

REAL DIVISORS ALGEBRAICALLY EQUIVALENT TO ZERO

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1. Introduction

Let X be a smooth projective variety over \mathbb{R} of dimension n (that is, an integral smooth projective scheme over \mathbb{R} of dimension n). We endow the set $X(\mathbb{R})$ of \mathbb{R} -rational points of X with the topology induced by the usual metric topology on \mathbb{R} and assume that $X(\mathbb{R})$ is nonempty. Thus $X(\mathbb{R})$ is a closed n -dimensional manifold of class \mathcal{C}^∞ . We have the cycle homomorphism

$$c\ell_{\mathbb{R}} : Z^k(X) \rightarrow H^k(X(\mathbb{R}), \mathbb{Z}/2)$$

defined on the group $Z^k(X)$ of algebraic cycles on X of codimension k : for any closed subvariety V of X of codimension k , the cohomology class $c\ell_{\mathbb{R}}(V)$ is Poincaré dual to the homology class in $H_{n-k}(X(\mathbb{R}), \mathbb{Z}/2)$ represented by $V(\mathbb{R})$ if $\dim V(\mathbb{R}) = n - k$ (cf. [4] or [2, Section 11.3] for the definition of this homology class) and $c\ell_{\mathbb{R}}(V) = 0$ if $\dim V(\mathbb{R}) < n - k$. Of fundamental interest in real algebraic geometry is the image

$$H_{\text{alg}}^k(X(\mathbb{R}), \mathbb{Z}/2) = c\ell_{\mathbb{R}}(Z^k(X))$$

of $Z^k(X)$ via $c\ell_{\mathbb{R}}$ (cf. [3] for a short survey of its properties and applications). Recently also the subgroup

$$\text{Alg}^k(X(\mathbb{R})) = c\ell_{\mathbb{R}}(Z_{\text{alg}}^k(X))$$

of $H_{\text{alg}}^k(X(\mathbb{R}), \mathbb{Z}/2)$, where $Z_{\text{alg}}^k(X)$ is the subgroup of $Z^k(X)$ consisting of all cycles algebraically equivalent to 0, proved to be very useful (cf. [7, Chapter 10] for the theory of algebraic equivalence). The group $\text{Alg}^k(X(\mathbb{R}))$ is often highly nontrivial, as illustrated for example by Theorem 1.3 below, dealing with $\text{Alg}^1(X(\mathbb{R}))$. The behavior of $H_{\text{alg}}^{n-k}(X(\mathbb{R}), \mathbb{Z}/2)$ is strongly influenced by $\text{Alg}^k(X(\mathbb{R}))$, cf. [1, 12]. Some crucial constructions of [14], at the borderline between real algebraic geometry and differential topology, depend on $\text{Alg}^k(-)$. In [13] it is proved, among other things, that $\text{Alg}^1(X(\mathbb{R}))$ is a birational invariant of X . In general, it is very difficult to compute explicitly $\text{Alg}^k(X(\mathbb{R}))$ for $0 < k < n$. The present paper shows that $\text{Alg}^1(X(\mathbb{R}))$ can be described in a purely topological way; in fact, every cohomology class in $\text{Alg}^1(X(\mathbb{R})) \subseteq H^1(X(\mathbb{R}), \mathbb{Z}/2)$ comes from a conjugation invariant cohomology class in $H^1(X(\mathbb{C}), \mathbb{Z})$ by restriction

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to $X(\mathbb{R})$ and reduction modulo 2. In order to state our result we need some preparation.

The set $X(\mathbb{C})$ of all \mathbb{C} -rational points of X will be regarded as a complex manifold. The Galois group $G = \text{Gal}(\mathbb{C}/\mathbb{R}) = \{1, \sigma\}$ of \mathbb{C} over \mathbb{R} acts on $X(\mathbb{C})$ and we let

$$\sigma_X : X(\mathbb{C}) \rightarrow X(\mathbb{C})$$

denote the antiholomorphic involution corresponding to σ . Moreover, G acts on $H^1(X(\mathbb{C}), \mathbb{Z})$, the action of σ being given by the induced isomorphism

$$\sigma_X^* : H^1(X(\mathbb{C}), \mathbb{Z}) \rightarrow H^1(X(\mathbb{C}), \mathbb{Z}).$$

We have

$$H^2(G, H^1(X(\mathbb{C}), \mathbb{Z})) = \text{Ker}(1 - \sigma_X^*) / \text{Im}(1 + \sigma_X^*),$$

where 1 stands for the identity map of $H^1(X(\mathbb{C}), \mathbb{Z})$. Since in this paper we do not use properties of the group cohomology, the reader may consider the equality above as the definition of $H^2(G, H^1(X(\mathbb{C}), \mathbb{Z}))$. Note that $\text{Ker}(1 - \sigma_X^*)$ is equal to the subgroup $H^1(X(\mathbb{C}), \mathbb{Z})^G$ of $H^1(X(\mathbb{C}), \mathbb{Z})$ consisting of all elements fixed by G .

