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DISTINGUISHED REPRESENTATIONS FOR $SL(2)$

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Abstract. Let E/F be a quadratic extension of p-adic fields. We compute the multiplicity of the space of $SL_2(F)$-invariant linear forms on a representation of $SL_2(E)$. This multiplicity varies inside an L-packet similar in spirit to the multiplicity formula for automorphic representations due to Labesse and Langlands.

1. Introduction

A representation π of a group G is said to be distinguished with respect to a subgroup H of G if it admits a non-trivial H-invariant linear form. More generally if χ is a character of H, we say that π is χ-distinguished if there is a non-trivial linear form ℓ on the space of π on which H operates via χ, i.e. $\ell : \pi \longrightarrow \mathbb{C}$ such that

$$\ell(\pi(h)v) = \chi(h)\ell(v)$$

for all $h \in H, v \in \pi$. This concept is specially useful when H is the fixed points of an involution on G. Classifying distinguished representations for both local and global fields has been an important and very active area of research in the last few years, cf. [A-T], [F], [H], [H-Ma], [H-Mu1], [H-Mu2], [J-Y1], [J-Y2], [J-R], [P3], [P4]. All these works are for the case $G = GL(n)$, and H one of the classical groups.

In what follows we investigate distinguished irreducible admissible representations for the pair (G, H), $G = SL_2(E)$, $H = SL_2(F)$, where E/F is a quadratic extension of p-adic fields. This represents the first case, in which complete results can be obtained, when the ambient group is not the general linear group.

Our results depend on the structure of L-packets due to Labesse and Langlands [L-L] (see [S] for the summary of results of [L-L]). We find that unlike the $(GL_n(E), GL_n(F))$ situation studied by Flicker, Jacquet and others, the space of $SL_2(F)$-invariant linear forms on a representation of $SL_2(E)$ need not be at most one dimensional (even for supercuspidal representations). Furthermore, the dimension of $SL_2(F)$-invariant linear forms varies inside an L-packet, similar in spirit to the multiplicity formula for automorphic representations due originally to Labesse and Langlands [L-L], developed further by Arthur and Kottwitz.

An important role in this work will be played by the subgroup $GL_2(E)^+$ of $GL_2(E)$ defined by

$$GL_2(E)^+ = \{g \in GL_2(E) \mid \det g \in F^* E^{*2}\}.$$

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Let us fix a non-trivial additive character ψ of E which has trivial restriction to F. Let π be an irreducible admissible (infinite dimensional) representation π of $GL_2(E)$. Consider the restriction of π to $GL_2(E)^+$, and write it as a direct sum of irreducible representations:

$$\pi|_{GL_2(E)^+} = \pi^+ \oplus \sum_{i\in I} \pi_i$$

where π^+ is ψ-generic and π_i are not ψ-generic. Note that the finite set I can possibly be empty. Observe that any two non-trivial characters ψ_1 and ψ_2 of E which are trivial on F are related as $\psi_1(x) = \psi_2(ax)$ for all $x \in E$ (for some $a \in F^*$). Thus as the determinant of elements in $GL_2(E)^+$ contains F^*, π^+ does not depend on the choice of the additive character ψ.

The representation theory of the special linear group is understood in terms of the representation theory of the general linear group [G-K, L-L, S]. In particular, an irreducible admissible representation of $GL_2(E)$ decomposes into a multiplicity free direct sum of irreducible representations of $SL_2(E)$ and any irreducible admissible representation of $SL_2(E)$ is obtained in this way. On the other hand, distinguishedness is well understood for the pair $(GL_2(E), GL_2(F))$ [A-T, F, H, P3]. Suppose an irreducible admissible representation V of $SL_2(E)$ is an irreducible component of the restriction to $SL_2(E)$ of an irreducible representation π of $GL_2(E)$. The representation π restricted to $SL_2(E)$ can be seen to be distinguished with respect to $SL_2(F)$ if and only if π is χ-distinguished for some character χ of F^*. A careful analysis of the distinguishedness criterion for $GL_2(E)$ together with the detailed analysis from [L-L] about restriction from $GL_2(E)$ to $SL_2(E)$ yields the following theorems about $SL(2)$.

Theorem 1.1. *Let π be an irreducible admissible representation of $GL_2(E)$. Then π is distinguished with respect to $SL_2(F)$ if and only if π^+ is distinguished with respect to $SL_2(F)$. Moreover, if V denotes an irreducible admissible representation of $SL_2(E)$ contained in a distinguished π, then V is distinguished with respect to $SL_2(F)$ if and only if V occurs in the restriction of π^+ to $SL_2(E)$.*

The above theorem enables us to compute the dimension of the space of $SL_2(F)$-invariant linear forms on an irreducible admissible representation V of $SL_2(E)$. We state the theorems for discrete series and principal series separately.

