Mathematical Research Letters **10**, 523–533 (2003)

SOLUTION OF THE $\bar{\partial}$-NEUMANN PROBLEM ON A BI-DISC

Dariush Ehsani

Abstract. In this paper we study the behavior of the solution to the $\bar{\partial}$-Neumann problem for $(0,1)$-forms on a bi-disc in $\mathbb{C}^2$. We show singularities which arise at the distinguished boundary are of logarithmic and arctangent type.

1. Introduction

Let $\Omega \subset \mathbb{C}^n$ be a bounded, pseudoconvex domain, equipped with the standard Hermitian metric. The $\bar{\partial}$-Neumann problem, on domains with a C^2 defining function, takes the form of the boundary value problem

$$\Box u = f \text{ in } \Omega,$$

for f in $L^2_{p,q}(\Omega)$, and

$$u \rfloor \bar{\partial}\rho = 0, \tag{1.1}$$

$$\bar{\partial}u \rfloor \bar{\partial}\rho = 0, \tag{1.2}$$

on $\partial\Omega$, where $\Box$ is the complex Laplacian, $\bar{\partial}\bar{\partial}^* + \bar{\partial}\bar{\partial}^*$.

In the past decade, considerable attention has been given to the study of the $\bar{\partial}$-Neumann problem on non-smooth domains. We point to the papers of Henkin and Iordan [4], Henkin, Iordan, and Kohn [5], Michel and Shaw [7, 8], and Straube [9], in which properties, compactness and subelliptic estimates, hold for the Neumann operator, N, the inverse to the $\bar{\partial}$-Neumann problem, on certain non-smooth domains.

In [2], the author studied the $\bar{\partial}$-Neumann problem for $(0,1)$-forms on a model domain, the product of two half-planes in $\mathbb{C}^2$. We continue here the study of the problem for $(0,1)$-forms on model domains, focusing on the bi-disc, $\Omega = \mathbb{D}_1 \times \mathbb{D}_2 \in \mathbb{C}^2$, where $\mathbb{D}_1 \subset \mathbb{C}$ and $\mathbb{D}_2 \subset \mathbb{C}$ are defined by the equations $r_1 < 1$ and $r_2 < 1$, respectively, where $r_j = |z_j|$, $j = 1,2$. The existence of a solution in $L^2(\Omega)$ is given by Hörmander [6]. We shall see singularities only occur on the distinguished boundary, $\partial\mathbb{D}_1 \times \partial\mathbb{D}_2$. Our main result is the

Theorem 1.1. *Let $\Omega \in \mathbb{C}^2$ be the bi-disc, $\mathbb{D}_1 \times \mathbb{D}_2$, where $\mathbb{D}_j$ is the disc $\{z_j : |z_j| < 1\}$ for $j = 1,2$. Let $f = f_1 d\bar{z}_1 + f_2 d\bar{z}_2$ be a $(0,1)$-form such that $f \in C^\infty_{(0,1)}(\overline{\Omega})$, the family of $(0,1)$-forms whose coefficients are in $C^\infty(\overline{\Omega})$, and $u = u_1 d\bar{z}_1 + u_2 d\bar{z}_2$ the $(0,1)$-form which solves the $\bar{\partial}$-Neumann problem with data the*

Received February 19, 2003.

2000 *Mathematics Subject Classification.* Primary 32W05; Secondary 35B65.

$(0,1)$-form f on Ω. Then, with $z_j = r_j e^{i\theta_j}$, near $r_1 = r_2 = 1$, u_j can be written as

$$u_j = \alpha_j \log\left((\log r_1)^2 + (\log r_2)^2\right) + \beta_j + \gamma_j \arctan\left(\frac{\log r_1}{\log r_2}\right) \qquad j = 1, 2,$$

where α_j, β_j, γ_j are smooth functions of $r_1, r_2, \theta_1, \theta_2$.

We point out the interesting phenomenon that, although the Neumann operator on $(0,1)$-forms, $N_{(0,1)}$, is not regular, $\bar{\partial}^* N_{(0,1)}$ is. Regularity of $\bar{\partial}^* N_{(0,1)}$ follows from regularity of the Bergman projection on the bi-disc and the formula for the Bergman projection, P,

$$Pg = g - \bar{\partial}^* N_{(0,1)} \bar{\partial} g$$

for $g \in L^2(\Omega)$.

