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EXTREME POINTS IN SPACES OF POLYNOMIALS

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Abstract. We determine the extreme points of the unit ball in spaces of complex polynomials (of a fixed degree), living either on the unit circle or on a subset of the real line and endowed with the supremum norm.

Introduction

Let $\mathcal{P}_n$ stand for the space of all polynomials with complex coefficients of degree not exceeding n. Given a compact set $E \subset \mathbb{C}$, one may treat $\mathcal{P}_n$ as a subspace of $C(E)$, the space of continuous functions on E, and equip it with the maximum norm

$$\|P\|_\infty = \|P\|_{\infty,E} := \max_{z\in E} |P(z)| \qquad (P \in \mathcal{P}_n).$$

The resulting space will be denoted by $\mathcal{P}_n(E)$. We write

$$\operatorname{ball}(\mathcal{P}_n(E)) := \{P \in \mathcal{P}_n : \|P\|_{\infty,E} \le 1\}$$

for the unit ball of $\mathcal{P}_n(E)$, and we shall be concerned with the extreme points of this ball. (As usual, an element of a convex set S is said to be its *extreme point* if it is not the midpoint of any nondegenerate segment contained in S.)

In this paper, we explicitly characterize the extreme points of $\operatorname{ball}(\mathcal{P}_n(E))$ in the case where E is either the circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ or a perfect compact subset of the real line $\mathbb{R}$. The description obtained is, perhaps, a bit more complicated than one could at first expect; however, the complexity seems to be in the nature of things.

Let us begin by recalling that the extreme points of the unit ball in $L^\infty(\mathbb{T})$ – or in $C(\mathbb{T})$ – are precisely the functions of modulus 1. (The same applies to other sets in place of $\mathbb{T}$.) Further, in the space H^∞ of bounded analytic functions on $\{|z| < 1\}$, as well as in the disk algebra $H^\infty \cap C(\mathbb{T})$, the extreme points are known to be the unit-norm functions f with $\int_{\mathbb{T}} \log(1 - |f(z)|^2)|dz| = -\infty$; see [H, Chap. 9].

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Yet another relevant example is provided by a theorem of Konheim and Rivlin [KR], dealing with the space $\mathcal{P}_n^{\mathbb{R}}(I)$ of all *real* polynomials of degree $\le n$ on the segment $I := [-1, 1]$. The theorem states that a unit-norm polynomial P is an extreme point of ball$\left(\mathcal{P}_n^{\mathbb{R}}(I)\right)$ if and only if $\mathcal{N}_I(1 - P^2) > n$; here $\mathcal{N}_I(f)$ is the total number of zeros (multiplicities included) that f has on I. A similar result holds for *real* trigonometric polynomials on $\mathbb{T}$; see [R] or Proposition 1 in Section 1 below.

With these examples in mind, one might be tempted to believe that, in order to recognize the extreme points among all unit-norm elements P of the *complex* space $\mathcal{P}_n(E)$ (say, with $E = \mathbb{T}$ or $E = I$), one only needs to know "how often" $|P|$ takes the extremal value 1 on E. In other words, one might seek to characterize the extreme points P in terms of the zeros – and their multiplicities – of the polynomial $1 - |P|^2$. (Strictly speaking, $1 - |P|^2$ is a trigonometric polynomial for $P \in \mathcal{P}_n(\mathbb{T})$ and a true polynomial when P lives on $\mathbb{R}$.)

However, no such thing can be done. Indeed, along with solving the two versions of the problem in Sections 1 and 2 (one of these deals with the circle, and the other with subsets of $\mathbb{R}$), we also construct in each case a pair of unit-norm polynomials P_1, P_2 in $\mathcal{P}_n(E)$ satisfying

$$1 - |P_1|^2 = 2\left(1 - |P_2|^2\right),$$

so that P_1 is a non-extreme point of ball$(\mathcal{P}_n(E))$, while P_2 is extreme. In fact, the construction is carried out for the smallest possible value of n, which equals 2 when $E = \mathbb{T}$, and 3 when E is a real segment.

In conclusion, we briefly mention the L^1 counterpart of the problem, i.e., the problem of determining the extreme points of the unit ball in certain L^1-spaces of polynomials. Here, the real case was settled by Garkavi [G] and the complex case by the author [D]. Garkavi's, as well as Konheim and Rivlin's results were then rediscovered – or reproved – by Parnes in [P], where the current problem (the case of complex polynomials on $\mathbb{T}$ with the sup-norm) was also considered, but not solved.