Theorem 1.2. *Let V be an irreducible admissible representation of $SL_2(E)$ that occurs in the restriction of a discrete series representation π of $GL_2(E)$. Suppose V is distinguished with respect to $SL_2(F)$. Then we have,*

1. $\dim_{\mathbb{C}} \mathrm{Hom}_{SL_2(F)}(V, 1) = 1$, *if π is either an exceptional supercuspidal or a special representation.*
2. $\dim_{\mathbb{C}} \mathrm{Hom}_{SL_2(F)}(V, 1) = 1$, *if π is a Weil representation which is a sum of two irreducibles upon restriction to $GL_2(E)^+$, and a sum of four irreducibles when restricted to $SL_2(E)$.*
3. $\dim_{\mathbb{C}} \mathrm{Hom}_{SL_2(F)}(V, 1) = 2$, *if π is a Weil representation which restricts to two irreducibles on $GL_2(E)^+$ and $SL_2(E)$.*

4. $\dim_{\mathbb{C}} \mathrm{Hom}_{SL_2(F)}(V,1) = 4$, *if* π *restricts to four irreducibles on* $GL_2(E)^+$.

The last case arises only when the residue characteristic of F *is two.*

Before we state the next theorem, let us introduce the quadratic character $\omega_{E/F}$ of F^* associated to the quadratic extension E/F by the local class field theory, i.e. $\omega_{E/F}$ is the non-trivial character of F^* which is trivial on $N_{E/F}(E^*)$ where $N_{E/F}$ is the norm map from E^* to F^*.

Theorem 1.3. *Let* V *be an irreducible admissible representation of* $SL_2(E)$ *that occurs in the restriction of a principal series representation* $\pi = Ps(\chi_1, \chi_2)$ *of* $GL_2(E)$. *Suppose that* V *is distinguished with respect to* $SL_2(F)$ *(and therefore* $\chi_1\chi_2^{-1}|_{F^*} = 1$ *or* $\chi_1\chi_2^{-1} = (\chi_1\chi_2^{-1})^\sigma$*). Then we have,*

1. $\dim_{\mathbb{C}} \mathrm{Hom}_{SL_2(F)}(V,1) = 1$, *if* $\chi_1\chi_2^{-1}|_{F^*} = 1$, *and* $\chi_1^2 \neq \chi_2^2$. *The L-packet of* $SL_2(E)$ *determined by* π *has only one element.*
2. $\dim_{\mathbb{C}} \mathrm{Hom}_{SL_2(F)}(V,1) = 1$, *if* $\chi_1\chi_2^{-1}|_{F^*} = \omega_{E/F}$, $\chi_1^2 = \chi_2^2$, $\chi_1 \neq \chi_2$. *The L-packet of* $SL_2(E)$ *determined by* π *has two elements, and both are distinguished by* $SL_2(F)$.
3. $\dim_{\mathbb{C}} \mathrm{Hom}_{SL_2(F)}(V,1) = 2$, *if either* $\chi_1\chi_2^{-1} = (\chi_1\chi_2^{-1})^\sigma$ *and* $\chi_1^2 \neq \chi_2^2$, *or* $\chi_1 = \chi_2$. *The L-packet of* $SL_2(E)$ *determined by* π *has only one element.*
4. $\dim_{\mathbb{C}} \mathrm{Hom}_{SL_2(F)}(V,1) = 3$, *if* $\chi_1\chi_2^{-1}|_{F^*} = 1$, $\chi_1^2 = \chi_2^2$, $\chi_1 \neq \chi_2$. *Further* $\mathrm{Hom}_{SL_2(F)}(V',1) = 0$ *for the unique other member* V' *in the L-packet of* $SL_2(E)$ *determined by* π.

In the next theorem, which can be verified from the above three theorems, we give an explicit formula for the dimension of $SL_2(F)$-invariant linear forms on an irreducible admissible representation V of $SL_2(E)$. This resembles the multiplicity formula for automorphic representations due to Labesse and Langlands [L-L]. We refer to [P2] for a similar formula about restriction from $SL(2)$ to $SO(2)$.

Let us introduce the notations required to make the precise statement. Let π denote an irreducible admissible representation of $GL_2(E)$ whose restriction to $SL_2(E)$ determines the L-packet of an irreducible admissible representation V of $SL_2(E)$. We need to consider the following three sets.