2. Setup

We set up the $\bar{\partial}$-Neumann problem for $(0,1)$-forms on the bi-disc, $\Omega = \mathbb{D}_1 \times \mathbb{D}_2 \in \mathbb{C}^2$ and prove regularity results away from the distinguished boundary.

$$\bar{\partial}\bar{\partial}^* u + \bar{\partial}^* \bar{\partial} u = f$$

gives equations for u_1 and u_2 based on the Laplacian:

$$\begin{aligned} \Delta u_1 &= -2f_1, \\ \Delta u_2 &= -2f_2, \end{aligned} \tag{2.1}$$

which, in polar coordinates (r_1, θ_1), (r_2, θ_2), are

$$\begin{aligned} &\frac{\partial^2 u_j}{\partial r_1^2} + \frac{1}{r_1}\frac{\partial u_j}{\partial r_1} + \frac{1}{r_1^2}\frac{\partial^2 u_j}{\partial \theta_1^2} + \\ &\frac{\partial^2 u_j}{\partial r_2^2} + \frac{1}{r_2}\frac{\partial u_j}{\partial r_2} + \frac{1}{r_2^2}\frac{\partial^2 u_j}{\partial \theta_2^2} = -2f_j. \end{aligned} \tag{2.2}$$

The boundary conditions (1.2), which were defined for C^2 domains, may be adapted to our case of the bi-disc to yield the conditions

$$\begin{aligned} u_1 &= 0 \qquad \text{when } r_1 = 1, \\ u_2 &= 0 \qquad \text{when } r_2 = 1, \end{aligned} \tag{2.3}$$

and

$$\frac{\partial u_2}{\partial \bar{z}_1} - \frac{\partial u_1}{\partial \bar{z}_2} = 0 \tag{2.4}$$

when $r_1 = 1$ or $r_2 = 1$. However, since $u_1 = 0$ when $r_1 = 1$, we must have $\frac{\partial u_1}{\partial \bar{z}_2} = 0$, and on the boundary $r_1 = 1$, (2.4) is

$$\frac{\partial u_2}{\partial \bar{z}_1} = 0.$$

Similarly, for $r_2 = 1$, (2.4) is

$$\frac{\partial u_1}{\partial \bar{z}_2} = 0. \tag{2.5}$$

Lemma 2.1. *Let u be a solution to the $\bar{\partial}$-Neumann problem on $\Omega = \mathbb{D}_1 \times \mathbb{D}_2$. Then u is smooth in any neighborhood, $V \subset \overline{\Omega}$ not intersecting $\partial\mathbb{D}_1 \times \partial\mathbb{D}_2$.*

Proof. We consider u_1, the solution to equation (2.1) with the boundary conditions given by (2.3) and (2.5).

Interior regularity follows from the strong ellipticity of the Laplacian.

Also, general regularity at the boundary arguments for the Dirichlet problem can be applied to the case in which V is a neighborhood such that $V \bigcap \partial\Omega = V \bigcap \partial\mathbb{D}_1 \neq \emptyset$ (see [3]).

Lastly, suppose V is a neighborhood such that $V \bigcap \partial\Omega = V \bigcap \partial\mathbb{D}_2 \neq \emptyset$. Define $v = \frac{\partial u_1}{\partial \bar{z}_2}$ and consider the related problem

$$\triangle v = -2\frac{\partial f_1}{\partial \bar{z}_2}$$

on Ω, with the conditions

$$\begin{aligned} v &= 0 \qquad \text{on } r_1 = 0, \\ v &= 0 \qquad \text{on } r_2 = 0. \end{aligned}$$

We know, from above, that v is smooth on all neighborhoods not intersecting $\partial\mathbb{D}_1 \bigcap \partial\mathbb{D}_2$, hence in V. Let $z' = (z_1', z_2') \in V \bigcap \partial\mathbb{D}_2$. We will work in the neighborhood $\mathbb{D}_1 \times V_2$, where V_2 is a bounded neighborhood of z_2' in $\overline{\mathbb{D}}_2$ such that $V_2 \bigcap \mathbb{D}_2$ has smooth boundary. Let $\chi \in C_0^\infty(\overline{V_2})$ such that $\chi \equiv 1$ near z_2'. Define

$$u' = \frac{1}{2\pi i}\int_{V_2} \frac{\chi(\zeta_2)v(z_1,\zeta_2)}{\zeta_2 - z_2} d\zeta_2 \wedge d\bar{\zeta}_2.$$

u' has the properties $\frac{\partial u'}{\partial \bar{z}_2} = v$ near z' and $u' \in C^\infty(\mathbb{D}_1 \times \overline{V_2})$ [1].