We identify $X(\mathbb{R})$ with the subset of $X(\mathbb{C})$ consisting of all points fixed by G , that is $X(\mathbb{R}) = \{x \in X(\mathbb{C}) \mid \sigma_X(x) = x\}$. Denoting by $i_X : X(\mathbb{R}) \hookrightarrow X(\mathbb{C})$ the inclusion map, one readily verifies that the induced homomorphism $i_X^* : H^1(X(\mathbb{C}), \mathbb{Z}) \rightarrow H^1(X(\mathbb{R}), \mathbb{Z})$ satisfies

$$i_X^*(\text{Im}(1 + \sigma_X^*)) \subseteq 2H^1(X(\mathbb{R}), \mathbb{Z}).$$

Hence we can define a homomorphism

$$i_X^2 : H^2(G, H^1(X(\mathbb{C}), \mathbb{Z})) \rightarrow H^1(X(\mathbb{R}), \mathbb{Z}/2),$$

$$i_X^2(u + \text{Im}(1 + \sigma_X^*)) = i^*(u)_2 \text{ for all } u \text{ in } H^1(X(\mathbb{C}), \mathbb{Z})^G,$$

where $i^*(u)_2$ denotes the image of $i^*(u)$ under the reduction modulo 2 homomorphism $H^1(X(\mathbb{R}), \mathbb{Z}) \rightarrow H^1(X(\mathbb{R}), \mathbb{Z}/2)$.

Theorem 1.1. *If X is a smooth projective variety over \mathbb{R} with $X(\mathbb{R})$ nonempty, then*

$$\text{Alg}^1(X(\mathbb{R})) = i_X^2(H^2(G, H^1(X(\mathbb{C}), \mathbb{Z}))).$$

Furthermore, i_X^2 is injective if X is an Abelian variety over \mathbb{R} .

As a consequence we obtain that $\text{Alg}^1(X(\mathbb{R}))$ is completely determined by the action of G on $X(\mathbb{C})$. In particular, we have the following result.

Corollary 1.2. *Let X and Y be smooth projective varieties over \mathbb{R} with $X(\mathbb{R})$ and $Y(\mathbb{R})$ nonempty. If $X(\mathbb{C})$ and $Y(\mathbb{C})$ are G -equivariantly homeomorphic, then the groups $\text{Alg}^1(X(\mathbb{R}))$ and $\text{Alg}^1(Y(\mathbb{R}))$ are isomorphic.*

Proof. Let $h : X(\mathbb{C}) \rightarrow Y(\mathbb{C})$ be a G -equivariant homeomorphism, and let $h_0 : X(\mathbb{R}) \rightarrow Y(\mathbb{R})$ be the restriction of h . We have the following commutative diagram:

$$\begin{CD} H^2(G, H^1(Y(\mathbb{C}), \mathbb{Z})) @>>> H^2(G, H^1(X(\mathbb{C}), \mathbb{Z})) \\ @V i_Y^2 VV @VV i_X^2 V \\ H^1(Y(\mathbb{R}), \mathbb{Z}/2) @>h_0^*>> H^1(X(\mathbb{R}), \mathbb{Z}/2) \end{CD}$$

where the upper horizontal arrow is the isomorphism induced by $h^* : H^1(Y(\mathbb{C}), \mathbb{Z}) \rightarrow H^1(X(\mathbb{C}), \mathbb{Z})$. Hence, in view of Theorem 1.1, h_0^* establishes an isomorphism from $\text{Alg}^1(Y(\mathbb{R}))$ onto $\text{Alg}^1(X(\mathbb{R}))$. \square

Theorem 1.1 will be proved in Section 2. Its proof depends on some classical results [6], whose modern treatment is contained in [5, 9, 11, 20].

In order to demonstrate that $\text{Alg}^1(-)$ is an interesting invariant we shall now cite a result of [15]. The k th Stiefel-Whitney class of a closed \mathcal{C}^∞ manifold M will be denoted by $w_k(M)$, while $[M]$ will stand for the fundamental class of M in $H_m(M, \mathbb{Z}/2)$, $m = \dim M$. As usual we will use \cup and \langle, \rangle to denote the cup product and scalar (Kronecker) product.

Theorem 1.3. *Let M be a closed \mathcal{C}^∞ manifold of dimension m with $m \geq 2$. Given a subgroup G of $H^1(M, \mathbb{Z}/2)$, the following conditions are equivalent:*

- (a) *There exist a smooth projective variety X over \mathbb{R} and a \mathcal{C}^∞ diffeomorphism $\varphi : X(\mathbb{R}) \rightarrow M$ such that $\varphi^*(G) = \text{Alg}^1(X(\mathbb{R}))$,*
- (b) *G is contained in the image of the reduction modulo 2 homomorphism $H^1(M, \mathbb{Z}) \rightarrow H^1(M, \mathbb{Z}/2)$, and for each integer ℓ , $1 \leq \ell \leq m$, and all u_1, \dots, u_ℓ in G , one has*

$$\langle u_1 \cup \dots \cup u_\ell \cup w_{i_1}(M) \cup \dots \cup w_{i_r}(M), [M] \rangle = 0$$

for all nonnegative integers i_1, \dots, i_r with $i_1 + \dots + i_r = m - \ell$.

Reference for the proof. [15].

In [15] the reader also can find some results similar to Theorem 1.3 for $\text{Alg}^k(-)$ with $k \geq 2$.

2. Proof of Theorem 1.1

We begin by introducing notation and terminology. As usual, if the Galois group $G = \text{Gal}(\mathbb{C}/\mathbb{R}) = \{1, \sigma\}$ acts on an Abelian group M , we denote by M^G the subgroup of M consisting of all elements fixed by G .

