Mathematical Research Letters **10**, 501–513 (2003)

HYPERKÄHLER MANIFOLDS WITH TORSION OBTAINED FROM HYPERHOLOMORPHIC BUNDLES

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Abstract. We construct examples of compact hyperkähler manifolds with torsion (HKT manifolds) which are not homogeneous and not locally conformal hyperkähler. Consider a total space T of a tangent bundle over a hyperkähler manifold M. The manifold T is hypercomplex, but it is never hyperkähler, unless M is flat. We show that T admits an HKT-structure. We also prove that a quotient of T by a $\mathbb{Z}$-action $v \longrightarrow q^n v$ is HKT, for any real number $q \in \mathbb{R}$, $q > 1$. This quotient is compact, if M is compact. A more general version of this construction holds for all hyperholomorphic bundles with holonomy in $Sp(n)$.

1. Introduction

Hyperkähler manifolds with torsion (HKT-manifolds) were introduced by P. S. Howe and G. Papadopoulos ([HP]) and much discussed in physics literature since then. For an excellent survey of these works written from a mathematician's point of view, the reader is referred to the paper of G. Grantcharov and Y. S. Poon [GP]. In physics, HKT-manifolds appear as moduli of brane solitons in supergravity and M-theory ([GP2], [P]). HKT-manifolds also arise as moduli space of some special black holes in N=2 supergravity ([GP1], [GPS]).

The term "hyperkähler manifold with torsion" is actually quite misleading, because an HKT-manifold is not hyperkähler. This is why we prefer to use the abbreviation "HKT-manifold".

HKT-manifolds are hypercomplex manifolds equipped with a special kind of Riemannian metrics.

A *hypercomplex manifold* ([Bo]) is a C^∞-manifold M endowed with a triple of almost complex structures $I, J, K \in \operatorname{End}(TM)$ which are integrable and satisfy the quaternionic relations $I \circ J = -J \circ I = K$. If, in addition, M is equipped with a Riemannian structure g preserved by I, J, K, then M is called *hypercomplex Hermitian*. If (M, g) is Kähler with respect to I, J, K, then (M, g, I, J, K) is called *hyperkähler*.

An HKT-manifold is a hypercomplex Hermitian manifold which satisfies a similar, but weaker condition (1.1).

Received March 16, 2003.

The author is partially supported by CRDF Grant RM1-2354-MO02 and EPSRC Grant GR/R77773/01.

Let (M, g, I, J, K) be a hypercomplex Hermitian manifold. Write the standard Hermitian forms on M as follows:

$$\omega_I := g(\cdot, I\cdot), \quad \omega_J := g(\cdot, J\cdot), \quad \omega_K := g(\cdot, K\cdot).$$

By definition, M is hyperkähler iff these forms are closed. The HKT condition is weaker:

$$\partial(\omega_J + \sqrt{-1}\,\omega_K) = 0. \tag{1.1}$$

Notice that $\Omega = \frac{1}{2}(\omega_J + \sqrt{-1}\omega_K)$ is a $(2,0)$-form, for any hypercomplex Hermitian manifold, as an elementary linear-algebraic calculation insures. This form is called *the canonical (2,0)-form associated with the hypercomplex Hermitian structure*. As we shall see (Proposition 3.2), the metric can be recovered from the hypercomplex structure and the form Ω.

Originally, the HKT-manifolds were defined in terms of a quaternionic invariant connection with totally antisymmetric torsion (see [HP], [GP]).

Many homogeneous examples of compact HKT-manifolds were obtained in [HP] and [GP]. In [I] it was shown that any locally conformally hyperkähler manifold also admits an HKT-structure (see [Or]). Converse result was obtained in [OPS]: it was found when an HKT-manifold is locally conformally hyperkähler manifold, in terms of symmetry conditions.

Locally, the HKT metrics can be studied using potential functions ([GP]) in the same fashion as one uses plurisubharmonic functions to study Kähler metrics. This way one obtains many examples of HKT-structures on a sufficiently small open hypercomplex manifolds.

If $\dim_{\mathbb{R}} M = 4$, every hypercomplex Hermitian metrics is also HKT (the condition (1.1) is satisfied vacuously because the left hand side of (1.1) is a $(3,0)$-form).