1. $X_\pi = \left\{ \chi \in \widehat{F^*} \mid \begin{array}{c} \pi \text{ is } \chi\text{-distinguished} \\ \text{with respect to } GL_2(F) \end{array} \right\}$.
2. $Y_\pi = \{\mu \in \widehat{E^*} \mid \pi \otimes \mu \cong \pi; \mu|_{F^*} = 1\}$.
3. $Z_\pi = \{\mu \in \widehat{E^*} \mid \pi \otimes \mu \cong \pi\}$.

Remark 1.1. *We note that* Y_π *and* Z_π *are abelian groups, and* Y_π *is a subgroup of* Z_π. *Further* X_π *is a set on which* Z_π *operates,* $\phi : Z_\pi \times X_\pi \to X_\pi$, *defined by* $\phi(\mu, \chi) = \mu|_{F^*} \cdot \chi$. *This is actually an action of* Z_π/Y_π *on* X_π *which is in fact a free action, therefore the cardinality of* Z_π/Y_π *divides that of* X_π.

Theorem 1.4. *Let* V *be an irreducible admissible representation of* $SL_2(E)$ *which occurs in the restriction of an irreducible admissible representation* π *of* $GL_2(E)$. *Let* ψ *be a character of* E *which has trivial restriction to* F. *Let* $< \, , >$

denote the pairing between Z_π and the L-packet of the restriction of π to $SL_2(E)$ defined as follows;

$$< \mu, V >= \mu(a) \qquad \text{if } V \text{ is } \psi_a\text{-generic.}$$

Then we have the following multiplicity formula.

$$\dim_{\mathbb{C}} \operatorname{Hom}_{SL_2(F)}(V,1) = \frac{|X_\pi|}{|Z_\pi|/|Y_\pi|} \cdot \left[\frac{1}{|Y_\pi|} \sum_{\mu \in Y_\pi} < \mu, V > \right].$$

Remark 1.2. *By remark 1.1, $|X_\pi|/(|Z_\pi|/|Y_\pi|)$ is an integer; clearly $\frac{1}{|Y_\pi|}\sum_{\mu\in Y_\pi} < \mu, V >$ is an integer which is either 0 or 1.*

Remark 1.3. *It can be seen that the pairing in Theorem 1.4 is well-defined. Indeed, if an irreducible admissible representation V of $SL_2(E)$ is both ψ_a-generic and ψ_b-generic with $a, b \in E^*$, then there exists $g \in GL_2(E)$ with $\det g = a^{-1}b$ such that $V^g \cong V$. Thus if $\mu \in Z_\pi$, then $\mu(a) = \mu(b)$ (cf. Corollary 2.7, Lemma 2.8,* [L-L]*).*

Remark 1.4. *Note that though the pairing in Theorem 1.4 is ψ-dependent, the multiplicity formula is independent of the fixed ψ. This is so since the sum is over characters in Y_π which are trivial on F^*.*

Remark 1.5. *The cardinality of X_π is the dimension of the space of $SL_2(F)$-invariant linear forms on π. The cardinality of Y_π is same as the number of irreducible constituents in the restriction of π to $GL_2(E)^+$. The cardinality of Z_π is equal to the cardinality of the L-packet of V.*

In the next section, we summarize the relevant results regarding the representation theory of $SL(2)$ and distinguishedness for $GL(2)$. The theorems are proved in section 3.

2. Preliminaries

First we recall the main results from the representation theory of $SL(2)$ that we will need in the sequel. More details can be found in [G-K], [L-L] or [S].

Let π be an irreducible admissible representation of $GL_2(E)$. Then, $\pi|_{SL_2(E)}$ is a finite multiplicity free direct sum of irreducible admissible representations of $SL_2(E)$, i.e.,

$$\pi|_{SL_2(E)} = \oplus_{i=1}^{r} V_i$$

with V_i irreducible and inequivalent. Let $V = V_1$. Consider the subgroup G_V defined by

$$G_V = \{g \in GL_2(E) \mid V^g \cong V\}$$

where $V^g(h) = V(g^{-1}hg)$. Then G/G_V permutes the V_i's simply transitively. Let

$$Z(\pi) = \{\mu \in \widehat{E^*} \mid \pi \otimes \mu \cong \pi\}$$

where a character of E^* is thought of as a character of $GL_2(E)$ by composition with the determinant map. Then the cardinality of $Z(\pi)$ is r. If a character of E^* is trivial on G_V, then it is in $Z(\pi)$. In fact G_V consists of those elements of $GL_2(E)$ which are in the kernel of all the characters in $Z(\pi)$. Now let V be an irreducible admissible representation of $SL_2(E)$. Then there exists an irreducible admissible representation π of $GL_2(E)$ such that $\pi|_{SL_2(E)}$ contains V. Moreover, if π_1 and π_2 are two irreducible admissible representations of $GL_2(E)$ whose restrictions contain V, then $\pi_2 \cong \pi_1 \otimes \mu$ for a character μ of E^*.