I thank Evgeny Abakumov for bringing Parnes' work to my attention.

1. Polynomials on the circle

Among the unit-norm polynomials in $\mathcal{P}_n(\mathbb{T})$, we single out the class of *monomials*; these are of the form cz^k, where $c \in \mathbb{C}$, $|c| = 1$ and $0 \le k \le n$. Of course, every monomial is an extreme point of ball$(\mathcal{P}_n(\mathbb{T}))$.

Now if $P \in \mathcal{P}_n(\mathbb{T})$ satisfies $\|P\|_\infty = 1$ and is distinct from a monomial, let $z_1, \dots, z_N$ be an enumeration of the (nonempty) set $\{z \in \mathbb{T} : |P(z)| = 1\}$. The points $z_1, \dots, z_N$ are thus the distinct zeros of $1 - |P|^2$ lying on $\mathbb{T}$, and the multiplicities of these zeros will be denoted by $2\mu_1, \dots, 2\mu_N$. The μ_j's are positive integers, and their sum

$$\mu := \sum_{j=1}^{N} \mu_j$$

does not exceed n. To see why, note that the function $z \mapsto 1 - |P(z)|^2$ (living on $\mathbb{T}$) is a nonnegative trigonometric polynomial of degree $\le n$, not vanishing identically. Therefore, its zeros lying on $\mathbb{T}$ are necessarily of even order, while the total number of its zeros (multiplicities included) is at most $2n$. Hence $2\mu_1 + \dots + 2\mu_N \le 2n$, so that $\mu \le n$, as claimed above.

Next, for $z = e^{it} \in \mathbb{T}$ and $k \in \mathbb{N}$, consider the Wronski-type matrix

$$W(z;k) = \begin{pmatrix} \overline{z}^{\mu/2}P(z) & \overline{z}^{\mu/2+1}P(z) & \dots & \overline{z}^{n-\mu/2}P(z) \\ \left(\overline{z}^{\mu/2}P(z)\right)' & \left(\overline{z}^{\mu/2+1}P(z)\right)' & \dots & \left(\overline{z}^{n-\mu/2}P(z)\right)' \\ \dots & \dots & \dots & \dots \\ \left(\overline{z}^{\mu/2}P(z)\right)^{(k-1)} & \left(\overline{z}^{\mu/2+1}P(z)\right)^{(k-1)} & \dots & \left(\overline{z}^{n-\mu/2}P(z)\right)^{(k-1)} \end{pmatrix}.$$

The exponent $n - \mu/2$ in the last column should be viewed as $\mu/2 + (n - \mu)$; thus, $W(z;k)$ is a $k \times (n - \mu + 1)$ matrix. The derivatives involved are with respect to the real variable $t = \arg z$.

Let $W_{\mathfrak{R}}(z;k)$ and $W_{\mathfrak{I}}(z;k)$ stand for the real and imaginary parts of $W(z;k)$, respectively. Finally, we need the block matrix

$$W_P = \begin{pmatrix} W_{\mathfrak{R}}(z_1;\mu_1) & W_{\mathfrak{I}}(z_1;\mu_1) \\ W_{\mathfrak{R}}(z_2;\mu_2) & W_{\mathfrak{I}}(z_2;\mu_2) \\ \dots & \dots \\ W_{\mathfrak{R}}(z_N;\mu_N) & W_{\mathfrak{I}}(z_N;\mu_N) \end{pmatrix}.$$

Here, each "entry" $W_{\mathfrak{R}}(z_j;\mu_j)$ or $W_{\mathfrak{I}}(z_j;\mu_j)$ is actually a $\mu_j \times (n - \mu + 1)$ submatrix, as defined above, where everything is computed at the point z_j. In particular, W_P is a $\mu \times 2(n - \mu + 1)$ matrix, and its rank is therefore bounded by $\min(\mu, 2(n - \mu + 1))$.

Theorem 1. *Let $P \in \mathcal{P}_n(\mathbb{T})$, $\|P\|_\infty = 1$. The following are equivalent.*

(i) *P is an extreme point of* $\operatorname{ball}(\mathcal{P}_n(\mathbb{T}))$.

