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ON SECTIONS WITH ISOLATED SINGULARITIES OF TWISTED BUNDLES AND APPLICATIONS TO FOLIATIONS BY CURVES

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Dedicated to the memory of A.N. Tyurin

Abstract. Let $E \longrightarrow M$ be a holomorphic rank n vector bundle over a compact Kähler manifold of dimension n, having a positive (or ample) line bundle $L \longrightarrow M$ and consider a global section s, with isolated singularities, of the twisted bundle $E \otimes L^{\otimes r}$, where r is an integer.

We prove that if r is large enough, then s is uniquely determined, up to a global endomorphism of the bundle E, by its subscheme of singular points (which we call the singular subscheme of s).

If in particular E is simple, then s is uniquely determined, up to a scalar factor, by its singular subscheme.

We recall that the last statement holds in case s is a holomorphic foliation by curves, with isolated singularities, on a projective manifold M with stable tangent bundle, so it holds in particular if M is a compact irreducible Hermitian symmetric space or a Calabi-Yau manifold.

If $L \longrightarrow \mathbb{P}^n$ is the hyperplane bundle, we show that it holds for every $r \geq 1$.

1. Introduction

In the previous paper [5], the authors have shown that an algebraic foliation of degree at least 2 in the projective plane is determined by the subscheme of its singular points (or its *singular subscheme*, after Definition 2.1 below). This result was known to hold for foliations by curves in projective spaces having a *reduced* singular subscheme (see [8]).

In this paper we study our previuos result in a general context, providing the following extension: A foliation by curves (with isolated singularities) of a sufficiently high degree on a compact projective manifold whose tangent bundle is simple, is determined by its singular subscheme (Corollary 3.2).

Since stable bundles are simple (Proposition 2.5 below), this holds in particular for the manifolds studied in [19] and [22]. We shall show (also in Corollary 3.2) that it holds as well for the classes of the so-called compact irreducible Hermitian symmetric spaces (see [10] or [1] for the definition and classification of

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these objects) and Calabi-Yau Manifolds (see Definition 3.1 for our convention on these).

For such foliations in projective spaces, we derive a precise generalization of the above-mentioned results from [5] and [8] in Theorem 3.5 below.

To our belief, the interest of such results relies on the fact that they allow to translate the algebro-geometrical, differential or even integral features of such a foliation into ones which deal only with the geometry of a scheme of points in the manifold. They provide, hence, a new scope to tackle the study of differential equations on projective manifolds, an area with several well-known open problems.

In particular, they may serve as a complementary tool for the classification of foliations on surfaces (see [3] for that of non-singular foliations and [4] for an account on the singular case).

Although our main interest is in foliations by curves, our results are based in one of a general character, that may be of interest in its own (Theorem 2.2 below). Thus, we will start by proving a general result for sections (with isolated singularities) of rank-n vector bundles over n-dimensional compact projective manifolds. Its precise statement, proof and applications are the content of the next section.

2. The general Theorem and its applications

Throughout the paper, $M = (M, g)$ will be a compact, connected Kähler manifold of dimension $n \geq 2$, with Kähler form Φ; L will denote an ample line bundle on M (so that M is projective algebraic), and $E = (E, h) \longrightarrow M$ will stand for an hermitian vector bundle of rank n. Its sheaves of sections will be denoted respectively by $\mathcal{L}$ and $\mathcal{E}$ and the structure sheaf of M, by $\mathcal{O}_M$. For an $\mathcal{O}_M$-sheaf $\mathcal{G}$, we will write $\mathcal{G}(r)$ for $\mathcal{G} \otimes \mathcal{L}^{\otimes r}$, if $r \geq 0$ and $\mathcal{G} \otimes (\mathcal{L}^*)^{\otimes |r|}$, if $r < 0$.

Definition 2.1. Let r be an integer and let s be a global section in $H^0(M, \mathcal{E}(r))$. The set $Z = Z_s$ of points p such that $s(p) = 0$ will be called the singular set of s. The closed subscheme $\mathrm{SingS}(s) = (Z, \mathcal{O}_Z)$ of M will be called the singular subscheme of s. The defining ideal sheaf $\mathcal{J} = \mathcal{J}_Z$ of Z_s will be called the singular ideal of s. It is related with the structure sheaves by the following short exact sequence:

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{O}_M \xrightarrow{\tilde{s_0}} \mathcal{O}_Z \longrightarrow 0.$$

As usual, we shall say that the section s has isolated singularities if Z is zero-dimensional (i.e. if it consists of isolated points).