We note that the facts listed in the previous paragraph hold true if we take an irreducible admissible representation of any totally disconnected group G, which has multiplicity free restriction to an open normal subgroup H of G with G/H finite abelian. The properties listed for $GL_2(E)$ and $SL_2(E)$ follow by taking $G = GL_2(E)$ and $H = Z(E)SL_2(E)$ where $Z(E)$ is the center of $GL_2(E)$.

If ϕ is an admissible homomorphism of theWeil-Deligne group W_E to$PGL_2(\mathbb{C})$, then attached to ϕ there is a finite set Π_ϕ of irreducible admissible representations of $SL_2(E)$ such that the following holds. If η denotes the natural map from $GL_2(\mathbb{C})$ to $PGL_2(\mathbb{C})$, then for any admissible homomorphism ϕ' of W_E to $GL_2(\mathbb{C})$, $\Pi_{\eta\circ\phi'}$ consists of the set of irreducible constituents of the representation $\Pi_{\phi'}$ of $GL_2(E)$ restricted to $SL_2(E)$. The set Π_ϕ is called the L-packet corresponding to the Langlands parameter ϕ. Thus for $SL_2(E)$, an L-packet simply consists of irreducible constituents of the restriction of an irreducible admissible representation of $GL_2(E)$ to $SL_2(E)$.

Let S_ϕ denote the component group attached to ϕ, which is the quotient of the centraliser of the image of ϕ in $PGL_2(\mathbb{C})$ by its identity component. Then the order of the L-packet attached to ϕ is same as the cardinality of S_ϕ. The cardinality of S_ϕ is known to be 1, 2 or 4. Thus an L-packet of representations of $SL_2(E)$ consists of 1, 2 or 4 elements. The L-packet attached to a special representation or an exceptional supercuspidal has only one element. For a principal series representation, the size of an L-packet is at most two. For a Weil representation associated to a character η of a quadratic extension L of E, the corresponding L-packet has cardinality two if $(\eta^{-1}\eta^\tau)^2 \neq 1$, and has cardinality four if $(\eta^{-1}\eta^\tau)^2 = 1$, $\eta \neq \eta^\tau$. Here τ is the non-trivial element of $Gal(L/E)$.

Now let us summarize a few results about representations of $GL_2(E)$ which are distinguished with respect to $GL_2(F)$. For an irreducible admissible representation π of $GL_2(E)$, let ω_π denote its central character. The contragredient of π is denoted by $\pi^\vee$. If σ denotes the non-trivial element of the Galois group $Gal(E/F)$ of E/F, then π^σ denotes the representation given by $\pi^\sigma(g) = \pi(g^\sigma)$.

First we have the following multiplicity one result (Proposition 11, [F]).

Proposition 2.1. *The space of $GL_2(F)$-invariant linear forms on an irreducible admissible representation of $GL_2(E)$ has dimension at most one.*

The next proposition characterizes $GL_2(F)$-distinguished representations of $GL_2(E)$ up to a twist by the quadratic character associated to the extension E/F (Theorem 7, [F], Theorem 1.3, [A-T], Proposition 3.1, [A-T]).

Proposition 2.2. *Let π be an irreducible admissible representation of $GL_2(E)$ such that its central character ω_π has trivial restriction to F^*. Then $\pi^\vee \cong \pi^\sigma$ if and only if π is either distinguished or $\omega_{E/F}$-distinguished with respect to $GL_2(F)$. Further for discrete series representations, exactly one of the possibilities occurs.*

We now introduce the basic exact sequence which can be used to prove results about distinguished representations for principal series. We recall that the projective line $\mathbb{P}^1(E)$ has two orbits for $GL_2(F)$; one which is the closed orbit $\mathbb{P}^1(F)$, and the other which is the open orbit $\mathbb{P}^1(E) - \mathbb{P}^1(F) \cong GL_2(F)/E^*$. Therefore by Mackey theory about restriction of an induced representation to a subgroup, we find

$$0 \longrightarrow \mathrm{ind}_{E^*}^{GL_2(F)}[\chi_1\chi_2^\sigma] \longrightarrow Ps(\chi_1,\chi_2)|_{GL_2(F)} \longrightarrow Ps(\chi_1|_{F^*}|\,|^{\frac{1}{2}}, \chi_2|_{F^*}|\,|^{\frac{-1}{2}}) \longrightarrow 0.$$

