{1/2}\|u\|^2_{L^4_T L^\infty_x}) \\ &\qquad \cdot\exp(cT^{1/2}\|\partial_x u\|^2_{L^4_T L^\infty_x}).\end{aligned}$$

Fix $\delta < 1$. We have shown that

$$\begin{aligned}\Psi(T) &:= \max\{\|\partial_x u\|_{L^4_T L^\infty_x}, \|u\|_{L^4_T L^\infty_x}, \Lambda(T)\} \\ &\lesssim \|u_0\|_{H^1} + (\|u_0\|_{H^1} + T^{\delta/3}\Lambda(T))(\|u_0\|_{H^1} + T^{\delta/6}\Lambda(T))(1 + \|u_0\|_{H^1} + T^{\delta/2}\Lambda(T)) \\ &\qquad \cdot(1+\|u_0\|_{H^1})\exp(cT^{1/2}\|u\|^2_{L^4_T L^\infty_x})\exp(cT^{1/2}\|\partial_x u\|^2_{L^4_T L^\infty_x}) \\ &\quad + (\|u_0\|^2_{H^1} + T^\delta\Lambda(T)^2)(1 + \|u_0\|_{H^1} + T^{3/4}\|\partial_x u\|_{L^4_T L^\infty_x})\,\|u_0\|_{H^1}\,\cdot \\ &\qquad \cdot\exp(cT^{1/2}\|u\|^2_{L^4_T L^\infty_x})\exp(cT^{1/2}\|\partial_x u\|^2_{L^4_T L^\infty_x}) \\ &\quad + T^{1/4}\|u_0\|^2_{H^1}\,\|\partial_x u\|_{L^4_T L^\infty_x}\exp(cT^{1/2}\|u\|^2_{L^4_T L^\infty_x})\exp(cT^{1/2}\|\partial_x u\|^2_{L^4_T L^\infty_x}).\end{aligned}$$

Set

$$\mu(T) := \max\{T^{\delta/6}\Lambda(T), T^{1/2}\|u\|^2_{L^4_T L^\infty_x}, T^{1/2}\|\partial_x u\|^2_{L^4_T L^\infty_x}, T^{1/4}\|\partial_x u\|_{L^4_T L^\infty_x}\}.$$

Note that $\mu(T)$ is continuous and $\mu(0) = 0$. If $\mu(T) \leq 1$ for all $T \leq 1$, we control the H^s norm at $T = 1$. If $\mu(1) > 1$, choose $T_0 \in (0, 1)$ such that $\mu(T_0) = 1$. Then

$$\Psi(T_0) \leq C + C\|u_0\|^4_{H^1}$$

and one of the following inequalities holds:

$$\begin{aligned}1 &= T_0^{\delta/6}\Lambda(T_0) \leq T_0^{\delta/6}(C + C\|u_0\|^4_{H^1}) \\ 1 &= T_0^{1/2}\|u\|^2_{L^4_{T_0} L^\infty_x} \leq T_0^{1/2}(C + C\|u_0\|^4_{H^1})^2 \\ 1 &= T_0^{1/2}\|\partial_x u\|^2_{L^4_{T_0} L^\infty_x} \leq T_0^{1/2}(C + C\|u_0\|^4_{H^1})^2 \\ 1 &= T_0^{1/4}\|\partial_x u\|_{L^4_{T_0} L^\infty_x} \leq T_0^{1/4}(C + C\|u_0\|^4_{H^1})\end{aligned}$$

and hence $T_0 \gtrsim (1 + \|u_0\|^4_{H^1})^{-6^+}$. At this stage, the existence, uniqueness, and continuous dependence on initial data follows from the standard compactness and Bona-Smith approximation arguments (see for example [I], [KPV1], [P]).

References

[B] T. B. Benjamin, *Internal waves of permanent form in fluids of great depth,* J. Fluid Mech. **29** (1967), 559–592.

[ChKi] M. Christ and A. Kiselev, *Maximal functions associated to filtrations,* J. Funct. Anal. **179** (2001), 409–425.

[CKS] J. Colliander, C. E. Kenig, and G. Staffilani, *Local well-posedness for dispersion generalized Benjamin-Ono equations,* Preprint (2002).

[GV] J. Ginibre and G. Velo, *Smoothing properties and existence of solutions for the generalized Benjamin-Ono equation,* J. Differential Equations **93** (1991), 150–232.

[I] R. J. Iorio, *On the Cauchy problem for the Benjamin-Ono equation,* Comm. Partial Differential Equations **11** (1986), 1031–1081.

[KPV1] C. E. Kenig, G. Ponce, and L. Vega, *Well-posedness of the initial value problem for the Korteweg-de Vries equation,* J. Amer. Math. Soc. **4** (1991), 323–347.

[KPV2] C. E. Kenig, G. Ponce, and L. Vega, *Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle,* Comm. Pure Appl. Math. **46** (1993), 527–620.

[KPV3] C. E. Kenig, G. Ponce, and L. Vega, *On the generalized Benjamin-Ono equation,* Trans. Amer. Math. Soc. **342** (1994), 155–172.

[KoTz] H. Koch and N. Tzvetkov, *On the local well-posedness of the Benjamin-Ono equation in $H^s(\mathbf{R})$,* Int. Math. Res. Not. **26** (2003), 1449–1464.

[MR] L. Molinet and F. Ribaud, *Well-posedness results for the generalized Benjamin-Ono equation with small initial data,* Preprint (2003).

[MSaTz] L. Molinet, J. C. Saut, and N. Tzvetkov, *Ill-posedness issues for the Benjamin-Ono and related equations,* SIAM J. Math. Anal. **33** (2001), 982–988.

[O] H. Ono, *Algebraic solitary waves in stratified fluids,* J. Phys. Soc. Japan **39** (1975), 1082–1091.

[P] G. Ponce, *On the global well-posedness of the Benjamin-Ono equation,* Differential Integral Equations **4** (1991), 527–542.

[Sa] J.-C. Saut, *Sur quelques généralisations de l'équation de Korteweg-de Vries,* J. Math. Pures Appl. **58** (1979), 21–61.

[SmSo] H. F. Smith and C. D. Sogge, *Global Strichartz estimates for nontrapping perturbations of the Laplacian,* Comm. Partial Differential Equations **25** (2000), 2171–2183.

[St] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions,* Princeton University Press, Princeton, 1970.

[Ta] T. Tao, *Global well-posedness of the Benjamin-Ono equation in $H^1(\mathbf{R})$,* Preprint (2003), http://arxiv.org/abs/math.AP/0307289.

[To] M. Tom, *Smoothing properties of some weak solutions to the Benjamin-Ono equation,* Differential Integral Equations **3** (1990), 683–694.

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