Recall that on an open trivialization $U \subset M$ of the bundle E, the singular ideal $\mathcal{J}$ of s is given by the ideal $\mathcal{J}(U) \subset \mathcal{O}_M(U)$ generated by the coefficients of s, relative to the trivialization.

Consider the projective space $\mathrm{Proj}H^0(M, \mathcal{E}(r))$ of lines through 0 in $H^0(M, \mathcal{E}(r))$. Clearly, *if* $s' \in [s] \in \mathrm{Proj}H^0(M, \mathcal{E}(r))$ *then* $\mathrm{SingS}(s') = \mathrm{SingS}(s)$. Our general theorem provides conditions for the converse to this remark to hold. Recalling that a bundle E is said to be *simple* if $H^0(M, \mathcal{E} \otimes \mathcal{E}^*) = H^0(M, \mathcal{E}nd(\mathcal{E})) = \mathbb{C} \cdot \mathrm{Id}$, it is the following:

Theorem 2.2. *Let M be a compact Kähler manifold of dimension $n \geq 2$, with an ample line bundle $L \longrightarrow M$ and let $E \longrightarrow M$ be a holomorphic vector bundle of rank n. There exists an integer r_0 such that, for every integer $r \geq r_0$, the following condition holds:*

Given a global section $s \in H^0(M, \mathcal{E}(r))$ with isolated singularities:

1. *A global section $s' \in H^0(M, \mathcal{E}(r))$ satisfies that* $\mathrm{SingS}(s') \supseteq \mathrm{SingS}(s)$ *if and only if $s' = \varphi(s)$, for some global endomorphism $\varphi \in H^0(M, \mathcal{E} \otimes \mathcal{E}^*)$.*
2. *In particular, if the bundle E is simple, then* $\mathrm{SingS}(s') \supseteq \mathrm{SingS}(s)$ *if and only if $s' = k \cdot s$, for some $k \in \mathbb{C}^*$. To wit, the class $[s] \in$* $\mathrm{Proj}\, H^0(M, \mathcal{E}(r))$ *is uniquely determined by the singular subscheme* $\mathrm{SingS}(s)$*.*

Proof. Fix $s \in H^0(M, \mathcal{E}(r))$ with isolated singularities and consider the Koszul resolution

$$\mathcal{C}_* \longrightarrow \begin{cases} \mathcal{O}_M \to \mathcal{O}_Z \to 0 \\ \mathcal{J}_Z \to 0 \end{cases}$$

associated to $Z = Z_s$ (see [9]). It follows from standard arguments (see [11], for instance) that it is given by the following complex of sheaves:

(2.1)
$$0 \to \Lambda^n \mathcal{E}^*(-nr) \overset{\tilde{s_n}}{\to} \cdots \overset{\tilde{s_3}}{\to} \Lambda^2 \mathcal{E}^*(-2r) \overset{\tilde{s_2}}{\to} \Lambda^1 \mathcal{E}^*(-r) \overset{\tilde{s_1}}{\to} \begin{cases} \mathcal{O}_M \overset{\tilde{s_0}}{\to} \mathcal{O}_Z \to 0 \\ \mathcal{J}_Z \to 0 \end{cases} .$$

Now, consider the complex $\mathcal{C}_* \otimes \mathcal{E}(r)$ obtained from the upper sequence in (2.1). To simplify the notation, let

$$\mathcal{G}^q = \Lambda^q \mathcal{E}^*(-qr) \otimes \mathcal{E}, \quad \text{for } q = 1, \dots, n, \tag{2.2}$$

so that

$$\mathcal{G}^q(r) = \Lambda^q \mathcal{E}^* \otimes \mathcal{E}(r - qr) = \Lambda^q \mathcal{E}^*(-qr) \otimes \mathcal{E}(r), \quad \text{for } q = 1, \dots, n.$$

In particular $\mathcal{G}^1(r) = \Lambda^1 \mathcal{E}^*(-r) \otimes \mathcal{E}(r) = \mathcal{E}^* \otimes \mathcal{E} = \mathcal{G}^1$. Since $\mathcal{O}_Z \otimes \mathcal{E}(r) = \mathcal{E}(r)|_Z$ (the sheaf restricted to the scheme), it follows that $\mathcal{C}_* \otimes \mathcal{E}(r)$ is given by

$$0 \to \mathcal{G}^n(r) \overset{s_n}{\to} \cdots \overset{s_3}{\to} \mathcal{G}^2(r) \overset{s_2}{\to} \mathcal{E}^* \otimes \mathcal{E} \overset{s_1}{\to} \mathcal{E}(r) \overset{s_0}{\to} \mathcal{E}(r)|_Z \to 0. \tag{2.3}$$