We define the operators $\triangle_j$ to be $\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2}$. Then, computing $\triangle u'$ in $\mathbb{D}_1 \times V_2$, we find

$$\triangle u' = -\frac{1}{\pi i}\int_{V_2} \frac{\chi(\zeta_2)\frac{\partial f_1}{\partial \zeta_2}}{\zeta_2 - z_2} d\zeta_2 \wedge d\bar{\zeta}_2 + \phi(z_1, z_2),$$

which is in $C^\infty(\mathbb{D}_1 \times \overline{V_2})$, where, with $\rho = |\zeta_2|$,

$$\begin{aligned} \phi(z_1,z_2) =& \frac{1}{2\pi i}\int_{V_2} \frac{(\triangle_2\chi)v(z_1,\zeta_2)}{\zeta_2 - z_2} d\zeta_2 \wedge d\bar{\zeta}_2 + \frac{2}{\pi i}\int_{V_2} \frac{\frac{\partial \chi}{\partial \zeta_2}\frac{\partial v}{\partial \bar{\zeta}_2} + \frac{\partial \chi}{\partial \bar{\zeta}_2}\frac{\partial v}{\partial \zeta_2}}{\zeta_2 - z_2} d\zeta_2 \wedge d\bar{\zeta}_2 \\ & - \frac{1}{2\pi i}\int_{\partial V_2 \cap\ \partial\mathbb{D}_2} \frac{\chi(\zeta_2)\frac{\partial v(z_1,\zeta_2)}{\partial \rho}}{\zeta_2 - z_2} d\zeta_2. \end{aligned}$$

We set $w = u_1 - u'$ and show $w \in C^\infty(\mathbb{D}_1 \times \overline{V_2})$. For z_2 near z_2', $\frac{\partial w}{\partial \bar{z}_2} = 0$, in which case

(2.6)

$$\triangle_1 w = \triangle w = -2f_1 - \triangle u' = -2f_1 + \frac{1}{\pi i}\int_{V_2} \frac{\chi(\zeta_2)\frac{\partial f_1}{\partial \bar{\zeta}_2}(z_1,\zeta_2)}{\zeta_2 - z_2} d\zeta_2 \wedge d\bar{\zeta}_2 - \phi(z_1,z_2).$$

We also have the boundary condition $w = 0$ when $r_1 = 0$. Hence w is the solution to a Dirichlet problem on the unit disc,

$$w = \int_{\mathbb{D}_1} G_1(z_1,\zeta_1)\Phi(\zeta_1,z_2) d\zeta_1 \wedge d\bar{\zeta}_1,$$

where G_1 is the Green's function for $\mathbb{D}_1$,

$$G_1 = \frac{1}{2\pi}\log|z_1 - \zeta_1| - \frac{1}{2\pi}\log\left||z_1|^{-1}z_1 - |\zeta_1|^{-1}\zeta_1\right|,$$

and Φ is defined to be the right hand side of equation 2.6. Because $\Phi \in C^\infty(\mathbb{D}_1 \times \overline{V_2})$, so is w, and $u_1 \in C^\infty(\mathbb{D}_1 \times \overline{V_2})$ follows from the fact that $u' \in C^\infty(\mathbb{D}_1 \times \overline{V_2})$.

The same reasoning applies to u_2, and this proves the lemma. □

We may simplify our calculations if we consider the equations

$$\triangle v_i = g_i$$

with boundary conditions

$$v_i = 0 \qquad \text{on } \partial\Omega$$

for $i = 1, 2$, where $v_i = \frac{\partial u_i}{\partial \bar{z}_j}$ and $g_i = -2\frac{\partial f_i}{\partial \bar{z}_j}$ $(j \neq i)$.

We expand v_1 and g_1 into Fourier series:

(2.7)
$$v_1 = \sum_{m_1,m_2=-\infty}^{\infty} a_{m_1m_2}(r_1,r_2)e^{im_1\theta_1}e^{im_2\theta_2}$$
$$g_1 = \sum_{m_1,m_2=-\infty}^{\infty} c_{m_1m_2}(r_1,r_2)e^{im_1\theta_1}e^{im_2\theta_2}.$$

Using these expansions in (2.2), (2.3), and (2.5), we see the family of equations

$$\frac{\partial^2 a_{m_1m_2}}{\partial r_1^2} + \frac{1}{r_1}\frac{\partial a_{m_1m_2}}{\partial r_1} - \frac{m_1^2}{r_1^2}a_{m_1m_2} + \frac{\partial^2 a_{m_1m_2}}{\partial r_2^2} + \frac{1}{r_2}\frac{\partial a_{m_1m_2}}{\partial r_2} - \frac{m_2^2}{r_2^2}a_{m_1m_2} = c_{m_1m_2} \qquad m_1, m_2 = 0, \pm 1, \ldots$$

are satisfied with the boundary conditions

$$a_{m_1m_2}(1,r_2) = 0,$$
$$a_{m_1m_2}(r_1,1) = 0.$$

We have analogous equations for v_2.