Let V be a complex manifold endowed with an antiholomorphic involution σ_V . The group G acts on the Picard group $\text{Pic}(V)$ of isomorphism classes of holomorphic line bundles on V . If an element ξ of $\text{Pic}(V) = H^1(V, \mathcal{O}_V^*)$ is represented by a Čech cocycle $\{U_i, g_{ij}\}$, then ξ^σ is represented by the Čech cocycle $\{\sigma_V(U_i), \sigma \circ g_{ij} \circ \sigma_V\}$. Note that the action of G on $\text{Pic}(V)$ can be restricted to

the subgroup $\text{Pic}^0(V)$ of $\text{Pic}(V)$ consisting of the isomorphism classes of holomorphic line bundles that are topologically trivial. In particular, $\text{Pic}(V)^G$ and $\text{Pic}^0(V)^G$ are defined. We say that a holomorphic line bundle (L, π, V) on V is *defined over \mathbb{R}* if it comes with an antiholomorphic involution $\sigma_L : L \rightarrow L$ such that $\pi \circ \sigma_L = \sigma_V \circ \pi$, and for each point x in V , the restriction $\sigma_L : L_x \rightarrow L_{\sigma_V(x)}$ is anti-linear, that is $\sigma_L(zv) = \sigma(z)\sigma_L(v)$ for all z in \mathbb{C} and v in L_x . Holomorphic line bundles defined over \mathbb{R} (on V) form a category in which morphisms are homomorphisms of holomorphic line bundles that commute with the given involutions on the total spaces. If L is defined over \mathbb{R} , then its class in $\text{Pic}(V)$ (ignore the involution on L) belongs to $\text{Pic}(V)^G$. We will regard $(L(\mathbb{R}), \pi(\mathbb{R}), V(\mathbb{R}))$, where

$$L(\mathbb{R}) = \{v \in L \mid \sigma_L(v) = v\}, \quad V(\mathbb{R}) = \{x \in V \mid \sigma_V(x) = x\},$$

and $\pi(\mathbb{R}) : L(\mathbb{R}) \rightarrow V(\mathbb{R})$ is the restriction of $\pi : L \rightarrow V$, as a topological real line bundle on $V(\mathbb{R})$.

Now let X be a smooth projective variety over \mathbb{R} with $X(\mathbb{R})$ nonempty. Given an invertible sheaf \mathcal{L} on X , we denote by $\mathcal{L}(\mathbb{R})$ the topological real line bundle on $X(\mathbb{R})$ determined by \mathcal{L} in the usual way. The correspondence which assigns to \mathcal{L} the first Stiefel-Whitney class $w_1(\mathcal{L}(\mathbb{R}))$ of $\mathcal{L}(\mathbb{R})$ gives rise to a canonical homomorphism

$$w_X : \text{Pic}(X) \rightarrow H^1(X(\mathbb{R}), \mathbb{Z}/2),$$

defined on the Picard group $\text{Pic}(X)$ of isomorphism classes of invertible sheaves on X . If $\mathcal{O}(D)$ is the invertible sheaf associated with a Weil divisor D on X , then $w_1(\mathcal{O}(D)(\mathbb{R})) = c\ell_{\mathbb{R}}(D)$, cf. [4, p. 498] (obviously, $Z^1(X)$ is the group of Weil divisors on X). Hence

$$(2.1) \quad \text{Alg}^1(X(\mathbb{R})) = w_X(\text{Pic}^0(X)),$$

where $\text{Pic}^0(X)$ is the subgroup of $\text{Pic}(X)$ consisting of the isomorphism classes of invertible sheaves of the form $\mathcal{O}(D)$ for D in $Z_{\text{alg}}^1(X)$, cf. Section 1.

Recall that the complex manifold $X(\mathbb{C})$ is endowed with the antiholomorphic involution σ_X and $X(\mathbb{R}) = \{x \in X(\mathbb{C}) \mid \sigma_X(x) = x\}$. In particular, $\text{Pic}(X(\mathbb{C}))$ and $\text{Pic}(X(\mathbb{C}))^G$ are defined, and it makes sense to talk about holomorphic line bundles on $X(\mathbb{C})$ defined over \mathbb{R} . For any invertible sheaf \mathcal{L} on X , we let $\mathcal{L}(\mathbb{C})$ denote the associated holomorphic line bundle on $X(\mathbb{C})$. Clearly, $\mathcal{L}(\mathbb{C})$ is defined over \mathbb{R} (with the natural antiholomorphic involution on the total space). Moreover, $\mathcal{L}(\mathbb{R})$ and $\mathcal{L}(\mathbb{C})(\mathbb{R})$ can be canonically identified. The assignment $\mathcal{L} \rightarrow \mathcal{L}(\mathbb{C})$ gives rise to a canonical homomorphism

$$\iota_X : \text{Pic}(X) \rightarrow \text{Pic}(X(\mathbb{C}))^G.$$

Since X is projective and $X(\mathbb{R}) \neq \emptyset$, the homomorphism ι_X is an isomorphism, and

$$\iota_X(\text{Pic}^0(X)) = \text{Pic}^0(X(\mathbb{C}))^G.$$

Furthermore, two holomorphic line bundles on $X(\mathbb{C})$ defined over \mathbb{R} represent the same element in $\text{Pic}(X(\mathbb{C}))$ if and only if they are isomorphic as holomorphic line bundles defined over \mathbb{R} . Indeed, by the GAGA principle [19], holomorphic

line bundles on $X(\mathbb{C})$ can be identified with invertible sheaves on the variety $X \times_{\mathbb{R}} \mathbb{C}$ over \mathbb{C} . A holomorphic line bundle L on $X(\mathbb{C})$ is topologically trivial if and only if it corresponds to a divisor on $X \times_{\mathbb{R}} \mathbb{C}$ that is algebraically equivalent to 0, cf. [8]. Therefore the assertions above follow from descent theory [10]. Consequently, the composed homomorphism