(ii) *Either P is a monomial, or* $\operatorname{rank} W_P = 2(n - \mu + 1)$.

The proof will be preceded by a brief discussion.

First of all, the condition $\operatorname{rank} W_P = 2(n - \mu + 1)$ can only be met if $\mu \ge 2(n - \mu + 1)$, i.e., if $\mu \ge \frac{2}{3}(n + 1)$. (The weaker condition $\mu > n/2$ was pointed out in [P] as necessary in order that P be an extreme point.) The inequalities $\frac{2}{3}(n+1) \le \mu \le n$ being incompatible for $n = 0$ and $n = 1$, there are no nontrivial extreme points for these n. (Here and below, "nontrivial" means "distinct from a monomial".) Now for $n = 2, 3, 4$, the two inequalities – in conjunction with the fact that $\mu \in \mathbb{N}$ – reduce to the condition $\mu = n$, which must be therefore fulfilled by each nontrivial extreme point P of $\operatorname{ball}(\mathcal{P}_n(\mathbb{T}))$.

On the other hand, for $n \ge 2$, the nontrivial extreme points P with $\mu = n$ are characterized by the condition $\operatorname{rank} W_P = 2$, which means that the two columns of W_P are linearly independent. This, in turn, is equivalent to saying that there is no straight line in $\mathbb{C}$ passing through the origin and containing the set

$$\bigcup_{j=1}^{N}\left\{\overline{z}_j^{n/2}P(z_j),\ \left(\overline{z}^{n/2}P\right)'(z_j),\ \dots,\ \left(\overline{z}^{n/2}P\right)^{(\mu_j-1)}(z_j)\right\}.$$

Now let us consider an example.

Example 1. Put $P_0(z) := \frac{1}{2}(z+z^{-1})$, so that $P_0(e^{it}) = \cos t$; then define

$$P_1(z) := zP_0(z) = \frac{1}{2}(z^2+1)$$

and

$$P_2(z) := \frac{z}{\sqrt{2}}\left(P_0(z)+i\right) = \frac{1}{2\sqrt{2}}(z^2+2iz+1).$$

Clearly, P_1 and P_2 are unit-norm polynomials in $\mathcal{P}_2(\mathbb{T})$. In fact, for $z = e^{it} \in \mathbb{T}$,

$$|P_1(z)|^2 = P_0^2(z) = \cos^2 t$$

and

$$|P_2(z)|^2 = \frac{1}{2}\left(P_0^2(z)+1\right) = \frac{1}{2}\left(\cos^2 t+1\right).$$

In particular,

$$1-|P_1(z)|^2 = 2\left(1-|P_2(z)|^2\right), \qquad z \in \mathbb{T}$$

(indeed, both sides equal $\sin^2 t$), and so the two polynomials have the same z_j's and μ_j's. Specifically, these are $z_1 = 1$, $z_2 = -1$ (or vice versa) and $\mu_1 = \mu_2 = 1$, so that $N = \mu = n = 2$.

However, while P_1 is the arithmetic mean of two monomials, z^2 and 1, and hence a *non-extreme* point of ball$(\mathcal{P}_2(\mathbb{T}))$, it turns out that P_2 is an *extreme* point thereof. This last fact follows by Theorem 1, since the matrix

$$W_{P_2} = \begin{pmatrix} \Re\left(P_2(1)\right) & \Im\left(P_2(1)\right) \\ \Re\left(-P_2(-1)\right) & \Im\left(-P_2(-1)\right) \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

has rank 2.

The proof of Theorem 1 will rely on two elementary observations. The first of these, stated for an arbitrary compact set $E \subset \mathbb{C}$, will also be used when proving Theorem 2 in the next section.

Observation 1. Clearly, a given unit-norm polynomial $P \in \mathcal{P}_n$ is an extreme point of ball$(\mathcal{P}_n(E))$ if and only if the only polynomial $Q \in \mathcal{P}_n$ satisfying

$$\|P+Q\|_\infty \le 1 \quad \text{and} \quad \|P-Q\|_\infty \le 1 \tag{1.1}$$

is $Q \equiv 0$. Rewriting (1.1) as

$$|P \pm Q|^2 = |P|^2 \pm 2\Re(\overline{P}Q) + |Q|^2 \leq 1$$

and noting that $\max(x, -x) = |x|$ for all $x \in \mathbb{R}$, we see that P is extreme iff there is no nontrivial $Q \in \mathcal{P}_n$ for which

$$2|\Re(\overline{P}Q)| + |Q|^2 \leq 1 - |P|^2 \tag{1.2}$$

everywhere on E.