By Frobenius reciprocity, $\mathrm{ind}_{E^*}^{GL_2(F)}[\chi_1(x)\chi_2^\sigma(x)]$ is distinguished with respect to $GL_2(F)$ if and only if $\chi_1^{-1} = \chi_2^\sigma$. It can be seen that if $Ps(\chi_1|_{F^*}|\,|^{1/2}, \chi_2|_{F^*}|\,|^{-1/2})$ is not already distinguished, then if $\chi_1^{-1} = \chi_2^\sigma$, the $GL_2(F)$-invariant linear form on $\mathrm{ind}_{E^*}^{GL_2(F)}1$ can be extended to $Ps(\chi_1,\chi_2)$. This analysis yields the following proposition (see also Proposition B17, [F-H]).

Proposition 2.3.
1. *The principal series representation $Ps(\mu^{-1}, \mu^\sigma)$ of $GL_2(E)$ is distinguished and $\omega_{E/F}$-distinguished.*
2. *The principal series representation $Ps(\mu_1,\mu_2)$, $\mu_1^\sigma \neq \mu_2^{-1}$, is distinguished precisely when $\mu_i|_{F^*} = 1$ $(i = 1, 2)$.*

The next proposition shows how distinguishedness is reflected on the Galois side under the local Langlands correspondence. To this end, let W_F (resp. W_E) denote the Weil-Deligne group of F (resp. E). If ρ is a two dimensional representation of W_E, we denote by $r(\rho)$ the four dimensional representation of W_F obtained by multiplicative induction, cf., §7 of [P1]. The following proposition is obtained by combining Proposition 8.4 and 8.4.2 of [P1]; it relies eventually on a theorem from [F].

Proposition 2.4. *Let ρ_π be the two dimensional representation of W_E corresponding to a discrete series representation π of $GL_2(E)$. Then π is distinguished with respect to $GL_2(F)$ if and only if $r(\rho_\pi)$ contains the trivial representation of W_F.*

3. The basic Lemmas

The theorems in this paper are proved using the following two lemmas about restriction of an irreducible representation of $GL_2(E)$ to $GL_2(E)^+$, and then further restriction from $GL_2(E)^+$ to $SL_2(E)$.

Lemma 3.1. *If an irreducible admissible representation of $GL_2(E)^+$ is distinguished by $SL_2(F)$, then it has a Whittaker model with respect to a (non-trivial) character of E which is trivial on F.*

Proof. Let π^0 denote an irreducible admissible representation of $GL_2(E)^+$ that is distinguished with respect to $SL_2(F)$. Let π be an irreducible admissible representation of $GL_2(E)$ such that π^0 occurs in the restriction of π to $GL_2(E)^+$. Without loss of generality we can assume that π^0 and π are distinguished with respect to $GL_2(F)$. Let ψ be a non-trivial additive character of E that is trivial on F. Let $\mathcal{W}(\pi, \psi)$ denote its ψ-Whittaker model, considered as a space of locally constant functions f on $GL_2(E)$ such that $f(gn) = \psi(n)f(g)$ where $n \in E$ is identified to the standard upper triangular unipotent matrix. Now the unique non-trivial $GL_2(F)$-invariant linear form ℓ on π given by Proposition 2.1 can be realized on the ψ-Whittaker model by

$$\ell(W) = \int_{F^*} W\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) d^\times a$$

where $d^\times a$ is a Haar measure on F^* (see Proposition 4.2, [H]). Since there is a unique ψ-Whittaker functional on the space of π, exactly one constituent of the restriction of π to $GL_2(E)^+$ is ψ-generic. Thus if $W \in \mathcal{W}(\pi, \psi)$ is such that it belongs to a non ψ-generic part, then $W \equiv 0$ on $GL_2(E)^+$. Therefore the $GL_2(F)$-invariant functional can be non-zero only on the ψ-generic part of the restriction of π to $GL_2(E)^+$. Therefore π^0 is ψ-generic. $\square$

Lemma 3.2. *An irreducible admissible representation π of $GL_2(E)^+$ restricted to $SL_2(F)$ is a sum of irreducible representations all of which are conjugate under the inner-conjugation action of $GL_2(F)$ on $SL_2(F)$. Hence,* $\dim_{\mathbb{C}} \mathrm{Hom}_{SL_2(F)}(\pi', 1)$ *is independent of the irreducible representation π' of $SL_2(E)$ if π' is contained in π.*

Proof. This is clear as $GL_2(F)SL_2(E)E^* = GL_2(E)^+$ where E^* sits inside $GL_2(E)$ as the scalar matrices. $\square$

4. Main arguments

We start with some propositions in this section, and take up the proof of the main theorems at the end of the section. Recall that $GL_2(E)^+ = \{g \in GL_2(E) \mid \det g \in F^*E^{*2}\}$.