$$w_X^G = w_X \circ \iota_X^{-1} : \text{Pic}(X(\mathbb{C}))^G \rightarrow H^1(X(\mathbb{R}), \mathbb{Z}/2)$$

can be described as follows: given ξ in $\text{Pic}(X(\mathbb{C}))^G$, choose a holomorphic line bundle L on $X(\mathbb{C})$ defined over \mathbb{R} and representing ξ , and then

$$(2.2) \quad w_X^G(\xi) = w_1(L(\mathbb{R})).$$

In view of (2.1), we get

$$(2.3) \quad w_X^G(\text{Pic}^0(X(\mathbb{C}))^G) = \text{Alg}^1(X(\mathbb{R})).$$

Formulas (2.2) and (2.3) will be starting points in the proof of Theorem 1.1.

It will be convenient to adopt the following convention. Let W be a topological space and let U be a subspace of W . For $\Pi = \mathbb{Z}$ or $\Pi = \mathbb{Z}/2$, the homomorphism $H^1(W, \Pi) \rightarrow H^1(U, \Pi)$, induced by the inclusion map $U \hookrightarrow W$, will be called the restriction homomorphism. The homomorphism $H^1(W, \mathbb{Z}) \rightarrow H^1(U, \mathbb{Z}/2)$, which is the composite of the restriction homomorphism $H^1(W, \mathbb{Z}) \rightarrow H^1(U, \mathbb{Z})$ and the reduction modulo 2 homomorphism $H^1(U, \mathbb{Z}) \rightarrow H^1(U, \mathbb{Z}/2)$, also will be called the restriction homomorphism.

Let A be an Abelian variety over \mathbb{R} . We regard $A(\mathbb{R})$ as a real Lie group. For any point x in $A(\mathbb{R})$, let $t_x : A \rightarrow A$ denote translation by x . The induced map $t_x(\mathbb{R}) : A(\mathbb{R}) \rightarrow A(\mathbb{R})$ is translation by x on $A(\mathbb{R})$. Observe that

$$H^1(A(\mathbb{R}), \mathbb{Z}/2)^{\text{inv}} = \{\xi \in H^1(A(\mathbb{R}), \mathbb{Z}/2) \mid t_x(\mathbb{R})^*(\xi) = \xi \text{ for all } x \in A(\mathbb{R})\}$$

is a subgroup of $H^1(A(\mathbb{R}), \mathbb{Z}/2)$.

Lemma 2.4. *With notation as above,*

$$\text{Alg}^1(A(\mathbb{R})) \subseteq H^1(A(\mathbb{R}), \mathbb{Z}/2)^{\text{inv}}.$$

Proof. Let x be a point in $A(\mathbb{R})$. If \mathcal{L} is an invertible sheaf on A whose isomorphism class is in $\text{Pic}^0(A)$, then the sheaves $t_x^*\mathcal{L}$ and \mathcal{L} are isomorphic, cf. [17, Proposition 9.2] and hence $w_1((t_x^*\mathcal{L})(\mathbb{R})) = w_1(\mathcal{L}(\mathbb{R}))$. Since $(t_x^*\mathcal{L})(\mathbb{R})$ is isomorphic to $t_x(\mathbb{R})^*\mathcal{L}(\mathbb{R})$, we complete the proof by applying (2.1). \square

Denote by $A(\mathbb{R})_0$ the connected component of $A(\mathbb{R})$ containing the identity element of the group $A(\mathbb{R})$. Let

$$\begin{aligned} R_A &: H^1(A(\mathbb{C}), \mathbb{Z}) \rightarrow H^1(A(\mathbb{R}), \mathbb{Z}/2) \\ r_A &: H^1(A(\mathbb{C}), \mathbb{Z}) \rightarrow H^1(A(\mathbb{R})_0, \mathbb{Z}/2) \\ \rho_A &: H^1(A(\mathbb{R}), \mathbb{Z}/2) \rightarrow H^1(A(\mathbb{R})_0, \mathbb{Z}/2) \end{aligned}$$

be the restriction homomorphisms, and let

$$\rho_A^{\text{inv}} : H^1(A(\mathbb{R}), \mathbb{Z}/2)^{\text{inv}} \rightarrow H^1(A(\mathbb{R})_0, \mathbb{Z}/2)$$

be the restriction of ρ_A .

Lemma 2.5. *With notation as above,*

$$R_A(H^1(A(\mathbb{C}), \mathbb{Z})) = H^1(A(\mathbb{R}), \mathbb{Z}/2)^{\text{inv}},$$

and ρ_A^{inv} is an isomorphism.

Proof. First we show that ρ_A^{inv} is injective. Suppose that ξ is in $H^1(A(\mathbb{R}), \mathbb{Z}/2)^{\text{inv}}$ and $\rho_A^{\text{inv}}(\xi) = \rho_A(\xi) = 0$. Let S be a connected component of $A(\mathbb{R})$, $j_S : S \hookrightarrow A(\mathbb{R})$ the inclusion map, x a point in S , and $q_x : A(\mathbb{R})_0 \rightarrow S$ the restriction of $t_x(\mathbb{R})$. Note that $t_x(\mathbb{R}) \circ j = j_S \circ q_x$, where $j : A(\mathbb{R})_0 \hookrightarrow A(\mathbb{R})$ is the inclusion map. Hence we have the following equalities in $H^1(A(\mathbb{R})_0, \mathbb{Z}/2)$:

$$q_x^*(j_S^*(\xi)) = j^*(t_x(\mathbb{R})^*(\xi)) = j^*(\xi) = \rho_A(\xi) = 0.$$

Since q_x is a homeomorphism, $j_S^*(\xi) = 0$. This in turn implies $\xi = 0$ for S is an arbitrary connected component of $A(\mathbb{R})$. Thus ρ_A^{inv} is injective as asserted.