If $\dim_{\mathbb{R}} M > 4$, the HKT-condition becomes highly non-trivial. There are examples of hypercomplex manifolds not admitting an HKT-structure ([FG]). All known examples of compact HKT-manifolds are either homogeneous or locally conformally hyperkähler.

In the present paper, we construct HKT-structures on fibered spaces associated with hyperkähler manifolds. A typical example of our construction is the following

Theorem 1.1. *Let M be a hyperkähler manifold and*

$$T^\circ M = \operatorname{Tot}(TM)\backslash\{\textit{zero section}\}$$

the total space of non-zero vectors in TM. Given $q \in \mathbb{R}$, $|q| \neq 1$, let $\sim_q$ be the equivalence relation generated by $x \sim_q qx$, $x \in TM$. Consider the quotient $T^\circ M/\sim_q$. Then $T^\circ M/\sim_q$ is equipped with a natural HKT-structure.

Proof. See Theorem 8.1. □

Theorem 1.1 is a special case of a much more general construction performed in Section 8.

2. The q-Dolbeault bicomplex

In this Section, we introduce some notions of quaternionic linear algebra which will be used further on. A reader well versed in quaternions can safely skip this section. We follow [V5].

Let M be a hypercomplex manifold, and

$$\Lambda^0 M \xrightarrow{d} \Lambda^1 M \xrightarrow{d} \Lambda^2 M \xrightarrow{d} ...$$

its de Rham complex. Consider the natural action of $SU(2)$ on $\Lambda^* M$. Clearly, $SU(2)$ acts on $\Lambda^i M$, $i \leqslant \frac{1}{2}\dim_{\mathbb{R}} M$ with weights $i, i-2, i-4, \ldots$

We denote by Λ^i_+ the maximal $SU(2)$-subspace of Λ^i, on which $SU(2)$ acts with weight i.

The following linear-algebraic lemma allows one to compute Λ^i_+ explicitly

Lemma 2.1. *In the above assumptions, let I be an induced complex structure, and $\mathbb{H}_I$ the quaternion space, considered as a 2-dimensional complex vector space with the complex structure induced by I. Denote by $\Lambda^{p,0}_I(M)$ the space of $(p,0)$-forms on (M,I). The space $\mathbb{H}_I$ is equipped with a natural action of $SU(2)$. Consider $\Lambda^{p,0}_I(M)$ as a representation of $SU(2)$, with trivial group action. Then, there is a canonical isomorphism*

$$\Lambda^p_+(M) \cong S^p_{\mathbb{C}}\mathbb{H}_I \otimes_{\mathbb{C}} \Lambda^{p,0}_I(M), \tag{2.1}$$

where $S^p_{\mathbb{C}}\mathbb{H}_I$ denotes a p-th symmetric power of $\mathbb{H}_I$. Moreover, the $SU(2)$-action on $\Lambda^p_+(M)$ is compatible with the isomorphism (2.1).

Proof. This is [V5], Lemma 8.1. □

Consider an $SU(2)$-invariant decomposition

$$\Lambda^p(M) = \Lambda^p_+(M) \oplus V^p, \tag{2.2}$$

where V^p is the sum of all $SU(2)$-subspaces of $\Lambda^p(M)$ of weight less than p. Using the decomposition (2.2), we define the quaternionic Dolbeault differential $d_+ : \Lambda^*_+(M) \longrightarrow \Lambda^*_+(M)$ as a composition of de Rham differential and projection of to $\Lambda^*_+(M) \subset \Lambda^*(M)$. Since the de Rham differential cannot increase the $SU(2)$-weight of a form more than by 1, d preserves the subspace $V^* \subset \Lambda^*(M)$. Therefore, d_+ is a differential in $\Lambda^*_+(M)$.

Let M be a hypercomplex manifold, and I an induced complex structure. Consider the operator $\mathcal{I} : \Lambda^*(M) \longrightarrow \Lambda^*(M)$ mapping a (p,q)-form η to $\sqrt{-1}(p-q)\eta$. By definition, $\mathcal{I}$ belongs to the Lie algebra $\mathfrak{su}(2)$ acting on $\Lambda^*(M)$ in the standard way. Therefore, $\mathcal{I}$ preserves the subspace $\Lambda^*_+(M) \subset \Lambda^*(M)$. We obtain the Hodge decomposition