Observation 2. If $z = e^{it}$ and $\zeta_j = e^{it_j}$ $(j = 1, \ldots, N)$ are points of $\mathbb{T}$, and if $k_1, \ldots, k_N$ are positive integers with $\sum_{j=1}^{N} k_j = k$, then the identities

$$z - \zeta_j = 2ie^{it_j/2}e^{it/2} \sin \frac{t - t_j}{2}$$

yield

$$\prod_{j=1}^{N}(z - \zeta_j)^{k_j} = ce^{ikt/2} \prod_{j=1}^{N} \left(\sin \frac{t - t_j}{2}\right)^{k_j}, \tag{1.3}$$

where

$$c = (2i)^k \prod_{j=1}^{N} \exp\left(\frac{ik_j t_j}{2}\right).$$

Proof of Theorem 1. (ii) $\Longrightarrow$ (i). We shall assume that P is distinct from a monomial (otherwise, it is obviously extreme) and that $\operatorname{rank} W_P = 2(n - \mu + 1)$. Now suppose (1.2) holds for some $Q \in \mathcal{P}_n$. In particular, we have then

$$|Q(z)|^2 \leq 1 - |P(z)|^2, \qquad z \in \mathbb{T},$$

and so, for $j = 1, \ldots, N$, the polynomial Q vanishes at z_j with multiplicity at least μ_j. (Recall that the multiplicity of z_j as a zero of $1 - |P|^2$ is $2\mu_j$.) Hence Q is divisible by $\prod_{j=1}^{N}(z - z_j)^{\mu_j}$ and takes the form

$$Q(z) = Q_0(z) \cdot z^{\mu/2} \prod_{j=1}^{N} \left(\sin \frac{t - t_j}{2}\right)^{\mu_j}, \qquad z = e^{it} \in \mathbb{T}, \tag{1.4}$$

for some $Q_0 \in \mathcal{P}_{n-\mu}$; by t_j we now denote $\arg z_j$. Here, to arrive at (1.4), we have used (1.3) with z_j in place of ζ_j and with μ_j (resp., μ) in place of k_j (resp., k). From (1.4) we get

$$\Re\left(\overline{P}(z)Q(z)\right) = \prod_{j=1}^{N} \left(\sin \frac{t - t_j}{2}\right)^{\mu_j} \Re\left(z^{\mu/2}\overline{P}(z)Q_0(z)\right). \tag{1.5}$$

Combining this with the fact that

$$\left|\Re\left(\overline{P}(z)Q(z)\right)\right| \le 1-|P(z)|^2$$

(which is contained in (1.2)) yields

$$\left|\Re\left(z^{\mu/2}\overline{P}(z)Q_0(z)\right)\right| \le \left(1-|P(z)|^2\right)\prod_{j=1}^{N}\left|\sin\frac{t-t_j}{2}\right|^{-\mu_j}, \quad z=e^{it}\in\mathbb{T}. \tag{1.6}$$

The right-hand side of (1.6) being $O\left(|z-z_j|^{\mu_j}\right)$ as $z\to z_j$, the left-hand side must also vanish at each z_j with multiplicity at least μ_j. In other words, for each $j=1,\dots,N$, one has

$$\Re\left(z^{\mu/2}\overline{P}Q_0\right)^{(l)}(z_j)=0 \qquad (l=0,1,\dots,\mu_j-1). \tag{1.7}$$

Putting

$$Q_0(z)=\sum_{k=0}^{n-\mu}(c_k+id_k)z^k$$

and substituting this into (1.7), we obtain

$$\sum_{k=0}^{n-\mu}c_k\Re\left(z^{\mu/2+k}\overline{P}\right)^{(l)}(z_j)-\sum_{k=0}^{n-\mu}d_k\Im\left(z^{\mu/2+k}\overline{P}\right)^{(l)}(z_j)=0 \tag{1.8}$$
$$(j=1,\dots,N;\ l=0,1,\dots,\mu_j-1),$$

which can be viewed as a homogeneous system of $\mu_1+\dots+\mu_N=\mu$ linear equations with $2(n-\mu+1)$ real unknowns $c_0,\dots,c_{n-\mu},d_0,\dots,d_{n-\mu}$. The matrix of this system is precisely W_P, and the hypothesis $\operatorname{rank}W_P=2(n-\mu+1)$ implies that the only solution is

$$c_0=\dots=c_{n-\mu}=d_0=\dots=d_{n-\mu}=0.$$

Thus $Q_0\equiv 0$, whence also $Q\equiv 0$, and P is an extreme point.