Now we prove $R_A(H^1(A(\mathbb{C}), \mathbb{Z})) \subseteq H^1(A(\mathbb{R}), \mathbb{Z}/2)^{\text{inv}}$. If $i : A(\mathbb{R}) \hookrightarrow A(\mathbb{C})$ is the inclusion map, then for any point x in $A(\mathbb{R})$, we have $t_x(\mathbb{C}) \circ i = i \circ t_x(\mathbb{R})$, where $t_x(\mathbb{C}) : A(\mathbb{C}) \rightarrow A(\mathbb{C})$ is the translation induced by t_x . Since $t_x(\mathbb{C})$ is homotopic to the identity map of $A(\mathbb{C})$,

$$i^* = t_x(\mathbb{R})^* \circ i^* : H^1(A(\mathbb{C}), \mathbb{Z}/2) \rightarrow H^1(A(\mathbb{R}), \mathbb{Z}/2),$$

which implies $i^*(H^1(A(\mathbb{C}), \mathbb{Z}/2)) \subseteq H^1(A(\mathbb{R}), \mathbb{Z}/2)^{\text{inv}}$. The inclusion under consideration follows at once.

It remains to prove $\rho_A(R_A(H^1(A(\mathbb{C}), \mathbb{Z}))) = H^1(A(\mathbb{R})_0, \mathbb{Z}/2)$. This however follows immediately since $r_A = \rho_A \circ R_A$, the reduction modulo 2 homomorphism $\pi : H^1(A(\mathbb{C}), \mathbb{Z}) \rightarrow H^1(A(\mathbb{C}), \mathbb{Z}/2)$ is surjective ($A(\mathbb{C})$ is a complex torus), and there is a retraction $A(\mathbb{C}) \rightarrow A(\mathbb{R})_0$ (cf. [5, Theorem 3.1.5(ii)] for the last fact). □

Recall that the number of connected components of $A(\mathbb{R})$ is 2^c for some integer c satisfying $0 \leq c \leq \dim A$, cf. [5, Theorem 3.1.4]

Proposition 2.6. *With notation as above,*

$$R_A(H^1(A(\mathbb{C}), \mathbb{Z})^G) = \text{Alg}^1(A(\mathbb{R})),$$

$$\dim_{\mathbb{Z}/2} \text{Alg}^1(A(\mathbb{R})) = c,$$

where 2^c is the number of connected components of $A(\mathbb{R})$.

Proof. We have the following commutative diagram:

$$\begin{array}{ccc} & & \text{Pic}^0(A(\mathbb{C}))^G \\ & & \downarrow w_A^G \\ H^1(A(\mathbb{C}), \mathbb{Z}) & \xrightarrow{R_A} & H^1(A(\mathbb{R}), \mathbb{Z}/2) \\ \text{identity} \downarrow & & \downarrow \rho_A \\ H^1(A(\mathbb{C}), \mathbb{Z}) & \xrightarrow{r_A} & H^1(A(\mathbb{R})_0, \mathbb{Z}/2). \end{array}$$

Therefore, in view of (2.3) and Lemmas 2.4 and 2.5, it is sufficient to prove the following two equalities:

- (a) $r_A(H^1(A(\mathbb{C}), \mathbb{Z})^G) = \rho_A(w_A^G(\text{Pic}^0(A(\mathbb{C}))^G))$
- (b) $\dim_{\mathbb{Z}/2} r_A(H^1(A(\mathbb{C}), \mathbb{Z})^G) = c.$

We shall establish (a) and (b) by direct computation in terms of a suitable period matrix of A .

Set $n = \dim A$. Let Ω be a complex $n \times 2n$ matrix such that the \mathbb{Z} -submodule $[\Omega]$ of \mathbb{C}^n generated by the columns of Ω has rank $2n$ and is mapped onto itself by the complex conjugation on $\mathbb{C}^n \rightarrow \mathbb{C}^n, z \rightarrow \bar{z}$. The complex conjugation on \mathbb{C}^n gives rise to group action of G on the complex torus $\mathbb{C}^n/[\Omega]$. If there exists a G -equivariant isomorphism between the complex Lie groups $\mathbb{C}^n/[\Omega]$ and $A(\mathbb{C})$, then Ω is said to be a period matrix of A . It is well known that A admits a period matrix of the form

$$(1) \quad \Omega = (I_n, \frac{1}{2}I_{n,k} + \sqrt{-1}S),$$

where I_n is the identity $n \times n$ matrix, $0 \leq k \leq n$, $I_{n,k}$ is obtained from I_n by replacing columns $k + 1$ through n by the columns consisting of zeros, and S is a real $n \times n$ matrix, cf. [5, Theorem 3.1.5].

