Mathematical Research Letters **10**, 659–669 (2003)

ON IMPROVED SOBOLEV EMBEDDING THEOREMS

M. Ledoux

Abstract. We present a direct proof of some recent improved Sobolev inequalities put forward by A. Cohen, R. DeVore, P. Petrushev and H. Xu [C-DV-P-X] in their wavelet analysis of the space $BV(\mathbb{R}^2)$. These inequalities are parts of the Hardy-Littlewood-Sobolev theory, connecting Sobolev embeddings and heat kernel bounds. The argument, relying on pseudo-Poincaré inequalities, allows us to study the dependence of the constants with respect to dimension and to consider several extensions to manifolds and graphs.

As part of the classical Sobolev inequalities, it is well known that for every function f on $\mathbb{R}^n$ vanishing at infinity in some mild sense,

$$\|f\|_q \le C\,\|\nabla f\|_1, \tag{1}$$

where $q = \frac{n}{n-1}$ and $C > 0$ only depends on n. The Sobolev inequality (1) is invariant under the $ax+b$ ($a>0$, $b \in \mathbb{R}^n$) group action, but is not under the Weyl-Heisenberg group action. Namely, if $f(x) = f_\omega(x) = e^{i\omega\cdot x}\varphi(x)$ where φ is in the Schwartz class, then $\|\nabla f\|_1 = |\omega|\|\varphi\|_1 + O(1)$ when $|\omega| \to \infty$, so that (1) is not well-suited for such modulated functions.

In their study of the space $BV(\mathbb{R}^2)$, A. Cohen et al. [C-DV-P-X] (see also [C-M-O]) improved the Sobolev inequality (1) into

$$\|f\|_q \le C\,\|\nabla f\|_1^{1/q}\,\|f\|_B^{1-(1/q)}, \tag{2}$$

where $B = B^{-(n-1)}_{\infty,\infty}$ is the homogeneous Besov space of indices $(-(n-1), \infty, \infty)$ (see below). This improved Sobolev inequality is easily seen to be sharper than (1). Furthermore, if $f(x) = f_\omega(x) = e^{i\omega\cdot x}\varphi(x)$ as above for some φ with enough regularity, it may easily be shown that $\|f\|_B = |\omega|^{-(n-1)}\|\varphi\|_\infty + O(|\omega|^{-n})$ so that (2) amounts in this case to the trivial bound $\|\varphi\|_q \le \|\varphi\|_1^{1/q}\|\varphi\|_\infty^{1-(1/q)}$.

The proof of (2) in [C-DV-P-X] and [C-M-O] is based on wavelet decompositions together with weak-ℓ^1 type estimates and interpolation results. The purpose of this note is to propose a direct semigroup argument without any use of wavelet decomposition. The approach we suggest also relies on weak-type estimates, but emphasizes the use of pseudo-Poincaré inequalities (cf. [SC]) for

Received July 16, 2001.

families of operators (heat kernels for example). The argument easily extends to more general frameworks including manifolds or graphs, out of reach of wavelets methods. It furthermore allows us to establish inequalities with constants independent of the dimension (for a certain range of parameters).

The first section of this note is devoted to the detailed proof of the improved Sobolev inequality (2) in the Euclidean case. One step of the proof is reminiscent of the Marcinkiewicz interpolation theorem. The result is moreover much in the spirit of the Varopoulos theorem on the equivalence between heat kernel bounds and Sobolev inequalities. We then discuss various extensions of the argument to reach similar inequalities in more general frameworks, including abstract Hardy-Littlewood-Sobolev theory and Varopoulos' theorem, manifolds with non-negative Ricci curvature and Cayley graphs, and to study dependence of the constants with respect to dimension.