3. Solution

We are then led to study the equations

$$\frac{\partial^2 a_{m_1m_2}}{\partial r_1^2}+\frac{1}{r_1}\frac{\partial a_{m_1m_2}}{\partial r_1}-\frac{m_1^2}{r_1^2}a_{m_1m_2}+\frac{\partial^2 a_{m_1m_2}}{\partial r_2^2}+\frac{1}{r_2}\frac{\partial a_{m_1m_2}}{\partial r_2}-\frac{m_2^2}{r_2^2}a_{m_1m_2}=c_{m_1m_2} \tag{3.1}$$

in the space $r_1 < 1$, $r_2 < 1$. Here $a_{m_1m_2} = a_{m_1m_2}(r_1, r_2)$ and $c_{m_1m_2} = c_{m_1m_2}(r_1, r_2)$, and the boundary conditions

$$a_{m_1m_2}(1, r_2) = 0,$$
$$a_{m_1m_2}(r_1, 1) = 0$$

hold.

We make the transformation $y_j = -\log r_j$ for $j = 1, 2$ in (3.1), and multiply the resulting equation by $e^{2y_1}e^{2y_2}$. Then with $A_{m_1m_2} = a_{m_1m_2}(e^{-y_1}, e^{-y_2})$ and $C_{m_1m_2} = e^{-2y_1}e^{-2y_2}c_{m_1m_2}(e^{-y_1}, e^{-y_2})$, (3.1) becomes

$$e^{-2y_2}(D_1^2 - m_1^2)A_{m_1m_2} + e^{-2y_1}(D_2^2 - m_2^2)A_{m_1m_2} = C_{m_1m_2}$$

on the first quadrant in $\mathbb{R}^2$, where D_j stands for the differential operator $\frac{\partial}{\partial y_j}$, and the boundary conditions are

$$A_{m_1m_2}(0, y_2) = 0,$$
$$A_{m_1m_2}(y_1, 0) = 0.$$

We extend $A_{m_1m_2}$ and $C_{m_1m_2}$ by odd reflections in the variables y_1 and y_2, labelling the extended functions $\tilde{A}_{m_1m_2}$ and $\tilde{C}_{m_1m_2}$, respectively, and we look to solve

$$e^{-2|y_2|}(D_1^2 - m_1^2)\tilde{A}_{m_1m_2} + e^{-2|y_1|}(D_2^2 - m_2^2)\tilde{A}_{m_1m_2} = \tilde{C}_{m_1m_2}.$$

Let χ be a smooth compactly supported cutoff function in $\mathbb{R}^2$, symmetric about the origin, such that $\chi \equiv 1$ in a neighborhood of the origin. Then $\chi\tilde{A}_{m_1m_2}$ satisfies

$$e^{-2|y_2|}(D_1^2 - m_1^2)\chi\tilde{A}_{m_1m_2} + e^{-2|y_1|}(D_2^2 - m_2^2)\chi\tilde{A}_{m_1m_2} = h, \tag{3.2}$$

where h is a compactly supported, odd function of y_1 and y_2, which, when restricted to the first quadrant, is C^∞ up to the boundary, and, in a neighborhood of the origin, is equivalent to $\tilde{C}_{m_1m_2}$.

In what follows we use the notation $\hat{)}$ to denote the Fourier transform of that which is enclosed by the parentheses and $\check{)}$ to denote the inverse Fourier transform. Upon taking Fourier transforms of (3.2) we obtain

$$\left((\eta_1^2 + m_1^2)e^{-2|D_{\eta_2}|} + (\eta_2^2 + m_2^2)e^{-2|D_{\eta_1}|}\right)\left(\chi\tilde{A}_{m_1m_2}\hat{\right)} = -\hat{h}, \tag{3.3}$$

where $|D_{\eta_j}|$ is the positive square root of $-\frac{\partial^2}{\partial \eta_j^2}$ for $j = 1, 2$. We intend to invert the operator

$$(\eta_1^2 + m_1^2)e^{-2|D_{\eta_2}|} + (\eta_2^2 + m_2^2)e^{-2|D_{\eta_1}|}.$$

$$\text{(3.4)} \quad \Big((\eta_1^2 + m_1^2)e^{-2|D_{\eta_2}|} + (\eta_2^2 + m_2^2)e^{-2|D_{\eta_1}|}\Big)\Big(\chi\tilde{A}_{m_1m_2}\Big)^{\hat{}} =$$

$$(\eta^2 + m^2)\Big(\chi\tilde{A}_{m_1m_2}\Big)^{\hat{}} +$$

$$\Big((\eta_1^2 + m_1^2)\Big((e^{-2|y_2|} - 1)\chi\tilde{A}_{m_1m_2}\Big)^{\hat{}} + (\eta_2^2 + m_2^2)\Big((e^{-2|y_1|} - 1)\chi\tilde{A}_{m_1m_2}\Big)^{\hat{}}\Big) =$$