(i) $\Longrightarrow$ (ii). The above argument can be essentially reversed. Indeed, suppose that (ii) fails, so that P is distinct from a monomial and $\operatorname{rank}W_P<2(n-\mu+1)$. The homogeneous system (1.8) has then a nontrivial solution, and the equations (1.7) hold for $j=1,\dots,N$ with some $Q_0\in\mathcal{P}_{n-\mu}$, $Q_0\not\equiv 0$. Multiplying Q_0 by a number $\varepsilon>0$ (if necessary), we may assume in addition that the norm $\|Q_0\|_\infty$ is appropriately small; we shall specify our choice later.

Now that we have such a Q_0 at our disposal, let us define Q by (1.4), where, as before, it is understood that $z_j=e^{it_j}$. By Observation 2, we have $Q\in\mathcal{P}_n$; we also remark that $Q\not\equiv 0$, because $Q_0\not\equiv 0$, and that (1.5) holds true.

We further claim that

$$|Q(z)|^2 = O\left(1 - |P(z)|^2\right), \qquad z \in \mathbb{T}, \tag{1.9}$$

and

$$\left|\Re\left(\overline{P}(z)Q(z)\right)\right| = O\left(1 - |P(z)|^2\right), \qquad z \in \mathbb{T}. \tag{1.10}$$

Indeed, (1.9) is fulfilled because Q is divisible by $\prod_{j=1}^{N}(z-z_j)^{\mu_j}$, and so $|Q(z)|^2$ vanishes at those points of $\mathbb{T}$ (viz., $z_1, \dots, z_N$) where $1 - |P(z)|^2$ does, with at least the same multiplicities (viz., $2\mu_1, \dots, 2\mu_N$). Similarly, to verify (1.10), one checks that its left-hand side has a zero at each z_j of multiplicity $\ge 2\mu_j$; this is due to (1.5) and (1.7).

In view of the above discussion, we could have started with a Q_0 for which the quantity $\|Q_0\|_\infty$, and hence also the "big oh" constants in (1.9) and (1.10), are as small as desired. In particular, a suitable choice ensures that

$$|Q(z)|^2 \le \frac{1}{2}\left(1 - |P(z)|^2\right), \qquad z \in \mathbb{T},$$

and

$$2\left|\Re\left(\overline{P}(z)Q(z)\right)\right| \le \frac{1}{2}\left(1 - |P(z)|^2\right), \qquad z \in \mathbb{T}.$$

Summing, we arrive at (1.2) and conclude that P is not an extreme point. The proof is complete. □

One might also consider the space $\mathcal{T}_n$ of all *trigonometric polynomials* of degree $\le n$; these are, by definition, functions of the form $\sum_{k=-n}^{n} c_k z^k$ living on $\mathbb{T}$. A trigonometric polynomial $T \in \mathcal{T}_n$ is an extreme point of ball($\mathcal{T}_n$) if and only if $z^n T$ is an extreme point of ball($\mathcal{P}_{2n}$). Thus, the extreme points T of ball($\mathcal{T}_n$) are actually described by Theorem 1, where obvious adjustments are needed: one should first replace n by $2n$, and then P by $z^n T$. (Of course, the monomials in the theorem's statement should now include those with negative exponents, too.)

Finally, we briefly discuss the subspace $\mathcal{T}_n^{\mathbb{R}}$ of real-valued functions in $\mathcal{T}_n$; a trigonometric polynomial $\sum_{k=-n}^{n} c_k z^k$ is thus in $\mathcal{T}_n^{\mathbb{R}}$ iff $c_{-k} = \bar{c}_k$ for $|k| \le n$. As before, given a nonconstant $P \in \mathcal{T}_n^{\mathbb{R}}$ with $\|P\|_\infty = 1$, we let $z_j = e^{it_j}$ ($j = 1, \dots, N$) be the distinct zeros that the (nonnegative) trigonometric polynomial $1 - P^2$ happens to have on $\mathbb{T}$; the (even) multiplicity of the zero z_j is again denoted by $2\mu_j$, and we write $\mu = \sum_{j=1}^{N} \mu_j$. This time, however, $1 - P^2$ is of degree $\le 2n$, so the only *a priori* estimate on μ is that $\mu \le 2n$. As to the constant polynomials $P \equiv 1$ and $P \equiv -1$, for each of these we put $\mu = +\infty$.