1. Improved Sobolev inequality in $\mathbb{R}^n$

Equip $\mathbb{R}^n$ with Lebesgue measure λ, and denote by $\|\cdot\|_p$ the L^p-norms. For simplicity, we only work with real-valued functions. We make essential use of the thermic description of Besov spaces (cf. [Tr]). To this task, let $P_t = e^{t\Delta}$, $t \geq 0$, be the heat semigroup on $\mathbb{R}^n$. For $\alpha < 0$, define then the Besov space $B^{\alpha}_{\infty,\infty}$ as the space of tempered distributions f on $\mathbb{R}^n$ for which the Besov norm

$$\|f\|_{B^{\alpha}_{\infty,\infty}} = \sup_{t>0} t^{-\alpha/2} \|P_t f\|_\infty$$

is finite. The space $B^{\alpha}_{\infty,\infty}$ is the homogeneous Besov space of indices (α, ∞, ∞) for which various descriptions are available ([Tr], [Me], [C-M-0]...).

The following theorem has been established first in [C-DV-P-X] for $p = 1$, $q = 2$ $(\alpha = -1)$ in dimension 2, and for $p = 1$, $q = \frac{n}{n-1}$ $(\alpha = -(n-1))$, that corresponds to (2), in [C-M-O]. The case $p = 1$, $q = 2$ $(\alpha = -1)$ is extended in [C-D-D-DV] to in any dimension as a particular case of some further interpolation inequalities between Besov spaces and the space of bounded variations (not covered by the method presented here). It should be emphasized, according to the previous references, that Littlewood-Paley analysis may be developed to cover the case $p > 1$. The value $p = 1$, of most interest, is not covered by these tools and thus requires an independent investigation.

Theorem 1. *For every $1 \leq p < q < \infty$ and every function f in the Sobolev space $W^{1,p}$,*

$$\|f\|_q \leq C \, \|\nabla f\|_p^{\theta} \, \|f\|^{1-\theta}_{B^{\theta/(\theta-1)}_{\infty,\infty}},$$

where $\theta = \frac{p}{q}$ and $C > 0$ only depends on p and q (and n).

By the heat kernel embeddings $\|P_t\|_{r\to\infty} \leq C\, t^{-n/2r}$, $r \geq 1$, one easily recovers from Theorem 1 the classical Sobolev inequalities $\|f\|_q \leq C \, \|\nabla f\|_p$,

$\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$, $1 \leq p < n$, as well as the Gagliardo-Nirenberg inequalities

$$\|f\|_q \leq C \|\nabla f\|_p^{p/q} \, \|f\|_r^{1-(p/q)}, \quad \frac{1}{q} = \frac{1}{p} - \frac{r}{qn}\,.$$

Theorem 1, $p = 1$, may be extended by standard approximations to functions of bounded variations (cf. [C-DV-P-X], [C-D-D-DV]).

The rest of this section is devoted to the proof of Theorem 1. We divide the proof into three steps for the purpose of the extensions we discuss next.

Proof.

Step 1. We note first that the weak-type inequality

$$\|f\|_{q,\infty} \leq C \, \|\nabla f\|_p^{\theta} \, \|f\|_{B^{\theta/(\theta-1)}_{\infty,\infty}}^{1-\theta} \tag{3}$$

is very easy to prove. Assume by homogeneity that $\|f\|_{B^{\theta/(\theta-1)}_{\infty,\infty}} \leq 1$. For every $u > 0$, let $t = t_u = u^{2(\theta-1)/\theta}$ so that $|P_{t_u} f| \leq u$. Then,

$$u^q \lambda\big(\{|f| \geq 2u\}\big) \leq u^p \lambda\big(|f - P_{t_u} f| \geq u\big) \leq u^{q-p} \int |f - P_{t_u} f|^p d\lambda.$$

The main tool of the approach is the following pseudo-Poincaré inequality: for any f in $W^{1,p}$ and $t \geq 0$,

$$\|f - P_t f\|_p \leq C \, \sqrt{t} \, \|\nabla f\|_p \tag{4}$$

(where $C > 0$ only depends on p and n). Inequality (4) is classical and may be established in a rather easy way. For example, write

$$f - P_t f = \int \big[f(\cdot) - f(\cdot - y)\big] p_t(y) dy$$

where $p_t(y)$ is the Gaussian kernel. Therefore,

$$\|f - P_t f\|_p \leq \int \big\|f(\cdot) - f(\cdot - y)\big\|_p \, p_t(y) dy.$$

Since $\|f(\cdot) - f(\cdot - y)\|_p \leq |y| \, \|\nabla f|_p$ when $f \in W^{1,p}$, the conclusion follows with $C = \int |y| \, e^{-|y|^2/2} (2\pi)^{-n/2} dy \leq A\sqrt{n}$ where $A > 0$ is numerical.