$$\Lambda^*_+(M) = \oplus_{p,q}\Lambda^{p,q}_{+,I}(M).$$

Let M be a hypercomplex manifold, I an induced comlex structure, and $I, J, K \in \mathbb{H}$ the standard triple of induced complex structures. Clearly, J acts on the complexified co tangent space $\Lambda^1 M \otimes \mathbb{C}$ mapping $\Lambda^{0,1}_I(M)$ to $\Lambda^{1,0}_I(M)$. Consider a differential operator

$$\partial_J : C^\infty(M) \longrightarrow \Lambda^{1,0}_I(M),$$

mapping f to $J(\overline{\partial} f)$, where $\overline{\partial} : C^\infty(M) \longrightarrow \Lambda^{0,1}_I(M)$ is the standard Dolbeault differential on a Kähler manifold (M, I). We extend ∂_J to a differential

$$\partial_J : \Lambda^{p,0}_I(M) \longrightarrow \Lambda^{p+1,0}_I(M),$$

using the Leibniz rule.

Proposition 2.2. *Let M be a hypercomplex manifold, I an induced complex structure, I, J, K the standard basis in quaternion algebra, and*

$$\Lambda^*_+(M) = \oplus_{p,q} \Lambda^{p,q}_{I,+}(M)$$

the Hodge decomposition of the quaternionic Dolbeault complex. Then there exists a canonical isomorphism

$$\Lambda^{p,q}_{I,+}(M) \cong \Lambda^{p+q,0}_I(M). \tag{2.3}$$

Under this identification, the quaternionic Dolbeault differential

$$d_+ : \Lambda^{p,q}_{I,+}(M) \longrightarrow \Lambda^{p+1,q}_{I,+}(M) \oplus \Lambda^{p,q+1}_{I,+}(M)$$

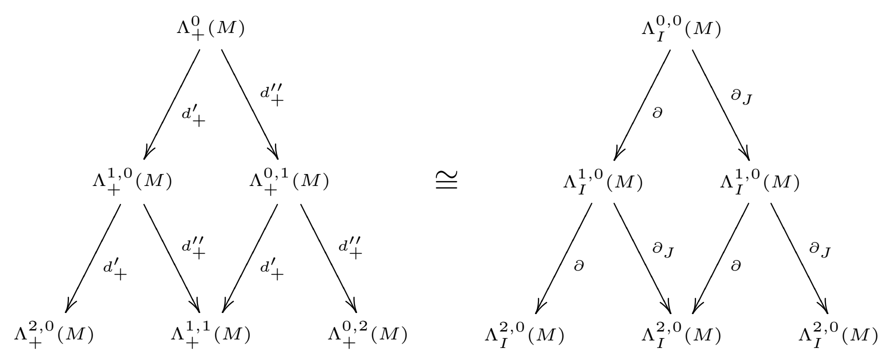
corresponds to a sum

$$\partial \oplus \partial_J : \Lambda^{p+q,0}_I(M) \longrightarrow \Lambda^{p+q+1,0}_I(M) \oplus \Lambda^{p+q+1,0}_I(M).$$

Proof. This is Proposition 8.13 of [V5]. ☐

The statement of Proposition 2.2 can be represented by the following diagram

(2.4)



where $d_+ = d'_+ + d''_+$ is the Hodge decomposition of the quaternionic Dolbeault differential.

Using the $SU(2)$-action, we may identify the bundles $\Lambda^{p,q}_+(M)$ with $\Lambda^{p+q,0}_+(M) = \Lambda^{p+q,0}_I(M)$ explicitly, as follows.

Let $\mathcal{J}, \mathcal{K}$ be the Lie algebra operators acting on differential forms and associated with J, K in the same way as $\mathcal{I}$ is associated with I. Consider the map $\mathcal{R}: \Lambda^*(M) \longrightarrow \Lambda^*(M)$,

$$\mathcal{R} := \frac{\mathcal{J} - \sqrt{-1}\,\mathcal{K}}{2}. \tag{2.5}$$

It is easy to check that the Lie algebra elements $\mathcal{R}, \mathcal{I}, \overline{\mathcal{R}}$ form an $SL(2)$-triple in the complexification of the standard $SU(2) \subset \operatorname{End}(\Lambda^*(M))$. Therefore, $\mathcal{R}$ maps $\Lambda_+^{p,q}(M)$ to $\Lambda_+^{p+1,q-1}(M)$. Since $\Lambda_+^m(M)$ is a representation of weight m, $\mathcal{R}$ induces an isomorphism

$$\mathcal{R}: \ \Lambda_{+,I}^{p,q}(M) \longrightarrow \Lambda_{+,I}^{p+1,q-1}(M),$$

for all $q > 0$.