$$(\eta^2 + m^2)(I - K)\Big(\chi\tilde{A}_{m_1m_2}\Big)^{\hat{}},$$

where $\eta^2 = \eta_1^2 + \eta_2^2$ and $m^2 = m_1^2 + m_2^2$, I is the identity operator, and K is the operator defined by

$$K\hat{\phi} = \frac{\eta_1^2 + m_1^2}{\eta^2 + m^2}\Big((1 - e^{-2|y_2|})\phi\Big)^{\hat{}} + \frac{\eta_2^2 + m_2^2}{\eta^2 + m^2}\Big((1 - e^{-2|y_1|})\phi\Big)^{\hat{}}$$

for $\phi \in L_0^2(\mathbb{R}^2)$.

Now let $\chi_1 = \chi$, and define cutoff functions, χ_j, which are symmetric about the origin, for $j = 1, 2 \ldots$, such that $\chi_j = 1$ on suppχ_{j-1}. Also, define $T_0 = I$ and $T_j\phi = \big(\, K(\chi_j T_{j-1}\phi)^{\hat{}}\,\big)^{\check{}}$ for $\phi \in L_0^2$ for $j = 1, 2, \ldots$. We may assume, after restricting the supports of the χ_j if necessary, that the following relations hold

$$\|T_j\phi\|_2 < \|\phi\|_2 \quad \forall\phi \in L^2 \text{ and } \forall j \in \mathbb{N};$$
$$\|T_jA_{m_1m_2}\|_2 \to 0 \quad \text{as } j \to \infty.$$

From (3.3) and (3.4) we have

$$\text{(3.5)} \qquad (I - K)\Big(\chi\tilde{A}_{m_1m_2}\Big)^{\hat{}} = \hat{\Phi},$$

where $\hat{\Phi} = -\frac{\hat{h}}{\eta^2 + m^2}$, and from (3.5) we obtain,

$$\text{(3.6)} \qquad \chi_{n+1}T_nA_{m_1m_2} - \chi_{n+2}T_{n+1}A_{m_1m_2} = \chi_{n+1}T_n\Phi + s_n,$$

where $s_n = (\chi_{n+1} - \chi_{n+2})T_{n+1}A_{m_1m_2} + \chi_{n+1}\big(K\hat{s}_{n-1}\big)^{\check{}}$ and $s_0 = (\chi_2 - \chi_1)\Phi$. Equation 3.6 gives terms of a telescoping series which converges in L^2 since $\|\chi_{n+2}T_{n+1}A_{m_1m_2}\|_2 \to 0$ as $n \to \infty$. For any $\epsilon > 0$ we may also choose the χ_j so that $\|\chi_{n+1} - \chi_n\|_2 < \frac{\epsilon}{2^{n+1}\|A_{m_1m_2}\|_2}$ for $n \geq 2$ and $\|\chi_2 - \chi_1\|_2 < \frac{\epsilon}{2\|\Phi\|_2}$ which implies $\|\sum_{n=0}^{\infty} s_n\|_2 < \epsilon$. Hence, we conclude

$$\text{(3.7)} \qquad \chi\tilde{A}_{m_1m_2} \overset{L^2}{=} \sum_{n=0}^{\infty} \chi_{n+1}T_n\Phi.$$

Remark 3.1. To proceed formally, we may take (3.7) as a starting point, using (3.7) to define a function $a_{m_1m_2}(r_1, r_2)$ from the transformations above. Then it is easy to show, working backwards, that v_1, as defined in (2.7), gives rise to a function u_1 which solves (2.1), (2.3), and (2.5). In fact, using Lemma 2.1, we can show the boundary conditions are satisfied in the classical sense.

4. Behavior at the distinguished boundary

Here we find the singular functions which are in the expansion, (3.7). We show, in particular,

Proposition 4.1. *$\forall N \in \mathbb{N}$, $\exists$ polynomials of degree N, A_N, B_N, and C_N, such that, near the origin, modulo terms which are in $C^N(\overline{\mathbb{R}_+ \times \mathbb{R}_+})$,*

$$A_{m_1m_2} = A_N \log(y_1^2 + y_2^2) + B_N + C_N \arctan \frac{y_1}{y_2}.$$

In the proof of the proposition we shall make use of functions constructed in [2]. Let

$$\Phi_1(y_1, y_2) = -\frac{i}{2} \log(y_1^2 + y_2^2)$$

and define Φ_{l+1} to be the unique solution of the form

$$p_1(y_1, y_2) \log(y_1^2 + y_2^2) + p_2(y_1, y_2),$$

where p_1 and p_2 are homogeneous polynomials of degree $2l - 2$ in y_1 and y_2 such that $p_2(y_1, 0) = 0$, to the equation