By means of (4), and since $q - p + p(\theta - 1)/\theta = 0$, we then get that

$$u^q \lambda\big(\{|f| \geq 2u\}\big) \leq C u^{q-p} \, t_u^{p/2} \int |\nabla f|^p d\lambda = C \int |\nabla f|^p d\lambda$$

which amounts to the weak-type inequality (3) by definition of $\|f\|_{q,\infty}$.

Step 2. We now would like to replace the weak L^q-norm in (3) by the strong one. Standard arguments involving cut-off functions are available to this task (see e.g. [B-C-L-SC]). One difficulty however in this case is that we may not reduce to non-negative functions. The argument we present is a refinement of the preceding step by means of some interpolation tools inspired from proof of the Marcinkiewicz theorem.

Start first with a function $f \in W^{1,p}$ such that $f \in \mathrm{L}^q$. We will see in the third step how to get rid of the latter. Assume as before that $\|f\|_{B^{\theta/(\theta-1)}_{\infty,\infty}} \leq 1$ and let us show that

$$\int |f|^q d\lambda \leq C \int |\nabla f|^p d\lambda \tag{5}$$

for some constant $C > 0$ (depending on q and n), which amounts to the inequality of the theorem. For $u > 0$, let again $t = t_u = u^{2(\theta-1)/\theta}$. Let $c \geq 5$ (depending on q and p) to be specified later. Write

$$\frac{1}{5^q}\,\|f\|_q^q = \int_0^\infty \lambda\big(\{|f| \geq 5u\}\big) d(u^q).$$

For every $u > 0$, set

$$\tilde{f}_u = (f-u)^+ \wedge \big((c-1)u\big) + (f+u)^- \vee \big(-(c-1)u\big).$$

On $\{|f| \geq 5u\}$, $|\tilde{f}_u| \geq 4u$. Therefore,

$$\begin{aligned}
\int_0^\infty \lambda\big(\{|f| \geq 5u\}\big) d(u^q) &\leq \int_0^\infty \lambda\big(\{|\tilde{f}_u| \geq 4u\}\big) d(u^q) \\
&\leq \int_0^\infty \lambda\Big(\big\{\big|\tilde{f}_u - P_{t_u}(\tilde{f}_u)\big| \geq u\big\}\Big) d(u^q) \\
&\quad + \int_0^\infty \lambda\Big(\big\{P_{t_u}\big(|f - \tilde{f}_u|\big) \geq 2u\big\}\Big) d(u^q)
\end{aligned}$$

where we used that $|P_{t_u}(f)| \leq u$.

By (4) applied to $\tilde{f}_u$,

$$\begin{aligned}
\lambda\Big(\big\{\big|\tilde{f}_u - P_{t_u}(\tilde{f}_u)\big| \geq u\big\}\Big) &\leq u^{-p} \int \big|\tilde{f}_u - P_{t_u}(\tilde{f}_u)\big|^p d\lambda \\
&\leq C\, u^{-p}\, t_u^{p/2} \int |\nabla \tilde{f}_u|^p d\lambda \\
&\leq C\, u^{-q} \int_{\{u \leq |f| \leq cu\}} |\nabla f|^p d\lambda.
\end{aligned}$$

Hence

$$\begin{aligned}
\int_0^\infty \lambda\Big(\big\{\big|\tilde{f}_u - P_{t_u}(\tilde{f}_u)\big| \geq u\big\}\Big) d(u^q) &\leq C\, q \int |\nabla f|^p \bigg(\int_{|f|/c}^{|f|} \frac{du}{u}\bigg) d\lambda \\
&= C\, q \log c \int |\nabla f|^p d\lambda.
\end{aligned}$$