The long exact sequence above breaks into the short exact sequences given by

$$0 \longrightarrow \mathcal{K}^0(r) \longrightarrow \mathcal{E}(r) \overset{s_0}{\longrightarrow} \mathcal{E}(r)|_Z \to 0, \quad \text{for } p = 0; \tag{2.4}$$

$$0 \longrightarrow \mathcal{K}^1(r) \longrightarrow \mathcal{E}^* \otimes \mathcal{E} \overset{s_1}{\longrightarrow} \mathcal{K}^0(r) \to 0, \quad \text{for } p = 1; \tag{2.5}$$

$$0 \longrightarrow \mathcal{K}^p(r) \longrightarrow \mathcal{G}^p(r) \overset{s_p}{\longrightarrow} \mathcal{K}^{p-1}(r) \to 0, \quad \text{for } p = 2, \dots, n-2; \tag{2.6}$$

$$0 \longrightarrow \mathcal{G}^n(r) \longrightarrow \mathcal{G}^{n-1}(r) \overset{s_p}{\longrightarrow} \mathcal{K}^{n-2}(r) \to 0, \quad \text{for } p = n-1; \tag{2.7}$$

after identifying the sheaves $\mathcal{K}^{n-1}(r)$ and $\mathcal{G}^n(r)$ through the injective map s_n.

The relevant parts of its associated exact cohomology sequences are given, for $p=0$, by

$$0 \longrightarrow H^0(M,\mathcal{K}^0(r)) \longrightarrow H^0(M,\mathcal{E}(r)) \xrightarrow{s_0^0} H^0(M,\mathcal{E}(r)|_Z) \longrightarrow \dots; \tag{2.8}$$

for $p=1$, by

$$\begin{aligned} 0 \longrightarrow H^0(M,\mathcal{K}^1(r)) \longrightarrow H^0(M,\mathcal{E}^*\otimes\mathcal{E}) \xrightarrow{s_1^0} H^0(M,\mathcal{K}^0(r)) \xrightarrow{\delta_1^0} \\ \xrightarrow{\delta_1^0} H^1(M,\mathcal{K}^1(r)) \longrightarrow H^1(M,\mathcal{E}^*\otimes\mathcal{E}) \longrightarrow H^1(M,\mathcal{K}^0(r)) \longrightarrow \dots; \end{aligned} \tag{2.9}$$

for $2 \leq p \leq n-2$, by

(2.10)

$$\begin{aligned} \dots \longrightarrow H^{p-2}(M,\mathcal{K}^p(r)) \longrightarrow H^{p-2}(M,\mathcal{G}^p(r)) \xrightarrow{s_p^{p-2}} H^{p-2}(M,\mathcal{K}^{p-1}(r)) \xrightarrow{\delta_p^{p-2}} \\ \xrightarrow{\delta_p^{p-2}} H^{p-1}(M,\mathcal{K}^p(r)) \longrightarrow H^{p-1}(M,\mathcal{G}^p(r)) \xrightarrow{s_p^{p-1}} H^{p-1}(M,\mathcal{K}^{p-1}(r)) \xrightarrow{\delta_p^{p-1}} \\ \xrightarrow{\delta_p^{p-1}} H^{p}(M,\mathcal{K}^p(r)) \longrightarrow H^{p}(M,\mathcal{G}^p(r)) \longrightarrow H^{p}(M,\mathcal{K}^{p-1}(r)) \longrightarrow \dots; \end{aligned}$$

and finally, for $p=n-1$, by

(2.11)

$$\begin{aligned} \dots \longrightarrow H^{n-3}(M,\mathcal{G}^n(r)) \longrightarrow H^{n-3}(M,\mathcal{G}^{n-1}(r)) \xrightarrow{s_{n-1}^{n-3}} H^{n-3}(M,\mathcal{K}^{n-2}(r)) \xrightarrow{\delta_{n-1}^{n-3}} \\ \xrightarrow{\delta_{n-1}^{n-3}} H^{n-2}(M,\mathcal{G}^n(r)) \longrightarrow H^{n-2}(M,\mathcal{G}^{n-1}(r)) \xrightarrow{s_{n-1}^{n-2}} H^{n-2}(M,\mathcal{K}^{n-2}(r)) \xrightarrow{\delta_{n-1}^{n-2}} \\ \xrightarrow{\delta_{n-1}^{n-2}} H^{n-1}(M,\mathcal{G}^n(r)) \longrightarrow H^{n-1}(M,\mathcal{G}^{n-1}(r)) \longrightarrow H^{n-1}(M,\mathcal{K}^{n-2}(r)) \longrightarrow \dots \end{aligned}$$