$$\frac{\partial \Phi_{l+1}}{\partial y_2} = \frac{1}{2l} y_2 \Phi_l$$

for $l \geq 1$. Then with Φ_l defined for $l \geq 1$, define $(\Phi_l)_0 = \Phi_l$ for $y_2 \geq 0$, and, for $j \geq 1$, $(\Phi_l)_j$ to be the unique solution of the form

$$p_1 \log(y_1^2 + y_2^2) + p_2 + p_3 \arctan\left(\frac{y_1}{y_2}\right)$$

on the half-plane $\{(y_1, y_2) : y_2 \geq 0\}$, where p_1, p_2, and p_3 are polynomials in y_1 and y_2 such that $p_2(0, y_2) = 0$, to the equation

$$\frac{\partial (\Phi_l)_j}{\partial y_1} = (\Phi_l)_{j-1}.$$

Also, define recursively for $k \geq 1$, on $y_2 \geq 0$,

$$(\Phi_l)_{jk} = \int_0^{y_2} \cdots \int_0^{t_2} \int_0^{t_1} (\Phi_l)_j(y_1, t) dt dt_1 \cdots dt_{k-1}.$$

Proof of the proposition. We shall prove that with T_n defined as above, $\forall\, N \in \mathbb{N}$, and $\forall\, n \geq 0$, on $\mathbb{R}_+ \times \mathbb{R}_+$, in a neighborhood of $(0, 0)$,

$$T_n \Phi = \sum_{\substack{a+b+2l-2+j+k=2 \\ l,j,k \geq 1}}^{N} c_{abljk} y_1^a y_2^b (\Phi_l)_{jk} + s, \tag{4.1}$$

where c_{abljk} depend on θ_1, θ_2, m_1, and m_2, and s is used to denote the Fourier transform of any function which, when restricted to $\mathbb{R}_+ \times \mathbb{R}_+$, is in $C^N(\overline{\mathbb{R}_+ \times \mathbb{R}_+})$ (plus terms which may be singular either along all of $y_1 = 0$ or along all of $y_2 = 0$). The proof is by induction. (4.1) holds true when $n = 0$, as shown in [2]. We use the Taylor expansion with remainder formula,

$$(4.2)\quad 1 - e^{-2y_k} = 2y_k - \frac{(2y_k)^2}{2!} + \cdots + (-1)^N \frac{(2y_k)^N}{(N+1)!} + \frac{(-2)^{N+1}}{(N+1)!} \int_0^{y_k} (y_k - t)^{N+1} e^{-2t} dt,$$

for $k = 1, 2$, in the integrands of the formula

$$(4.3)\quad \widehat{T_n\Phi} = (-2i)^2 \frac{\eta_1^2 + m_1^2}{\eta^2 + m^2} \int_0^\infty \int_0^\infty (1 - e^{-2y_2}) \chi_{n-1}(T_{n-1}\Phi) \sin(\eta_1 y_1) \sin(\eta_2 y_2) dy_1 dy_2 + (-2i)^2 \frac{\eta_2^2 + m_2^2}{\eta^2 + m^2} \int_0^\infty \int_0^\infty (1 - e^{-2y_1}) \chi_{n-1}(T_{n-1}\Phi) \sin(\eta_1 y_1) \sin(\eta_2 y_2) dy_1 dy_2.$$

Now for $y_2 \geq 0$,

$$(4.4)\qquad (\Phi_l)_{jk} = p_1 \log(y_1^2 + y_2^2) + p_2 + p_3 \arctan\left(\frac{y_1}{y_2}\right) + p_4 \log|y_1|,$$

where the p_m are homogeneous polynomials of degree $(2l - 2) + j + k$ in y_1 and y_2 for $m = 1, 2, 3, 4$, and we shall also denote by $(\Phi_l)_{jk}$ its extension to $\mathbb{R}^2 \setminus \{y_1 = 0, y_2 = 0\}$, where we use the branch from 0 to $-\infty$ to extend the arctan function.