Together with (2.4), this observation implies the following.

Claim 2.3. Let M be a hypercomplex manifold, I an induced complex structure, and $\eta \in \Lambda_I^{1,1}(M)$ a (1,1)-form. Then η is $SU(2)$-invariant if and only if $\mathcal{R}(\eta) = 0$. Moreover, for all functions ψ on M, we have

$$\mathcal{R}(\partial\overline{\partial}(\psi)) = \partial\partial_J(\psi).$$

Assume now that the manifold M is hypercomplex Hermitian. Consider the 3-dimensional space generated by the 2-forms ω_I, ω_J and ω_K. This is a weight 2 representationn of $SU(2)$. Moreover, that

$$\mathcal{R}(\omega_I) = \Omega, \tag{2.6}$$

where $\Omega = \frac{1}{2}(\omega_J + \sqrt{-1}\,\omega_K)$ is the canonical $(2,0)$-form.

3. The q-positive forms

Let M be a hypercomplex manifold, and $\Lambda_I^{p,q}(M)$ the bundle of (p,q)-forms on (M, I). Consider the map $J: \ \Lambda^*(M) \longrightarrow \Lambda^*(M)$,

$$J(dx_1 \wedge dx_2 \wedge ...) = J(dx_1) \wedge J(dx_2) \wedge ...$$

Clearly, on 2-forms we have $J^2 = 1$; more generally,

$$\left(J\Big|_{\Lambda^{\text{even}}(M)}\right)^2 = 1. \tag{3.1}$$

Since J and I anticommute, we have $J(\Lambda_I^{p,q}(M)) = \Lambda_I^{q,p}(M)$. By (3.1), the map $\eta \longrightarrow J(\overline{\eta})$ defines a real structure on $\Lambda_I^{2,0}(M)$.

Definition 3.1. Let $\eta \in \Lambda_I^{2,0}(M)$ be a (2,0)-form on a hypercomplex manifold M. We say that η is q-real if $\eta = J(\overline{\eta})$. We say that η is q-positive if η is q-real, and

$$\eta(v, J(\overline{v})) \geqslant 0 \tag{3.2}$$

for any $v \in T_I^{1,0}(M)$. We say that η is *strictly q-positive* if the inequality (3.2) is strict, for all $v \neq 0$.

The q-positive forms were introduced and studied at some length in [V4], under the name "K-positive forms". These forms were used to study the stability of certain coherent sheaves. Some properties of q-positive forms are remarkably close to that of the usual positive forms, studied in algebraic geometry in connection with Vanishing Theorems.

Proposition 3.2. *Let M be a hypercomplex manifold, and h a hypercomplex Hermitian metric. Consider the form*

$$\Omega := \omega_J + \sqrt{-1}\,\omega_K$$

(see (1.1)). Then Ω is strictly q-positive. Conversely, every strictly q-positive $(2,0)$-form is obtained from a unique hypercomplex Hermitian metric on M.

Proof. The form $\Omega := \omega_J + \sqrt{-1}\,\omega_K$ is q-positive as an elementary calculation insures. Indeed, write the orthonormal basis $\xi_1, \xi_2, ...\xi_{2n} \in \Lambda^{1,0}(M)$ in such a way that

$$J(\xi_{2i-1}) = \overline{\xi}_{2i}, \quad J(\xi_{2i}) = -\overline{\xi}_{2i-1}. \tag{3.3}$$

Then

$$\Omega = \xi_1 \wedge \xi_2 + \xi_3 \wedge \xi_4 + ... \tag{3.4}$$

This form is clearly q-real and strictly q-positive.

Conversely, let Ω be a q-real and strictly q-positive form on a hypercomplex manifold M. We can write Ω is coordinates as

$$\Omega = \alpha_1 \xi_1 \wedge \xi_2 + \alpha_3 \xi_3 \wedge \xi_4 + ...$$

where α_i are positive real numbers, and ξ_i satisfy (3.3).