Proposition 4.1. *Let π be an irreducible admissible representation of $GL_2(E)$. Then,*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{SL_2(F)}(\pi, 1) = \left|\left\{\chi \in \widehat{F^*} \mid \begin{array}{c} \pi \textit{ is } \chi\textit{-distinguished} \\ \textit{with respect to } GL_2(F) \end{array}\right\}\right|.$$

Proof. We assume, after twisting by a character of E^* if necessary, that the central character of π restricted to F^* is trivial. The space $\mathrm{Hom}_{SL_2(F)}(\pi, 1)$ has the structure of a $GL_2(F)$-module on which $F^* \cdot SL_2(F)$ acts trivially. Since $GL_2(F)/F^*SL_2(F) \cong F^*/F^{*2}$, a finite abelian group, this space decomposes into a direct sum of one dimensional spaces on which $GL_2(F)$ acts via characters. Each character appears with multiplicity at most one by Proposition 2.1, hence the proposition. □

Proposition 4.2. *Let π be a discrete series representation of $GL_2(E)$ which is distinguished with respect to $SL_2(F)$. Then the sets*

$$X = \left\{ \chi \in \widehat{F^*} | \begin{array}{c} \pi \text{ is } \chi\text{-distinguished} \\ \text{with respect to } GL_2(F) \end{array} \right\}$$

and

$$Y = \{\mu \in \widehat{E^*} \mid \pi \otimes \mu \cong \pi; \mu|_{F^*} = 1\}$$

have the same cardinality; in fact $\chi \longrightarrow \chi \circ N_{E/F}$ induces an isomorphism of X onto Y if π is $GL_2(F)$-distinguished.

Proof. We may (and will) assume without loss of generality that π is distinguished with respect to $GL_2(F)$. Then we give explicit maps from X to Y and from Y to X.

For $\chi \in X$, let $\tilde{\chi}$ be a character of E^* restricting to χ on F^*. By Proposition 2.2, we get,

1. $\pi^\vee \cong \pi^\sigma$.
2. $(\pi \otimes \tilde{\chi}^{-1})^\vee \cong (\pi \otimes \tilde{\chi}^{-1})^\sigma$.

Combining 1 and 2, we find that

$$\pi \cong \pi \otimes \chi \circ N_{E/F}.$$

Note that since π is both distinguished and χ-distinguished with respect to $GL_2(F)$, consideration of the central character implies that $\omega_\pi|_{F^*} = \omega_\pi|_{F^*}\chi^{-2} = 1$. Therefore $\chi^2 = 1$, thus $\chi \circ N_{E/F} \in Y$. This allows us to define a map from X to Y by sending χ to $\chi \circ N_{E/F}$.

If $\mu \in Y$, then, since $\mu|_{F^*} = 1$, and $\mu^2 = 1$, we have that μ factors through the norm map $N_{E/F}$. Let $\mu = \eta\eta^\sigma$ for a character η of E^*. Now consider the representation $\pi \otimes \eta$. Observe that

$$(\pi \otimes \eta)^\vee \cong (\pi \otimes \eta)^\sigma,$$

and that

$$\omega_{\pi\otimes\eta}|_{F^*} = 1.$$

Therefore by Proposition 2.2, $\pi\otimes\eta$ is either distinguished with respect to $GL_2(F)$ or $\omega_{E/F}$-distinguished with respect to $GL_2(F)$. We map μ to $\eta|_{F^*}$ or $\eta|_{F^*}\omega_{E/F}$ accordingly. Clearly the above two maps are inverses of each other and hence X and Y have the same cardinality, completing the proof of the proposition. □

Corollary 4.3. *Let π be a discrete series representation of $GL_2(E)$ which is distinguished with respect to $SL_2(F)$. Then the number of $SL_2(F)$-invariant linear forms on π equals the number of irreducible components of π restricted to $GL_2^+(E)$.*

Proposition 4.4. *Let π be a discrete series representation of $GL_2(E)$ which does not restrict irreducibly to $SL_2(E)$. Suppose π is distinguished with respect to $SL_2(F)$. Then π restricted to $GL_2(E)^+$ is not irreducible. Moreover, it is precisely the ψ-generic part of this direct sum that is distinguished with respect to $SL_2(F)$.*