The following proposition is a trigonometric version of the Konheim–Rivlin result that can be found in [R]; a short self-contained proof will be given here for the sake of completeness.

Proposition 1. *Let* $P \in \mathcal{T}_n^{\mathbb{R}}$, $\|P\|_\infty = 1$. *Then* P *is an extreme point of* $\text{ball}\left(\mathcal{T}_n^{\mathbb{R}}\right)$ *if and only if* $\mu > n$.

Proof. To prove the "if" part, assume that $\mu > n$ and that (1.1) holds for some $Q \in \mathcal{T}_n^{\mathbb{R}}$. We have then $\pm P \pm Q \le 1$ on $\mathbb{T}$, where the signs can be chosen in the four possible ways. Consequently,

$$|Q| \le 1 - |P| \le 1 - P^2.$$

Now since the right-hand side has in total 2μ $(> 2n)$ zeros on $\mathbb{T}$, while Q is of degree $\le n$, it follows that $Q \equiv 0$ and P is an extreme point.

To establish the "only if" part, assume that $\mu \le n$ and put

$$Q(e^{it}) := \varepsilon \prod_{j=1}^{N} \left(\sin \frac{t - t_j}{2} \right)^{2\mu_j},$$

with a suitable $\varepsilon > 0$. Then $Q \in \mathcal{T}_n^{\mathbb{R}}$, and making ε sufficiently small we can arrange it so that

$$|Q| \le \frac{1}{2}\left(1 - P^2\right) \le 1 - |P|.$$

From this, (1.1) follows immediately, and P fails to be extreme. $\square$

We remark, in conclusion, that every nonconstant trigonometric polynomial in $\text{ball}\left(\mathcal{T}_n^{\mathbb{R}}\right)$ is a non-extreme point of $\text{ball}(\mathcal{T}_n)$, the unit ball of the *complex* space $\mathcal{T}_n$.

2. Polynomials on subsets of $\mathbb{R}$

Let K be a perfect compact subset of $\mathbb{R}$ (as usual, "perfect" means "having no isolated points"), and let P be a nonconstant polynomial in $\mathcal{P}_n(K)$ with $\|P\|_\infty = 1$. Here and throughout this section, $\|P\|_\infty$ stands for $\|P\|_{\infty,K} := \max_{x\in K} |P(x)|$. (Likewise, some of the other symbols below should not be confused with their namesakes in Section 1.)

Further, let $x_1, \dots, x_N$ be the distinct elements of the set $\{x \in K : |P(x)| = 1\}$, and let $m_1, \dots, m_N$ denote the respective multiplicities of these points, regarded as zeros for $1 - |P|^2$. (We remark that m_j need not be even, unless x_j is an interior point for K.) The function $x \mapsto 1 - |P(x)|^2$ being a polynomial of degree $\le 2n$, we have $m_1 + \cdots + m_N \le 2n$. Next, we introduce the numbers

$$\mu_j := \left[\frac{m_j + 1}{2} \right] \qquad (j = 1, \dots, N),$$

where $[\cdot]$ denotes the integral part, and their sum $\mu := \sum_{j=1}^{N} \mu_j$. Finally, let M be the number of those j's for which $m_j \ge 2$. Thus $0 \le M \le N$, and we may assume that the inequality $m_j \ge 2$ holds precisely for $1 \le j \le M$.

For a constant polynomial $P \equiv c$ with $|c| = 1$, we put $\mu = +\infty$.