Let Ω be as in (1), $\Lambda = [\Omega]$, $T = \mathbb{C}^n/\Lambda$, and let $h : T \rightarrow A(\mathbb{C})$ be a G -equivariant isomorphism of complex Lie groups. The complex torus T is endowed with an antiholomorphic involution σ_T induced by the complex conjugation on \mathbb{C}^n . This allows us to use terminology and notation introduced at the beginning of this section. Denote by $T(\mathbb{R})_0$ the connected component of $T(\mathbb{R}) = \{x \in T \mid \sigma_T(x) = x\}$ containing the identity element of the group $T(\mathbb{R})$. We have the following commutative diagram:

$$\begin{CD} \text{Pic}^0(A(\mathbb{C}))^G @>h^*>> \text{Pic}^0(T)^G \\ @V\rho_A \circ w_A^G VV @VVwV \\ H^1(A(\mathbb{R})_0, \mathbb{Z}/2) @>h_0^*>> H^1(T(\mathbb{R})_0, \mathbb{Z}/2) \\ @AAr_A A @AAr A \\ H^1(A(\mathbb{C}), \mathbb{Z})^G @>h^*>> H^1(T, \mathbb{Z})^G \end{CD}$$

where $h_0 : T(\mathbb{R})_0 \rightarrow A(\mathbb{R})_0$ is the restriction of $h : T \rightarrow A(\mathbb{C})$, $w = h_0^* \circ \rho_A \circ w_A^G \circ (h^*)^{-1}$, and $r : H^1(T, \mathbb{Z}) \rightarrow H^1(T(\mathbb{R})_0, \mathbb{Z}/2)$ is the restriction homomorphism. It follows that (a) and (b) are equivalent to

- (a') $r(H^1(T, \mathbb{Z})^G) = w(\text{Pic}^0(T)^G)$
- (b') $\dim_{\mathbb{Z}/2} r(H^1(T, \mathbb{Z})^G) = c.$

We shall now give a more direct description of the homomorphism $w : \text{Pic}^0(T)^G \rightarrow H^1(T(\mathbb{R})_0, \mathbb{Z}/2)$. Given a holomorphic line bundle L on T , we write $[L]$ for its isomorphism class. Each element of $\text{Pic}^0(T)^G$ is of the form $[L]$ for some L defined over \mathbb{R} , and in view of (2.2),

$$(2) \quad w([L]) = w_1(L(\mathbb{R})|T(\mathbb{R})_0).$$

Our next step is an explicit description of $\text{Pic}^0(T)^G$ coming from the Appell-Humbert theorem. We regard the unit circle $\mathbb{C}_1 = \{v \in \mathbb{C} \mid |v| = 1\}$ as a multiplicative group. For any χ in $\text{Hom}(\Lambda, \mathbb{C}_1)$, define an action of Λ on $\mathbb{C}^n \times \mathbb{C}$ by

$$(\lambda, (z, v)) \rightarrow (z + \lambda, \chi(\lambda)v)$$

for all λ in Λ and (z, v) in $\mathbb{C}^n \times \mathbb{C}$. Denote by $L(\chi)$ the holomorphic line bundle on T obtained by taking the quotient of $\mathbb{C}^n \times \mathbb{C}$ by Λ . Then

$$P : \text{Hom}(\Lambda, \mathbb{C}_1) \rightarrow \text{Pic}^0(T), \quad P(\chi) = [L(\chi)]$$

is an isomorphism, cf. [16, p. 32, Theorem 2.3] or [18, p. 20]. There is a natural action of G on $\text{Hom}(\Lambda, \mathbb{C}_1)$, $\chi \rightarrow \chi^\sigma$, where $\chi^\sigma(\lambda) = \overline{\chi(\bar{\lambda})}$ for all λ in Λ . We have $[L(\chi^\sigma)] = [L(\chi)]^\sigma$ (cf. [5, Lemma 3.3.1]), and hence the isomorphism P is G -equivariant.

Note that $T(\mathbb{R})_0$ can be described in a very explicit way as well. Indeed, regarding \mathbb{R}^n as a subset of \mathbb{C}^n , we can also view $\mathbb{R}^n/\mathbb{Z}^n$ as a subtorus of $T = \mathbb{C}^n/\Lambda$ ($\Lambda = [\Omega]$ and Ω is as in (1)). Hence

$$T(\mathbb{R})_0 = \mathbb{R}^n/\mathbb{Z}^n.$$

The canonical isomorphisms $\Lambda \rightarrow H_1(T, \mathbb{Z})$ and $\mathbb{Z}^n \rightarrow H_1(T(\mathbb{R})_0, \mathbb{Z})$ induce canonical isomorphisms

$$H_\Lambda : H^1(T, \mathbb{Z}) = \text{Hom}(H_1(T, \mathbb{Z}), \mathbb{Z}) \rightarrow \text{Hom}(\Lambda, \mathbb{Z})$$

$$H : H^1(T(\mathbb{R})_0, \mathbb{Z}/2) = \text{Hom}(H_1(T(\mathbb{R})_0, \mathbb{Z}), \mathbb{Z}/2) \rightarrow \text{Hom}(\mathbb{Z}^n, \mathbb{Z}/2).$$

Defining an action of G on $\text{Hom}(\Lambda, \mathbb{Z})$ by $\varphi \rightarrow \varphi^\sigma$, where φ is in $\text{Hom}(\Lambda, \mathbb{Z})$ and $\varphi^\sigma(\lambda) = \varphi(\bar{\lambda})$ for all λ in Λ , we see that the isomorphism H_Λ is G -equivariant.

We shall now find connections between the isomorphisms P , H_Λ , H , which will reduce (a') and (b') to simple algebraic statements.