Proof. Suppose π restricted to $GL_2(E)^+$ is irreducible. Then as π restricted to $SL_2(E)$ is not irreducible, there is a non-trivial branching from $GL_2(E)^+$ to $SL_2(E)$, thus by Lemma 3.2, the number of $SL_2(F)$-invariant forms is more than 1, contrary to Corollary 4.3. Thus π restricts reducibly to $GL_2(E)^+$. $\square$

Proposition 4.5. *Let π be a principal series representation of $GL_2(E)$ which is distinguished with respect to $SL_2(F)$. Suppose the restriction of π to $SL_2(E)$ is reducible (and hence has two irreducible components). Then precisely one of the components of this restriction is distinguished with respect to $SL_2(F)$ if π restricted to $GL_2(E)^+$ breaks up into two irreducible components, and both the components are distinguished if π restricts irreducibly to $GL_2(E)^+$.*

Proof. This is clear from Lemma 3.1 and 3.2. $\square$

4.1. Proof of the Theorems.

Theorem 1.1 follows immediately from Lemma 3.1 and Lemma 3.2.

In Theorem 1.2, if the L-packet has cardinality one, from Corollary 4.3 it follows that the space of $SL_2(F)$-invariant linear forms on V is one dimensional. In case 2, when the L-packet has cardinality four, we have $|Y| = |X| = 2$, and by Lemma 3.2, the result follows. When the L-packet contains two representations, the ψ-generic part of the restriction of π to $GL_2(E)^+$ restricts irreducibly to V, and hence the space of invariant linear forms on V is two dimensional. The last case follows similarly. Moreover the last case arises only when F has even residue characteristic, as in odd residue characteristic, E^*/F^*E^{*2} is of size two.

We omit the rather formal computations that will prove Theorem 1.3.

5. Examples

Although Theorem 1.2 gives possible dimensions of the space of $SL_2(F)$-invariant linear forms, we have not so far proved the existence of the possibility for a discrete series representation to have higher multiplicity. In this section we construct examples of discrete series representations V of $SL_2(E)$ such that

1. $\dim_{\mathbb{C}} \mathrm{Hom}_{SL_2(F)}(V, 1) = 1$, and the L-packet containing V has size four.
2. $\dim_{\mathbb{C}} \mathrm{Hom}_{SL_2(F)}(V, 1) = 2$.
3. $\dim_{\mathbb{C}} \mathrm{Hom}_{SL_2(F)}(V, 1) = 4$.

Since we are dealing with discrete series representations, we have $|X_\pi| = |Y_\pi|$ by Proposition 4.2. Also by Proposition 2.4, $|X_\pi|$ is equal to the number of characters appearing in the direct sum decomposition of $r(\rho_\pi)$. We note in passing that the characters occurring in $r(\rho_\pi)$ are all distinct. This follows by Schur's lemma, as $r(\rho_\pi)|_{W_E} = \rho_\pi \otimes \rho_\pi^\sigma$.

All the three families of examples will be constructed in a similar fashion. To this end, let K be a quadratic extension of F different from E, and let η be a character of K^* with trivial restriction to F^*. Let L denote the compositum of E and K. Also assume that $\eta^2 \neq 1, \omega_{L/K}$. Let π_0 be the representation of $GL_2(F)$ such that $\rho_{\pi_0} = \mathrm{Ind}_{W_K}^{W_F} \eta$. By the assumptions on η, ρ_{π_0} is irreducible.

The representation π of $GL_2(E)$ that we are interested in is the base change lift of π_0 to $GL_2(E)$. Thus $\rho_\pi = \rho_{\pi_0}|_{W_E}$. Therefore, $\rho_\pi = \mathrm{Ind}_{W_L}^{W_E}(\eta \circ N_{L/K})$. Since $\eta^2 \neq 1, \omega_{L/K}$, it can be seen that ρ_π is irreducible. We now calculate $r(\rho_\pi)$:

$$\begin{aligned}
r(\rho_\pi) &= r(\rho_{\pi_0}|_{W_E}) \\
&= \mathrm{Sym}^2(\rho_{\pi_0}) \oplus \omega_{E/F} \wedge^2 (\rho_{\pi_0}) \\
&= \mathrm{Ind}_{W_K}^{W_F} \eta^2 \oplus \eta|_{F^*} \oplus \eta|_{F^*} \omega_{K/F} \omega_{E/F} \\
&= \mathrm{Ind}_{W_K}^{W_F} \eta^2 \oplus 1 \oplus \omega_{K/F} \omega_{E/F}.
\end{aligned}$$

Since $r(\rho_\pi)$ contains the trivial representation, π is distinguished by Proposition 2.4.