On the other hand, since

$$|f - \tilde{f}_u| = |f - \tilde{f}_u|\, 1_{\{|f| \le cu\}} + |f - \tilde{f}_u|\, 1_{\{|f|>cu\}} \le u + |f|\, 1_{\{|f|>cu\}},$$

it follows that

$$\begin{aligned}
\int_0^\infty \lambda\Big(\big\{P_{t_u}\big(|f - \tilde{f}_u|\big) \ge 2u\big\}\Big) d(u^q) &\le \int_0^\infty \lambda\Big(\big\{P_{t_u}\big(|f|\, 1_{\{|f|>cu\}}\big) \ge u\big\}\Big) d(u^q) \\
&\le \int_0^\infty \frac{1}{u}\bigg(\int |f|\, 1_{\{|f|>cu\}} d\lambda\bigg) d(u^q) \\
&= \frac{q}{q-1} \int |f| \bigg(\int_0^\infty 1_{\{|f|>cu\}} d(u^{q-1})\bigg) d\lambda \\
&= \frac{q}{q-1} \cdot \frac{1}{c^{q-1}}\, \|f\|_q^q.
\end{aligned}$$

Combining the preceding estimates,

$$\frac{1}{5^q}\, \|f\|_q^q \le C\, q \log c \int |\nabla f|^p d\lambda + \frac{q}{q-1} \cdot \frac{1}{c^{q-1}}\, \|f\|_q^q.$$

Choose then c large enough depending only on $q > 1$ to deduce (5).

Step 3. In this final step, one just would like to get rid of the assumption $f \in \mathrm{L}^q$ in the preceding argument. Some approximation argument should be enough, however we have not been able to produce a reasonably simple one. The following is a simple modification of the preceding proof that is enough for this purpose. Let thus f be such that $\|\nabla f\|_p < \infty$ and $\|f\|_{B^{\theta/(\theta-1)}_{\infty,\infty}} \le 1$ (by homogeneity). Since by the weak-type inequality (3) $\|f\|_{q,\infty} < \infty$, we may define for every ε, $0 < \varepsilon < 1$,

$$N_\varepsilon(f) = \int_\varepsilon^{1/\varepsilon} \lambda\big(\{|f| \ge 5u\}\big) d(u^q) < \infty.$$

The 2nd step shows at some point that

$$N_\varepsilon(f) \le C\, q \log c \int |\nabla f|^p d\lambda + \int_\varepsilon^{1/\varepsilon} \frac{1}{u}\bigg(\int |f|\, 1_{\{|f|>cu\}}) d\lambda\bigg) d(u^q).$$

It is easy to check by integration by parts and Fubini's theorem that

$$\begin{aligned}
\int_\varepsilon^{1/\varepsilon} \frac{1}{u}\bigg(\int |f|\, 1_{\{|f|>cu\}}) d\lambda\bigg) d(u^q) &\le \frac{cq}{q-1} \int_\varepsilon^{1/\varepsilon} \lambda\big(\{|f| \ge cu\}\big) d(u^q) \\
&\quad + \frac{cq}{q-1} \cdot \frac{1}{\varepsilon^{q-1}} \int_{1/\varepsilon}^\infty \lambda\big(\{|f| \ge cu\}\big) du.
\end{aligned}$$

One may then get the bound

$$\int_\varepsilon^{1/\varepsilon} \frac{1}{u}\bigg(\int |f|\,1_{\{|f|>cu\}}\big)d\lambda\bigg)d(u^q)$$
$$\leq \frac{q}{q-1}\cdot\frac{5^q}{c^{q-1}}\,N_\varepsilon(f) + \frac{q}{q-1}\cdot\frac{1}{c^{q-1}}\,\|f\|_{q,\infty}^q\bigg(q\log\Big(\frac{c}{5}\Big)+\frac{1}{q-1}\bigg).$$

Hence, for c large enough, it will follow that $\sup_{0<\varepsilon<1} N_\varepsilon(f) < \infty$, and thus that $\|f\|_q < \infty$. The previous argument then shows that the inequality of the theorem holds for any f such that $\nabla f \in \mathrm{L}^p$ and $f \in B^{\theta/(\theta-1)}_{\infty,\infty}$. The theorem is established. □