Now suppose P is a unit-norm polynomial in $\mathcal{P}_n(K)$ with the property $\mu \le n$. To such a P, we associate the Wronski-type matrix

$$W(x;k) = \begin{pmatrix} P(x) & xP(x) & \dots & x^{n-\mu}P(x) \\ P'(x) & (xP(x))' & \dots & (x^{n-\mu}P(x))' \\ \dots & \dots & \dots & \dots \\ P^{(k-1)}(x) & (xP(x))^{(k-1)} & \dots & (x^{n-\mu}P(x))^{(k-1)} \end{pmatrix},$$

where $x \in \mathbb{R}$ and $k \in \mathbb{N}$. The real and imaginary parts of $W(x;k)$ will be denoted by $W_{\mathfrak{R}}(x;k)$ and $W_{\mathfrak{I}}(x;k)$. This said, we form the block matrix

$$W_P = \begin{pmatrix} W_{\mathfrak{R}}(x_1;\, m_1 - \mu_1) & W_{\mathfrak{I}}(x_1;\, m_1 - \mu_1) \\ W_{\mathfrak{R}}(x_2;\, m_2 - \mu_2) & W_{\mathfrak{I}}(x_2;\, m_2 - \mu_2) \\ \dots & \dots \\ W_{\mathfrak{R}}(x_M;\, m_M - \mu_M) & W_{\mathfrak{I}}(x_M;\, m_M - \mu_M) \end{pmatrix},$$

which has $\sum_{j=1}^{N} m_j - \mu$ rows and $2(n-\mu+1)$ columns. In the case that $M = 0$ (i.e., when $m_j = \mu_j = 1$ for all j), it is understood that W_P is the zero matrix (of any order), so that $\operatorname{rank} W_P = 0$.

Theorem 2. *Let $P \in \mathcal{P}_n(K)$ and $\|P\|_\infty = 1$. The following are equivalent.*

(i) *P is an extreme point of* $\operatorname{ball}(\mathcal{P}_n(K))$.

(ii) *Either $\mu > n$, or* $\operatorname{rank} W_P = 2(n-\mu+1)$.

One easily checks that for $n \le 2$, condition (ii) reduces to just saying that $\mu > n$. It is for $n \ge 3$ that things become more complicated, as the following example shows.

Example 2. Let $K = [-1, 2]$, and put

$$P_1(x) := \frac{1}{2}(x^3 - 3x), \qquad P_2(x) := \frac{1}{\sqrt{2}}\left(P_1(x) + i\right).$$

One easily verifies that $|P_1(x)| \le 1$ for $x \in K$, the equality being attained at the points

$$x_1 = -1, \quad x_2 = 1, \quad x_3 = 2. \tag{2.1}$$

Then one deduces a similar fact for P_2 by noting that

$$|P_2(x)|^2 = \frac{1}{2}\left(P_1^2(x) + 1\right).$$

Thus, P_1 and P_2 are unit-norm elements of $\mathcal{P}_3(K)$. Furthermore,

$$1 - P_1^2(x) = 2\left(1 - |P_2(x)|^2\right) = -\frac{1}{4}(x+2)(x+1)^2(x-1)^2(x-2).$$

The zeros of this last polynomial belonging to K (i.e., the common x_j's for P_1 and P_2) are given by (2.1), and the corresponding (common) multiplicities are

$$m_1 = 2, \quad m_2 = 2, \quad m_3 = 1.$$

Hence $\mu_1 = \mu_2 = \mu_3 = 1$, so that $N = \mu = n = 3$ and $M = 2$. Theorem 2 tells us now that P_1 is a *non-extreme* point of $\operatorname{ball}(\mathcal{P}_3(K))$, while P_2 is *extreme*. Indeed, the polynomial P_1 being real-valued, the second column of the (2×2)-matrix W_{P_1} is null, whence $\operatorname{rank} W_{P_1} = 1$, whereas the matrix

$$W_{P_2} = \begin{pmatrix} \Re\,(P_2(-1)) & \Im\,(P_2(-1)) \\ \Re\,(P_2(1)) & \Im\,(P_2(1)) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

has rank 2.

Proof of Theorem 2. (ii) $\Longrightarrow$ (i). Suppose (1.1) is fulfilled for some $Q \in \mathcal{P}_n$. Then (1.2) holds everywhere on K, whence in particular

$$|Q(x)|^2 \le 1 - |P(x)|^2, \qquad x \in K.$$

Here, the right-hand side is $O\left(|x - x_j|^{m_j}\right)$ as $x \to x_j$, and so

$$Q(x) = O\left(|x - x_j|^{m_j/2}\right) \quad \text{as } x \to x_j,\ x \in K. \tag{2.2}$$

Since Q is a polynomial, while μ_j is the smallest integer in the interval $[m_j/2, \infty)$, it actually follows from (2.2) that Q has a zero of multiplicity $\ge \mu_j$ at x_j. Hence

$$Q(x) = Q_0(x) \prod_{j=1}^{N} (x - x_j)^{\mu_j} \tag{2.3}$$

for some polynomial Q_0.