For the first set of examples, we need to show that there is a representation π of $GL_2(E)$ such that $|Z_\pi| = 4$, $|Y_\pi| = 2$. That is, π should have three non-trivial self-twists by characters of E^*. Suppose now that F has odd residue characteristic and F is such that for K/F the quadratic unramified extension, we can choose an η such that $\eta^8 = 1$, $\eta^4 \neq 1$. This implies that $\eta^4 = \omega_{L/K}$. Hence ρ_π has four self-twists. i.e., $|Z_\pi| = 4$. Also $|X_\pi| = 2$, as $\mathrm{Ind}_{W_K}^{W_F} \eta^2$ is irreducible by our assumptions on η, and therefore by Proposition 4.2, $|Y_\pi| = 2$.

To get the second set of examples, we need to get a π such that $|Z_\pi| = |Y_\pi| = 2$. Again, let F be of odd residue characteristic. Let η be such that $\eta^8 \neq 1$. In particular, $\eta^4 \neq 1, \omega_{L/K}$. Thus it follows that the only non-trivial self-twist of ρ_π is $\omega_{L/E}$. This means that $|Z_\pi| = 2$, and therefore by Proposition 4.4, $|Y_\pi| = 2$. Thus if V comes in the restriction of π^+, then the space of $SL_2(F)$-invariant linear forms on V is two dimensional.

Finally let the residue characteristic of F be two. Choose η such that $\eta^4 = 1$, $\eta^2 \neq 1, \omega_{L/K}$. Note that such a choice is possible since K^*/F^*K^{*2} has cardinality more than two. Since $\eta^4 = 1$ (and $\eta|_{F^*} = 1$), $\mathrm{Ind}_{W_K}^{W_F} \eta^2$ is a sum of two characters. Thus $|X_\pi|$ (and hence $|Y_\pi|$ and $|Z_\pi|$) is of size four.

6. Some remarks on $SL(n)$

The L-packets of representations of $SL_n(E)$ can be understood exactly the same way they are understood for $SL_2(E)$. Therefore it is natural to ask if our results hold true for general n.

Define

$$GL_n(E)^+ = \{g \in GL_n(E) | \det g \in F^* E^{*n}\}.$$

Let π be a generic representation of $GL_n(E)$. Then we expect that the basic lemmas of section 3 are still valid. Whereas Lemma 3.2 is obviously true for general n, Lemma 3.1 should follow since the unique $GL_n(F)$-invariant functional on a $GL_n(F)$-distinguished representation π can be realized on the ψ-Whittaker model as an integral generalizing the integral functional quoted in the proof of Lemma 3.1. (For instance see Corollary 1.2, [A-K-T] where this is done when π is tempered.) As Theorem 1.1 depends only on these two lemmas, it is true for $SL(n)$ as well. By Lemma 3.2, all the constituents of the restriction of a representation of $GL_n(E)^+$ to $SL_n(E)$ admit the same number of linearly independent $SL_n(F)$-invariant linear functionals. If π^+ is a representation of $GL_n(E)^+$ which is distinguished with respect to $SL_n(F)$, let $a(\pi)$ be the number of $SL_n(F)$-invariant linear forms on each constituent of the restriction of π^+ to $SL_n(E)$. It is not hard to see that $a(\pi) = |X_\pi|/(|Z_\pi|/|Y_\pi|)$. Now it is clear from Remark 1.2 that the multiplicity formula of Theorem 1.4 holds true for any n. It will be of interest to see what form the multiplicity formula might take for a non-generic irreducible admissible representation V of $SL_n(E)$.

The key inputs which lead to the computations of Theorem 1.2 are Proposition 4.2 and the fine knowledge about the cardinality of the L-packet attached to representations of $GL_2(E)$. Proposition 4.2 can be proved by modifying the set Y as

$$Y' = \{\mu \in \widehat{E^*} | \pi \otimes \mu \cong \pi; \mu|_{E^1} = 1\},$$

where E^1 is the group of Norm 1 elements of E^*. Thus the subgroup of $GL_n(E)$ corresponding to Y' is

$$G'(E) = \{g \in GL_n(E) | \det g \in E^1 E^{*n}\}$$

So for multiplicity computations, it is the number of constituents of the restriction of a discrete series representation π to this group that will be of interest. Note that for $GL_2(E)$, this number is same as the number of constituents of the restriction of π to $GL_2(E)^+$.

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