2. Extensions. Dimension free bounds

Besides the 3rd (approximation) step, the proof in the Euclidean case emphasizes one main tool in the argument, namely the pseudo-Poincaré bound (4) on $f - P_t f$. The terminology pseudo-Poincaré is taken from the reference [SC] where it is applied for averages on balls (see below). D. Robinson [Ro] was the first to use this property to prove a Nash type inequality, and it has then been sucessfully applied in in various contexts [C-SC], [V-SC-C] (cf. [SC]). Under (4) and domination of $P_t f$ by the Besov norm, the weak-type inequality (3) immediately follows. The interpolation argument of the second step develops similarly in a rather large generality. Therefore, once we are given a family $(P_t)_{t\geq 0}$ of uniformly bounded operators on all L^p-spaces satisfying a pseudo-Poincaré inequality, and once we consider the corresponding Besov norm on $(P_t)_{t\geq 0}$, the main result of Theorem 1 immediately extends. We describe in this section various instances where (4), and thus the result of Theorem 1, generalizes. We furthermore inspect dependence of the constants upon dimension. For simplicity, we only state the inequalities for classes of smooth functions.

1) Sobolev embeddings and Varopoulos's theorem.

For the Laplace-Beltrami operator on a manifold, or a second order differential operator A (without constant term), the pseudo-Poincaré inequality (4) for the heat semigroup $P_t = e^{tA}$, $t \geq 0$, for $p = 2$ is essentially spectral and always satisfied. Indeed, more generally, if $P_t = e^{tA}$ is a Markov self-adjoint semigroup on $\mathrm{L}^2(E,\mu)$, then it is classical (cf. [Da], [SC]) that

$$\|f - P_t f\|_2^2 \leq 2\,t\,\mathcal{E}(f,f)$$

where $\mathcal{E}$ is the associated Dirichlet form $\mathcal{E}(f,f) = \int f(-Af)d\mu$. As a consequence of the method developed in Section 1, we may thus conclude to the following statement.

Corollary 2. *Under the preceding notation, for every q, $2 < q < \infty$ and every function f in the domain of A,*

$$\|f\|_q \leq C\,\mathcal{E}(f,f)^{\theta/2}\,\|f\|^{1-\theta}_{B^{\theta/(\theta-1)}_{\infty,\infty}},$$

where $\theta = \frac{2}{q}$ and where $C > 0$ only depends on q.

Corollary 2 is a self-contained version of one half of the celebrated Varopoulos theorem on the equivalence between Sobolev inequalities and heat kernel decay ([Va1], [Va2], [Da]): namely, under the heat kernel bound $\|P_t\|_{1\to\infty} \le C\,t^{-n/2}$, $t > 0$, the Sobolev inequality

$$\|f\|^2_{2n/(n-2)} \le C\,\mathcal{E}(f,f) = C\int f(-Af)d\mu$$

($n > 2$) holds. As such, the improved Sobolev inequality of Corollary 2 is part of the abstract Hardy-Littlewood-Sobolev theory (cf. [B-C-L-SC], [SC]).

2) Riemannian manifolds with non-negative Ricci curvature.

Let M be a Riemannian manifold with dimension n and non-negative Ricci curvature. Denote by Δ the Laplace-Beltrami operator on M and by $P_t = e^{t\Delta}$, $t \ge 0$, the associated heat semigroup. The following lemma shows how (4) extends to this context and furthermore describes the dependence of the constants with respect to dimension.

Lemma 3. *Under the previous notation and hypotheses, for every smooth function f on M and every $t \ge 0$,*

$$\|f - P_t f\|_p \le C\sqrt{t}\,\|\nabla f\|_p,$$

where $C > 0$ is numerical for $1 \le p \le 2$ and only depends on n for $2 < p \le \infty$.