We show, writing r_{Nk} for the remainder term in (4.2),

$$(4.5)\qquad \frac{\eta_i^2 + m_i^2}{\eta^2 + m^2} \int_0^\infty \int_0^\infty r_{Nk} \chi_{n-1}(T_{n-1}\Phi) \sin(\eta_1 y_1) \sin(\eta_2 y_2) dy_1 dy_2,$$

for $i = 1, 2$, is the Fourier transform of a function which may be included in a function s. We now use the induction hypothesis so that we may utilize the properties of the particular functions comprising $T_{n-1}\Phi$. $r_{Nk}(y_1^a y_2^b (\Phi_l)_{jk})$ vanishes to $(N+2)nd$ order along y_k hence its odd reflection about the y_k-axis will still be C^{N+1} on the appropriate half-plane. Then the regularity of the operator $D_1^2 + D_2^2 - m_1^2 - m_2^2$ shows

$$\frac{\eta_i^2 + m_i^2}{\eta^2 + m^2} \int_0^\infty \int_0^\infty r_{Nk} \chi_{n-1}(y_1^a y_2^b (\Phi_l)_{jk}) \sin(\eta_1 y_1) \sin(\eta_2 y_2) dy_1 dy_2$$

is in $C^N(\overline{\mathbb{R}_+ \times \mathbb{R}_+})$. Again, using the regularity of $D_1^2 + D_2^2 - m_1^2 - m_2^2$, when the remaining terms of $T_{n-1}\Phi$ are considered in the integral in (4.5), we can show that (4.5) may be included in a function s.

After using the induction hypothesis in (4.3), we consider

$$\widehat{\Psi} = \frac{\eta_i^2 + m_i^2}{\eta^2 + m^2} \int_0^\infty \int_0^\infty \chi_{n-1} y_1^p y_2^q (\Phi_l)_{jk} \sin(\eta_1 y_1) \sin(\eta_1 y_1) dy_1 dy_2.$$

Instead of looking at the odd function, Ψ, of both variables, we extend $\Psi|_{\mathbb{R}_+ \times \mathbb{R}_+}$, denoting the extended function $\widetilde{\Psi}$, in such a way that

$$\widehat{\widetilde{\Psi}} = \frac{\eta_i^2 + m_i^2}{\eta^2 + m^2} \big(\chi_{n-1} y_1^p y_2^q (\Phi_l)_{jk}\big)\hat{}.$$

Then using the relations

$$\begin{aligned}
\frac{\partial}{\partial y_1} (\Phi_l)_{jk} &= (\Phi_l)_{(j-1)k}, \\
\frac{\partial}{\partial y_2} (\Phi_l)_{jk} &= (\Phi_l)_{j(k-1)}, \\
\frac{\partial}{\partial y_1} \Phi_l &= y_1 \Phi_{l-1}, \\
\frac{\partial}{\partial y_2} \Phi_l &= y_2 \Phi_{l-1}, \\
\widehat{\chi_{n-1} (\Phi_l)}_{jk} &= \frac{1}{\eta_1^j} \frac{1}{\eta_2^k} \frac{1}{(\eta_1^2 + \eta_2^2)^l} + s,
\end{aligned}$$

where

$$\Phi_0 = \frac{1}{y_1^2 + y_2^2},$$

we may write $\widehat{\widetilde{\Psi}}$ as a sum of terms of the form

$$\Big(\varphi y_1^\alpha y_2^\beta (\Phi_l)_{jk}\Big)\hat{} + s,$$

where $\varphi \in C_0^\infty$ is equivalent to 1 in a neighborhood of the origin, α and β are non-negative integers and l, j, and k are positive integers.

Once (4.1) is proved, another induction argument shows

$$T_n \Phi\big|_{\mathbb{R}_+ \times \mathbb{R}_+} \in C^n(\overline{\mathbb{R}_+ \times \mathbb{R}_+})$$

(modulo a function to be included within s), and thus we may prove the proposition by looking at only the first N terms in (3.7), using Lemma 2.1 to argue the vanishing of singular terms along all of $y_1 = 0$ or $y_2 = 0$ arising from (4.1) or (4.4). □

After using the decay of $c_{m_1 m_2}$ with respect to m_1 and m_2 to sum over m_1 and m_2, and then transforming back to the variables z_1 and z_2, we deduce that, $\forall n \in \mathbb{N}$ v_1 may be written

$$v_1 = a_n \log\big(\log r_1)^2 + (\log r_2)^2\big) + b_n + c_n \arctan\left(\frac{\log r_1}{\log r_2}\right),$$

where a_n, b_n, and c_n are polynomials of degree n in $\log r_1$ and $\log r_2$ whose coefficients are smooth functions of θ_1 and θ_2.