Now if $\mu > n$, then (2.3) is only possible for $Q \equiv 0$, which implies that P is an extreme point.

It remains to consider the case where $\mu \le n$ and $\operatorname{rank} W_P = 2(n - \mu + 1)$. In this case, (2.3) holds for some $Q_0 \in \mathcal{P}_{n-\mu}$, and we write

$$Q_0(x) = \sum_{k=0}^{n-\mu} (c_k + i d_k) x^k \tag{2.4}$$

with $c_k, d_k \in \mathbb{R}$. Also, (2.3) yields

$$\Re\left(\overline{P}(x) Q(x)\right) = \prod_{j=1}^{N} (x - x_j)^{\mu_j} \Re\left(\overline{P}(x) Q_0(x)\right). \tag{2.5}$$

Substituting this into the inequality

$$\left|\Re\left(\overline{P}(x)Q(x)\right)\right| \le 1-|P(x)|^2, \qquad x\in K$$

(which is a consequence of (1.2)), we get

$$\prod_{j=1}^{N}|x-x_j|^{\mu_j}\left|\Re\left(\overline{P}(x)Q_0(x)\right)\right| \le 1-|P(x)|^2, \qquad x\in K. \tag{2.6}$$

The right-hand side of (2.6) being $O\left(|x-x_j|^{m_j}\right)$ as $x\to x_j$, we deduce that

$$\Re\left(\overline{P}(x)Q_0(x)\right)=O\left(|x-x_j|^{m_j-\mu_j}\right) \quad \text{as } x\to x_j,\ x\in K. \tag{2.7}$$

Here, the restriction $x\in K$ can be actually dropped (i.e., replaced by $x\in\mathbb{R}$), since $\Re\left(\overline{P}Q_0\right)$ is a polynomial. Thus (2.7) tells us that $\Re\left(\overline{P}Q_0\right)$ vanishes at x_j with multiplicity at least $m_j-\mu_j$; of course, this is only meaningful for $1\le j\le M$, since otherwise $m_j=\mu_j=1$. Therefore,

$$\Re\left(\overline{P}Q_0\right)^{(l)}(x_j)=0 \qquad (1\le j\le M,\ 0\le l\le m_j-\mu_j-1). \tag{2.8}$$

With (2.4) plugged in, (2.8) becomes a homogeneous system of linear equations with respect to the unknowns $c_0,\dots,c_{n-\mu},d_0,\dots,d_{n-\mu}$. The matrix of the system is W_P, and the hypothesis $\operatorname{rank}W_P=2(n-\mu+1)$ ensures that the only solution is the trivial one. Hence $Q_0\equiv 0$, which implies $Q\equiv 0$ and proves that P is an extreme point.

(i) $\Longrightarrow$ (ii). Conversely, if $\mu\le n$ and $\operatorname{rank}W_P<2(n-\mu+1)$, then the homogeneous system just mentioned has a nontrivial solution, so that (2.8) holds with some $Q_0\in\mathcal{P}_{n-\mu}$, $Q_0\not\equiv 0$. Now if the norm $\|Q_0\|_\infty$ is appropriately small (which can be safely assumed), then the nontrivial polynomial $Q\in\mathcal{P}_n$ defined by (2.3) will satisfy

$$|Q|^2\le\frac{1}{2}\left(1-|P|^2\right) \tag{2.9}$$

and

$$2\left|\Re\left(\overline{P}Q\right)\right|\le\frac{1}{2}\left(1-|P|^2\right) \tag{2.10}$$

everywhere on K. Indeed, for $j=1,\dots,N$, the left-hand sides of (2.9) and (2.10) vanish at x_j with multiplicity at least m_j each. (To see why, recall that $2\mu_j\ge m_j$ and use the relations (2.5) and (2.8).)

Taken together, (2.9) and (2.10) yield (1.2), and we conclude that P fails to be extreme in $\operatorname{ball}(\mathcal{P}_n(K))$. $\square$

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