Proof. For every smooth function g with $\|g\|_{p^*} \le 1$ where p^* is the conjugate of p,

$$\begin{aligned}\int g(f - P_t f)dx &= -\int_0^t \left(\int g\Delta P_s f dx\right)ds \\ &= \int_0^t \left(\int \nabla P_s g \cdot \nabla f dx\right)ds \\ &\le \|\nabla f\|_p \int_0^t \|\nabla P_s g\|_{p^*} ds.\end{aligned}$$

We next provide two distinct arguments to bound $\|\nabla P_s g\|_{p^*}$ according as $1 \le p^* \le 2$ or $2 \le p^* \le \infty$.

The Li-Yau inequality [L-Y] in manifolds with non-negative Ricci curvature asserts that for every smooth function $g > 0$, and every $s > 0$,

$$\frac{|\nabla P_s g|^2}{(P_s g)^2} - \frac{\Delta P_s g}{P_s g} \le \frac{n}{2s}\,. \tag{6}$$

Multiplying by $(P_s g)^{p^*}$,

$$(P_s g)^{p^*-2}|\nabla P_s g|^2 - (P_s g)^{p^*-1}\Delta P_s g \le \frac{n}{2s}\,(P_s g)^{p^*}.$$

By integration by parts,

$$p^* \int (P_s g)^{p^*-2} |\nabla P_s g|^2 dx \leq \frac{n}{2s} \int g^{p^*} dx.$$

Now, provided that $1 \leq p^* \leq 2$, we get from Hölder's inequality that

$$\|\nabla P_s g\|_{p^*} \leq \Big(\frac{n}{2p^* s}\Big)^{1/2} \|g\|_{p^*} \leq \Big(\frac{n}{2s}\Big)^{1/2} \|g\|_{p^*}$$

from which the lemma easily follows in this case.

When $2 \leq p^* \leq \infty$, we make use of the Poincaré type inequality for heat kernel measures in Riemannian manifolds with non-negative Ricci curvature (see e.g. [Le]) that indicates that, for every smooth function g, and at every point,

$$2s\, |\nabla P_s g|^2 \leq P_s(g^2) - (P_s g)^2.$$

Hence, by Hölder's inequality again,

$$\|\nabla P_s g\|_{p^*} \leq \Big(\frac{1}{2s}\Big)^{1/2} \|g\|_{p^*}$$

with thus a numerical constant. The proof of the lemma is then easily completed. □

As a consequence of this lemma and the proof of Theorem 1 in the previous section, we may state the following consequence, with some further aspects on the dependence of constants upon dimension. Define as in the Euclidean case the Besov norm, for $\alpha < 0$,

$$\|f\|_{B^{\alpha}_{\infty,\infty}} = \sup_{t>0} t^{-\alpha/2} \|P_t f\|_\infty.$$

Corollary 4. *Let M be a Riemannian manifold with non-negative Ricci curvature. Then, for every $1 \leq p < q < \infty$ and every function f in the Sobolev space $W^{1,p}(M)$,*

$$\|f\|_q \leq C\, \|\nabla f\|_p^\theta\, \|f\|^{1-\theta}_{B^{\theta/(\theta-1)}_{\infty,\infty}},$$

where $\theta = \frac{p}{q}$ and where $C > 0$ only depends on q and p whenever $1 \leq p \leq 2$, and on q, p and n whenever $2 < p < \infty$.

In particular, one recovers in this way that under the heat kernel bound $\|P_t\|_{1\to\infty} \leq C\, t^{-n/2}$, $t > 0$, on a manifold M with non-negative Ricci curvature, the classical Sobolev inequality (1) holds on M (cf. [C-L]). Moreover, in the context of Markov operators as in the preceding section, the functional dimension is only reflected at the level of the heat kernel bounds.

One may similarly consider localized versions of the preceding result. For example, on a compact Riemannian manifold, the Li-Yau inequality (6) holds (up to constants depending on the lower bound of the Ricci curvature) for all $0 < t \leq 1$. The preceding arguments then show similarly that for the localized Besov norm

$$\|f\|_{B^{\alpha}_{\infty,\infty}} = \sup_{0<t\leq 1} t^{-\alpha/2}\|P_t f\|_{\infty},$$

one has the localized improved Sobolev inequalities

$$\|f\|_q \leq C\left(\|\nabla f\|_p + \|f\|_p\right)^{\theta}\|f\|^{1-\theta}_{B^{\theta/(\theta-1)}_{\infty,\infty}}.$$

3) Averages on balls and Morrey spaces.