Now, to prove statement (1), let $s' \in H^0(M,\mathcal{E}(r))$ be a global section such that $\mathrm{Sing}(s') \supseteq \mathrm{Sing}(s)$. By construction, both sections s and s' belong to the kernel of the map s_0^0 in (2.8). Since the map s_1^0 in (2.9) is given by $s_1^0(\varphi)=\varphi(s)$, it suffices to show that there exists an integer r_0 such that s_1^0 is an isomorphism for every $r \geq r_0$. In view of (2.9), this is equivalent to the conditions

$$H^0(M,\mathcal{K}^1(r)) = H^1(M,\mathcal{K}^1(r)) = 0, \quad \text{for some } r \geq r_0.$$

A diagram chase in the cohomology sequences (2.10) and (2.11) shows that the conditions above hold, *if* there exists an r_0 such that

$$H^p(M,\mathcal{G}^q(r)) = 0, \quad \text{for} \quad r \geq r_0,\ 2 \leq q \leq n,\ q-2 \leq p \leq q-1, \tag{2.12}$$

where $\mathcal{G}^q$ was defned in (2.2).

The existence of such an r_0 is provided by the Serre-dual of the Cartan-Serre Theorem B (see [9]). This finishes the proof of statement (1), from which (2) follows at once. □

Remark 2.3. Let us point out in Theorem 2.2 that the section s' *is not* assumed to have isolated singularities.

Now, for the applications of Theorem 2.2 (2), recall the notion of (Mumford-Takemoto) L-stability of a bundle E (see [20]), its differential geometric counterpart of Φ-stability (see [14, V§7]) and the following results:

Proposition 2.4. [14, (V.8.3)] *Every irreducible Einstein-Hermitian vector bundle (E, h, M, g) is Φ-stable*

Proposition 2.5. [14, (V.7.14)] *and* [20, Corollary 1.8] *Every Φ (or L)-stable bundle E is simple.*

For a converse of Proposition 2.4, see [23] or [6] (and further discussion in [21], [22] or [17]). A converse of Proposition 2.5 for bundles on surfaces may be found in [15].

We shall apply the Propositions above in the following section.

3. On manifolds with simple tangent bundle

In this section we focus on the case $E = TM$, the tangent bundle of M. As usual in this context, we shall denote the sheaves of sections $\mathcal{E}$ and $\mathcal{E}^*$ respectively by Θ_M and Ω_M, and will say (following [14]) that M is Einstein-Kähler (resp. simple) whenever TM is Einstein-Hermitian (resp. simple).

Now recall from [7] that, given a line bundle $H \longrightarrow M$, the projective space

$$\operatorname{Proj} H^0(M, \Theta_M \otimes \mathcal{H}) = \operatorname{Proj} H^0(M, \mathcal{H}om(\mathcal{H}^*, \Theta_M)). \tag{3.1}$$

is *the space of holomorphic foliations by curves (or meromorphic vector fields) on M with tangent line bundle H*. We shall denote it by $\mathcal{F}ol(M, \mathcal{H})$ and in accordance, the class $[s] \in \mathcal{F}ol(M, \mathcal{H})$ of a global section s will be denoted by $\mathcal{F} = \mathcal{F}_s$.

It turns out (see [7]) that the Chern class $c_1(H)$ is a discrete numerical invariant in the space $\mathcal{F}ol(M)$ of foliations by curves on M and in particular that, if the first Betti number $b_1(M)$ of M vanishes, then $\mathcal{F}ol(M)$ is the disjoint union of the projective spaces given by (3.1), indexed by the admissible Chern classes $c_1(H) \in H^2(M, \mathbb{Z})$.

If moreover $\operatorname{Pic}(M) = \mathbb{Z}$, so that every line bundle H on M is of the form $L^{\otimes r}$, for some very ample line bundle $L = \mathcal{O}(1)$ and some integer $r \in \mathbb{Z}$, then we may write $\mathcal{F}ol(M, \mathcal{H}) = \mathcal{F}ol(M, \mathcal{L}^{\otimes r}) = \mathcal{F}ol_r(M, \mathcal{L})$ and hence

$$\mathcal{F}ol(M) = \coprod_{r \geq r_1 \in \mathbb{Z}} \mathcal{F}ol_r(M, \mathcal{L}).$$

It is well-known that this description holds, in particular, if M is a compact irreducible Hermitian symmetric space (in which case, L may be taken to be the canonical bundle, see [16] or [18], for instance).