We now obtain the singularities of u_1 from those of v_1. If

$$u_1 = 2 \sum_{m_1,m_2=-\infty}^{\infty} b_{m_1 m_2}(r_1, r_2) e^{im_1\theta_1} e^{im_2\theta_2},$$

then u_1 and v_1 are related by

$$\frac{\partial}{\partial r_2} b_{m_1 m_2} - m_2 \frac{b_{m_1 m_2}}{r_2} = a_{m_1(m_2+1)}. \tag{4.6}$$

We assume without loss of generality that, $\forall\ m_1,\ m_2 \geq 0$, $b_{m_1 m_2}$ and $a_{m_1 m_2}$ are supported in some neighborhood of $r_1 = r_2 = 1$. The solution to (4.6) is given by

$$b_{m_1 m_2} = r_2^{m_2} \int_0^{r_2} t^{-m_2} a_{m_1(m_2+1)}(r_1, t) dt.$$

We make the substitution $u = -\log t$ in the above integral to get

$$b_{m_1 m_2} = r_2^{m_2} \int_{-\log r_2}^{\infty} e^{u(m_2+1)} a_{m_1(m_2+1)}(r_1, e^{-u}) du. \tag{4.7}$$

The integral in (4.7) was considered in [2] and gives, after summing over m_1 and m_2, and using a theorem of Borel, with similar results on the form of u_2, Theorem 1.1.

We note that there are $f \in C^\infty(\overline{\Omega})$, for example those f whose components, f_1 and f_2, are equivalently equal to 1 in a neighborhood of $\partial\mathbb{D}_1 \times \partial\mathbb{D}_2$, which make Theorem 1.1 non-trivial, i.e. α_j and γ_j are not necessarily 0.

We may also determine a sufficient condition under which the solution exhibits any desired degree of regularity up to the boundary of the bi-disc.

Proposition 4.2. *If*

$$\frac{\partial^{2j}}{\partial r_1^{2j}} \frac{\partial^{2k}}{\partial r_2^{2k}} \left(\frac{\partial f_1}{\partial \bar{z}_2} \right) \bigg|_{r_1=r_2=0} = 0 \tag{4.8}$$

$\forall\ j,k \geq 0$ *such that* $j+k \leq n+2$, *then* $u_1 \in C^n(\overline{\Omega})$.

Proof. If (4.8) holds, then $\forall\ m_1, m_2$

$$\frac{\partial^{2j}}{\partial y_1^{2j}} \frac{\partial^{2k}}{\partial y_2^{2k}} C_{m_1 m_2} \bigg|_{y_1=y_2=0} = 0,$$

$\forall\ j,k \geq 0$ such that $j+k \leq n+2$, which implies $A_{m_1 m_2} \in C^n(\overline{\mathbb{R}_+ \times \mathbb{R}_+})$ (see [2]), $A_{m_1 m_2}$ and $C_{m_1 m_2}$ defined as above, and thus $v_1 = \frac{\partial u_1}{\partial \bar{z}_2} \in C^n(\overline{\Omega})$. Then, we can see u_1 is in $C^n(\overline{\Omega})$ by considering integrals as in (4.7), where now the integrands are in $C^n(\overline{\Omega})$. □

Acknowledgements

I wish to thank several people who helped me in the preparation of this article. I offer my gratitude to David Barrett under whose guidance and encouragement I first considered the problem on the bi-disc. I also wish to thank Harold Boas, Peter Kuchment, and Emil Straube with whom I could share and discuss various ideas in the course of my research.

References

[1] S. Bell, *The Cauchy transform, potential theory, and conformal mapping*, Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.

[2] D. Ehsani, *Solution of the $\bar{\partial}$-neumann problem on a non-smooth domain*, to appear in Indiana Univ. Math. J.

[3] G. Folland, *Introduction to partial differential equations*, Second edition. Princeton University Press, Princeton, NJ, 1995.

[4] G. Henkin, A. Iordan, *Compactness of the Neumann operator for hyperconvex domains with non-smooth B-regular boundary*, Math. Ann. **307** (1997), 151–168.

[5] G. Henkin, A. Iordan, J. Kohn, *Estimations sous-elliptiques pour le problème $\bar{\partial}$-Neumann dans un domaine strictement pseudoconvexe à frontière lisse par morceaux*, C. R. Acad. Sci. Paris Sér. I Math. **323** (1996), 17–22.

[6] L. Hörmander, *L^2 estimates and existence theorems for the $\bar{\partial}$ operator*, Acta. Math. **113** (1965), 89–152.

[7] J. Michel, M.-C. Shaw, *Subelliptic estimates for the $\bar{\partial}$-Neumann operator on piecewise smooth strictly pseudoconvex domains*, Duke Math. J. **93** (1998), 115–128.

[8] ———, *The $\bar{\partial}$-Neumann operator on Lipschitz pseudoconvex domains with plurisubharmonic defining functions*, Duke Math. J. **108** (2001), 421–447.

[9] E. J. Straube, *Plurisubharmonic functions and subellipticity of the $\bar{\partial}$-Neumann problem on non-smooth domains*, Math. Res. Lett. **4** (1997), 459–467.

Department of Mathematics, Texas A&M University, College Station, Texas 77843-3368, U.S.A.

E-mail address: ehsani@math.tamu.edu