The arguments developed in Section 1 work similarly with averages on balls. For a locally integrable function f on $\mathbb{R}^n$, set

$$f_r(x) = \frac{1}{V(x,r)}\int_{B(x,r)} f d\lambda, \quad r > 0,\ x \in \mathbb{R}^n,$$

where $B(x,r)$ is the ball with center x and radius r and $V(x,r)$ $(= V(0,1)r^n)$ its volume. One may then consider the norm

$$\|f\|_{M^{\alpha}_{\infty}} = \sup_{x\in\mathbb{R}^n, r>0} r^{-\alpha}\big|f_r(x)\big|$$

that corresponds to a Morrey space rather than a Besov space [Me], [Tr], and which is strictly smaller in the Euclidean case (cf. [Ta] and the references therein).

The proof of Theorem 1 may be repeated completely similarly with the averages f_r in place of the Gaussian convolutions $P_t f$. In analogy with Lemma 2, the key ingredient is a family of local Poincaré inequalities

$$\int_{B(x,r)} |f - f_r|^p d\lambda \leq C\, r^p \int_{B(x,r)} |\nabla f|^p d\lambda \tag{7}$$

for $p \geq 1$, some constant $C > 0$ and all $r > 0$ and f. Such families are known (cf. [C-SC], [SC]) to yield the pseudo-Poincaré inequality for the averages f_r

$$\|f - f_r\|_p \leq C\, r\, \|\nabla f\|_p \tag{8}$$

analogous to (4). With this tool, the proof of Theorem 1 develops completely analogously.

While weaker than the Besov type result of Theorem 1 in $\mathbb{R}^n$, it allows various extensions of independent interest. Namely, local Poincaré inequalities (7) and their associated pseudo-Poincaré inequalities (8) have been investigated in a number of various settings in connection with Sobolev type embeddings and Harnack inequalities (we refer to [V-SC-C], [SC] for an account on the subject).

In particular, (7) has been shown to hold in [Bu], by geometric tools, on manifolds of non-negative Ricci curvature. We thus get a Morrey version of Corollary 3, usually not directly comparable.

The Poincaré and pseudo-Poincaré inequalities have been investigated also on related structures such as Lie groups and graphs. For example, the pseudo-Poincaré inequality (8) is satisfied by the natural metrics on Lie groups associated with invariant vector fields [Ro], [V-SC-C], [C-SC], producing thus a version of the improved Sobolev embeddings in this context. If X is countable connected graph such that each vertex has at most a finite number N of neighbors, define by $d(x,y)$ the minimal number of edges that connect x to y. If f is a function (with finite support) on X, let for $x \in X$ and $k \geq 1$,

$$f_k(x) = \frac{1}{V(x,k)} \sum_{y \in B(x,k)} f(y),$$

where $B(x,k)$ is the ball with center x and radius k and $V(x,k)$ its volume (= cardinal). We also define the (length of the) gradient $|\nabla f|(x)$ of f at the point x as

$$|\nabla f|(x) = \sum_{y \sim x} \big| f(x) - f(y) \big|,$$

where the sum runs over all neighbors y of x.

If X is the Cayley graph of a finitely generated group, it has been shown in [D-SC] (see also [C-SC], [SC]), as a consequence of local Poincaré inequalities, that for every $p \geq 1$ and some $C > 0$,

$$\|f - f_k\|_p \leq Ck \, \|\nabla f\|_p \tag{9}$$

for every f and $k \geq 1$. L^p-norms are defined here with respect to the counting measure on X. As a consequence, we may state the following improved Sobolev inequalities in this case. We let

$$\|f\|_{M^\alpha_\infty} = \sup_{x \in X, k \geq 1} k^{-\alpha} \big| f_k(x) \big|.$$

Corollary 5. *Let X be the Cayley graph of a finitely generated group. Then, for $1 \leq p < q < \infty$, some $C > 0$ and any finitely supported function f on X,*

$$\|f\|_p \leq C \, \|\nabla f\|_p^\theta \, \|f\|^{1-\theta}_{M^{\theta/(\theta-1)}_\infty}$$

where $\theta = \frac{p}{q}$.