On the other hand, we shall adopt from [12] the following

Definition 3.1. A Calabi-Yau manifold is a compact connected Kähler manifold (M, J, g) of dimension $n \geq 2$, such that the holonomy group $\operatorname{Hol}(g) = \operatorname{SU}(n)$.

Now we claim that Theorem 2.2 (2) holds in this context, whenever M is a compact irreducible Hermitian symmetric space or a Calabi-Yau manifold:

Corollary 3.2. *Let M be a compact projective simple manifold and let $L \longrightarrow M$ be an ample line bundle. There exists an integer r_0 such that, for any integer $r \geq r_0$, the following condition holds: Given a holomorphic foliation by curves $\mathcal{F} \in \mathcal{F}ol_r(M, \mathcal{L})$ with isolated singularities, if $\mathcal{F}' \in \mathcal{F}ol_r(M, \mathcal{L})$ is another foliation such that $\mathrm{SingS}(\mathcal{F}') \supseteq \mathrm{SingS}(\mathcal{F})$, then $\mathcal{F}' = \mathcal{F}$.*

In particular, this holds true if M is a

(a) *Compact irreducible Hermitian symmetric space, or a*
(b) *Projective $K3$ surface or a Calabi-Yau manifold of dimension $n \geq 3$,*

Proof. It suffices to show, in both cases, that the metric g under consideration is Einstein-Kähler and has irreducible holonomy group, for it will then follow from Propositions 2.4 and 2.5 that TM is simple. We just recall how this is done:
(a) This is the content of (IV.6.2) and (IV.6.3) in [14]. For a direct proof of the stability of TM, see [16].
(b) To start with, recall that a Calabi-Yau manifold of dimension $n \geq 3$ is projective ([12, (6.2.7)]) and that the two-dimensional Calabi-Yau manifolds are precisely the $K3$ surfaces ([12, (7.3.13)]).

Now, being $\mathrm{Hol}(g) = \mathrm{SU}(n)$, it is irreducible and moreover, g is Ricci-flat [1, (10.29)] and its Ricci form $\rho = 0$ [1, (10.30)], hence g is an Einstein-Kähler metric [1, (11.12)]. □

Remark 3.3. By Proposition 2.5, Corollary 3.2 holds in case M is a complex projective manifold whose tangent bundle is H-stable, for some ample line bundle $H \longrightarrow M$. Examples of such manifolds can be found in [19], [22] or [14].

An accurate version of Theorem 2.2 (and in particular, of Corollary 3.2) can be given in case E is a homogeneous vector bundle over a homogeneous manifold M: Bott's Theorem [2] is an effective tool to compute the least integer r_0 satisfying (2.12).

If $L \longrightarrow \mathbb{P}^n$ is the hyperplane bundle on the n-dimensional complex projective space, this computation gives the value $r_0 = 1$:

Proposition 3.4. [8, (1.3 (2))] *Let $n \geq 2$ and let $L \longrightarrow \mathbb{P}^n$ be the hyperplane bundle. Let $\mathcal{G}^q(r) = (\Lambda^q \Omega_{\mathbb{P}^n}) \otimes \Theta_{\mathbb{P}^n}((1-q)r)$, where $r > 0$, $0 \leq q \leq n$; then $H^p(\mathbb{P}^n, \mathcal{G}^q(r)) = 0$ if $p < q$, except for $H^0(\mathbb{P}^n, \mathcal{G}^1(r))$, which is one dimensional.*

This, together with Corollary 3.2, gives the following

Theorem 3.5. *Let $n \geq 2$ and let $L \longrightarrow \mathbb{P}^n$ be the hyperplane bundle. Let $\mathcal{F} \in \mathcal{F}ol_r(\mathbb{P}^n, \mathcal{L})$ be a holomorphic foliation by curves of degree $r \geq 1$, with isolated singularities, and let $\mathcal{F}' \in \mathcal{F}ol_r(\mathbb{P}^n, \mathcal{L})$ be another foliation such that $\mathrm{SingS}(\mathcal{F}') \supseteq \mathrm{SingS}(\mathcal{F})$. Then $\mathcal{F}' = \mathcal{F}$.*

This result generalizes both Theorem 2.6 in [8], where $\mathrm{SingS}(\mathcal{F})$ is assumed to be reduced, and Theorem 3.5 in [5], where $n = 2$ and the degree r' used therein was taken to be that of a homogeneous polynomial vector field X in $\mathbb{C}^{n+1}$ giving rise to $\mathcal{F}$, so that $r = r' - 1$.

Actually, these two results were the starting point of the present paper.

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