If χ is in $\text{Hom}(\Lambda, \mathbb{C}_1)^G$, then $\chi(\Lambda) \subseteq \{-1, 1\}$, and hence we can define a homomorphism

$$\omega : \text{Hom}(\Lambda, \mathbb{C}_1)^G \rightarrow \text{Hom}(\mathbb{Z}^n, \mathbb{Z}/2)$$

by $\omega(\chi)(\nu) = e(\chi(\nu))$ for all ν in \mathbb{Z}^n , where $e : \{-1, 1\} \rightarrow \mathbb{Z}/2$ is the unique group isomorphism.

Define a homomorphism

$$\pi : \text{Hom}(\Lambda, \mathbb{Z}) \rightarrow \text{Hom}(\mathbb{Z}^n, \mathbb{Z}/2)$$

by $\pi(\varphi)(\nu) = \varphi(\nu) \pmod{2}$ for all φ in $\text{Hom}(\Lambda, \mathbb{Z})$ and ν in \mathbb{Z}^n .

We claim that the following diagram is commutative:

$$\begin{array}{ccc}
 \text{Hom}(\Lambda, \mathbb{C}_1)^G & \xrightarrow{P} & \text{Pic}^0(T)^G \\
 \omega \downarrow & & w \downarrow \\
 \text{Hom}(\mathbb{Z}^n, \mathbb{Z}/2) & \xleftarrow{H} & H^1(T(\mathbb{R})_0, \mathbb{Z}/2) \\
 \pi \uparrow & & r \uparrow \\
 \text{Hom}(\Lambda, \mathbb{Z})^G & \xleftarrow{H_\Lambda} & H^1(T, \mathbb{Z})^G.
 \end{array}
 \tag{3}$$

Since P and H_Λ are G -equivariant isomorphisms, the upper and lower horizontal arrows are well defined isomorphisms. It is clear that the lower square is commutative. In order to show that the upper square is commutative we need an auxiliary construction.

Given an element α in $\text{Hom}(\mathbb{Z}^n, \mathbb{Z}/2)$, define an action of \mathbb{Z}^n on $\mathbb{R}^n \times \mathbb{R}$ by

$$(\nu, (x, u)) \rightarrow (x + \nu, (-1)^{\alpha(\nu)} u)$$

for all ν in \mathbb{Z}^n and (x, u) in $\mathbb{R}^n \times \mathbb{R}$. Let $M(\alpha)$ be the real line bundle on $T(\mathbb{R})_0 = \mathbb{R}^n / \mathbb{Z}^n$ obtained by taking the quotient of $\mathbb{R}^n \times \mathbb{R}$ by \mathbb{Z}^n . We assert

$$H(w_1(M(\alpha))) = \alpha. \tag{4}$$

If $n = 1$, then (4) follows immediately. Consequently, we readily get $H(w_1(M(\alpha_i))) = \alpha_i$ for $1 \leq i \leq n$, where $\alpha_i(\nu) = \nu_i \pmod{2}$ and ν_i is the i th component of ν in \mathbb{Z}^n (think of $M(\alpha_i)$ as a pullback of a real line bundle on \mathbb{R}/\mathbb{Z}). Since the line bundles $M(\alpha + \beta)$ and $M(\alpha) \otimes M(\beta)$ are isomorphic for all α and β in $\text{Hom}(\mathbb{Z}^n, \mathbb{Z}/2)$, we conclude that (4) holds.

By construction, if χ is in $\text{Hom}(\Lambda, \mathbb{C}_1)^G$, then $L(\alpha)(\mathbb{R})|_{T(\mathbb{R})_0}$ is isomorphic to $M(\omega(\chi))$. Hence the upper square of (3) is commutative in view of (2), (4), and the definition of P .

Commutativity of (3) implies that (a') and (b') are equivalent to

$$(a'') \quad \omega(\text{Hom}(\Lambda, \mathbb{C}_1)^G) = \pi(\text{Hom}(\Lambda, \mathbb{Z})^G)$$

$$(b'') \quad \dim_{\mathbb{Z}/2} \pi(\text{Hom}(\Lambda, \mathbb{Z})^G) = c.$$

Recall that $\Lambda = [\Omega]$, where $\Omega = (I_n, \frac{1}{2}I_{n,k} + \sqrt{-1}S)$ is as in (1). Let $\epsilon_1, \dots, \epsilon_n$ (resp. $\gamma_1, \dots, \gamma_n$) be the columns of I_n (resp. S). Setting $\tau_j = \frac{1}{2}(\epsilon_j + \gamma_j)$ for $1 \leq j \leq k$, we obtain a \mathbb{Z} -basis

$$(\epsilon_1, \dots, \epsilon_n, \tau_1, \dots, \tau_k, \gamma_{k+1}, \dots, \gamma_n) \tag{5}$$

for the lattice Λ satisfying

$$\bar{\tau}_j = \epsilon_j - \tau_j \text{ for } 1 \leq j \leq k. \tag{6}$$

Obviously, $(\epsilon_1, \dots, \epsilon_n)$ is the standard \mathbb{Z} -basis for \mathbb{Z}^n . Setting

$$\Gamma = \{\alpha \in \text{Hom}(\mathbb{Z}^n, \mathbb{Z}/2) \mid \alpha(\epsilon_j) = 0 \text{ for } 1 \leq j \leq k\},$$

we assert

$$\pi(\text{Hom}(\Lambda, \mathbb{Z})^G) = \Gamma = \omega(\text{Hom}(\Lambda, \mathbb{C}_1)^G). \tag{7}$$