Acknowledgements

Thanks are due to A. Cohen and Y. Meyer for their interest and to P. Auscher, Th. Coulhon and L. Saloff-Coste for helpful comments. Thanks also to A. Ayache for making several references available.

References

[Ba] D. Bakry, *L'hypercontractivité et son utilisation en théorie des semigroupes*, Ecole d'Eté de Probabilités de St-Flour. Lecture Notes in Math. **1581**, (1994), 1–114.

[B-C-L-SC] D. Bakry, T. Coulhon, M. Ledoux, L. Saloff-Coste, *Sobolev inequalities in disguise*, Indiana J. Math. **44**, (1995), 1033–1074.

[Bu] P. Buser, *A note on the isoperimetric constant*, Ann. scient. Éc. Norm. Sup. **15**, (1982), 213–230.

[C-D-D-DV] A. Cohen, W. Dahmen, I. Daubechies, R. DeVore, *Harmonic analysis of the space BV*, Rev. Mat. Iberoamericana **19**, (2003), 235–263.

[C-DV-P-X] A. Cohen, R. DeVore, P. Petrushev, H. Xu, *Nonlinear approximation and the space $BV(\mathbb{R}^2)$*, Amer. J. Math. **121**, (1999), 587–628.

[C-M-O] A. Cohen, Y. Meyer, F. Oru, *Improved Sobolev inequalities*, Séminaires X-EDP, École Polytechnique (1998).

[C-L] Th. Coulhon, M. Ledoux, *Isopérimétrie, décroissance du noyau de la chaleur et transformations de Riesz : un contre-exemple*, Ark. Mat. **32**, (1994), 63–77.

[C-SC] Th. Coulhon, L. Saloff-Coste, *Isopérimétrie pour les groupes et les variétés*, Revista Math. Iberoamericana **9**, (1993), 293–314.

[Da] E. B. Davies, *Heat kernels and spectral theory*, Cambridge Tracts in Mathematics, **92**, (1990).

[D-SC] P. Diaconis, L. Saloff-Coste, *Comparison theorems for reversible Markov chains*, Ann. Applied Probab. **3**, (1993), 696–730.

[Le] M. Ledoux, *The geometry of Markov diffusion generators*, Ann. Fac. Sci. Toulouse IX, **9**, (2000), 305–366.

[L-Y] P. Li, S.-T. Yau, *On the parabolic kernel of the Schrödinger operator*, Acta Math. **156**, (1986), 153–201.

[Me] Y. Meyer, *Ondelettes et opérateurs*, Actualits Mathmatiques, (1990).

[Ro] D. Robinson, *Elliptic operators on Lie groups*, Oxford Mathematical Monographs, (1991).

[SC] L. Saloff-Coste, *Aspects of Sobolev-type inequalities*, London Math. Soc. Lecture Notes Series 289. Cambridge University Press (2002).

[Ta] M. Taylor, *Analysis on Morrey spaces and applications to Navier-Stokes and other evolution equations*, Comm. Partial Differential Equations **17**, (1992), 1407–1456.

[Tr] H. Triebel, *Theory of function spaces II*, Monographs in Mathematics, **84**, (1992).

[Va1] N. Varopoulos, *Hardy-Littlewood theory for semigroups*, J. Funct. Anal. **63**, (1985), 240–260.

[Va2] ———, *Analysis and geometry on groups*, Proceedings of the International Congress of Mathematicians, Kyoto (1990), vol. II, 951–957, (1991). Springer-Verlag.

[V-SC-C] N. Varopoulos, L. Saloff-Coste, T. Coulhon, *Analysis and geometry on groups*, Cambridge Tracts in Mathematics, **100**, (1993).

Institut de Mathématiques, Université Paul-Sabatier, 31062 Toulouse, France
E-mail address: ledoux@math.ups-tlse.fr