In order to prove (7) consider the following commutative diagram:

$$\begin{array}{ccc}
 \mathrm{Hom}(\Lambda, \mathbb{Z})^G & \xrightarrow{\Theta} & \mathrm{Hom}(\Lambda, \mathbb{C}_1)^G \\
 \searrow \pi & & \swarrow \omega \\
 & \mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z}/2) &
 \end{array}$$

where $\Theta(\varphi)(\lambda) = (-1)^{\varphi(\lambda)}$ for all φ in $\mathrm{Hom}(\Lambda, \mathbb{Z})^G$ and λ in Λ . In particular, we get

$$\pi(\mathrm{Hom}(\Lambda, \mathbb{Z})^G) \subseteq \omega(\mathrm{Hom}(\Lambda, \mathbb{C}_1)^G).$$

Since (5) is a basis for Λ , the inclusion $\Gamma \subseteq \pi(\mathrm{Hom}(\Lambda, \mathbb{Z})^G)$ is obvious. Thus to complete the proof of (7) it remains to show

$$\rho(\mathrm{Hom}(\Lambda, \mathbb{C}_1)^G) \subseteq \Gamma.$$

If χ is in $\mathrm{Hom}(\Lambda, \mathbb{C}_1)^G$, then $\chi(\Lambda) \subseteq \{-1, 1\}$, and hence, in view of (6), for each j satisfying $1 \leq j \leq k$, we have

$$\chi(\tau_j) = \chi(\bar{\tau}_j) = \chi(\epsilon_j - \tau_j) = \chi(\epsilon_j)\chi(\tau_j)^{-1} = \chi(\epsilon_j)\chi(\tau_j),$$

which implies $\chi(\epsilon_j) = 1$. Hence $\omega(\chi)$ belongs to Γ and the inclusion under consideration is proved. The proof of (7) is now complete.

Note that $\dim_{\mathbb{Z}/2} \Gamma = n - k$. Since (5) is a \mathbb{Z} -basis for Λ , it follows that $T(\mathbb{R})$ has 2^{n-k} connected components, cf. [5, Theorems 3.1.4, 3.1.5]. Thus $A(\mathbb{R})$ also has 2^{n-k} connected components, and hence $c = n - k$. Combining this with (7), we conclude that (a'') and (b'') are satisfied. The proposition is proved. \square

Proof of Theorem 1.1. Step 1. Assume that $X = A$ is an Abelian variety. With notation as in Section 1 and Proposition 2.6,

$$i_A^2(H^2(G, H^1(A(\mathbb{C}), \mathbb{Z}))) = R_A(H^1(A(\mathbb{C}), \mathbb{Z})^G) = \mathrm{Alg}^1(A(\mathbb{R})).$$

Furthermore, $\dim_{\mathbb{Z}/2} \mathrm{Alg}^1(A(\mathbb{R})) = c$, where 2^c is the number of connected components of $A(\mathbb{R})$. On the other hand, it is also known that

$$\dim_{\mathbb{Z}/2} H^2(G, H^1(A(\mathbb{C}), \mathbb{Z})) = c,$$

(cf. [5, proof of Theorem 2.7.9], note that $H^2(G, H^1(A(\mathbb{C}), \mathbb{Z})) = H^1(G, H^1(A(\mathbb{C}), \mathbb{Z}(-1)))$, where the latter group is introduced in [5]). Hence i_A^2 is injective.

Step 2. We shall now deal with arbitrary X . Let A be the Albanese variety of X and let $\alpha : X \rightarrow A$ be the Albanese morphism corresponding to some point in $X(\mathbb{R})$. In particular, A and α are defined over \mathbb{R} . Let $\alpha(\mathbb{C}) : X(\mathbb{C}) \rightarrow A(\mathbb{C})$ and $\alpha(\mathbb{R}) : X(\mathbb{R}) \rightarrow A(\mathbb{R})$ be the maps determined by α . It is well known that

$$\alpha(\mathbb{C})^* : H^1(A(\mathbb{C}), \mathbb{Z}) \rightarrow H^1(X(\mathbb{C}), \mathbb{Z})$$

is an isomorphism. Since $\alpha(\mathbb{C})^*$ is G -equivariant, it induces an isomorphism

$$\alpha(\mathbb{C})^2 : H^2(G, H^1(A(\mathbb{C}), \mathbb{Z})) \rightarrow H^2(G, H^1(X(\mathbb{C}), \mathbb{Z})),$$

included in the following commutative diagram:

$$\begin{array}{ccc} H^2(G, H^1(A(\mathbb{C}), \mathbb{Z})) & \xrightarrow{\alpha(\mathbb{C})^2} & H^2(G, H^1(X(\mathbb{C}), \mathbb{Z})) \\ i_A^2 \downarrow & & i_X^2 \downarrow \\ H^1(A(\mathbb{R}), \mathbb{Z}/2) & \xrightarrow{\alpha(\mathbb{R})^*} & H^1(X(\mathbb{R}), \mathbb{Z}/2). \end{array}$$

As observed in [13, Lemma 2.1], $\alpha(\mathbb{R})^*(\text{Alg}^1(A(\mathbb{R}))) = \text{Alg}^1(X(\mathbb{R}))$, and hence Step 1 implies

$$i_X^2(H^2(G, H^1(X(\mathbb{C}), \mathbb{Z}))) = \text{Alg}^1(X(\mathbb{R})).$$

The proof is complete. \square

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