Write a hypercomplex Hermitian form h as

$$\begin{aligned} h =& \alpha_1((\operatorname{Re}\xi_1)^2 + (\operatorname{Im}\xi_1)^2 + (\operatorname{Re}\xi_2)^2 + (\operatorname{Im}\xi_2)^2) \\ &+ \alpha_3((\operatorname{Re}\xi_3)^2 + (\operatorname{Im}\xi_3)^2 + (\operatorname{Re}\xi_4)^2 + (\operatorname{Im}\xi_4)^2) + ... \end{aligned} \tag{3.5}$$

Clearly, the corresponding canonical $(2,0)$-form is equal Ω.

The Hermitian metric (3.5) can be reconstructed from Ω directly as follows:

$$h(x,y) = \Omega(x^{1,0}, J(y^{0,1})),$$

for all $x, y \in T_{\mathbb{R}}M$, where $x^{1,0}$, $y^{0,1}$ denotes the $(1,0)$ and $(0,1)$-parts of x, y. We proved that the hypercomplex Hermitian structure is uniquely determined by the strictly q-positive form Ω. □

The following Corollary gives an interpretation of HKT-structures in terms of the canonical $(2,0)$-form.

Corollary 3.3. *Let M be a hypercomplex manifold, and $\Omega \in \Lambda^{2,0}(M)$ a ∂-closed strictly q-positive $(2,0)$-form. Then M is an HKT-manifold, and Ω is obtained as a canonical $(2,0)$-form of an HKT-metric h. Moreover, h is uniquely determined by Ω.*

Proof. By Proposition 3.2, $\Omega = \omega_J + \sqrt{-1}\Omega_K$, for some hypercomplex Hermitian metric h. Since $\partial\Omega = 0$, (M,h) is an HKT-manifold. □

4. Hyperholomorphic bundles

Hyperholomorphic bundles were introduced and studied at some length in [V1]. Let B be a holomorphic vector bundle over a complex manifold X, ∇ a connection in B and $\Theta \in \Lambda^2 \otimes \operatorname{End}(B)$ be its curvature. This connection is called *compatible with the holomorphic structure* if $\nabla_\gamma(\zeta) = 0$ for any holomorphic section ζ and any antiholomorphic tangent vector field $\gamma \in T^{0,1}(X)$. If there exists a holomorphic structure compatible with the given Hermitian connection then this connection is called *integrable.*

Theorem 4.1. *Let ∇ be a Hermitian connection in a complex vector bundle B over a complex manifold X. Then ∇ is integrable if and only if $\Theta \in \Lambda^{1,1}$ $(X, \operatorname{End}(B))$, where $\Lambda^{1,1}(X, \operatorname{End}(B))$ denotes the forms of Hodge type (1,1). Also, the holomorphic structure compatible with ∇ is unique.*

Proof. This is Proposition 4.17 of [Kob], Chapter I. □

This proposition is a version of Newlander-Nirenberg theorem. For vector bundles, it was proven by M. Atiyah and R. Bott.

Definition 4.2. Let B be a Hermitian vector bundle with a connection ∇ over a hypercomplex manifold M. Then ∇ is called *hyperholomorphic* if ∇ is integrable with respect to each of the complex structures induced by the hypercomplex structure.

As follows from Theorem 4.1, ∇ is hyperholomorphic if and only if its curvature Θ is of Hodge type (1,1) with respect to any of the complex structures induced by a hypercomplex structure.

An easy calculation shows that ∇ is hyperholomorphic if and only if Θ is an $SU(2)$-invariant differential form.

Hyperholomorphic bundles are quite ubiquitous. Clearly, the tangent bundle to a hyperkähler manifold and all its tensor powers are hyperholomorphic. There are many other examples

Example 4.3. Let M be a compact hyperkähler manifold, B a holomorphic bundle. Then B admits a unique hyperholomorphic connection, if B is stable and the cohomology classes $c_1(B)$ and $c_2(B)$ are $SU(2)$-invariant. Moreover, if M is generic in its deformation class, then all stable bundles admit a hyperholomorphic connection.

5. $\mathbb{H}$-hyperholomorphic bundles

Definition 5.1. Let M be a hypercomplex manifold, and (B, ∇) a hyperholomorphic bundle on M, $\dim_{\mathbb{C}} B = 2n$. The bundle B is called $\mathbb{H}$-*hyperholomorphic* if ∇ preserves a $\mathbb{C}$-linear symplectic structure on B. In other words, B is $\mathbb{H}$-*hyperholomorphic* if the holonomy of ∇ is contained in $Sp(n)$.

The following examples are obvious.

Example 5.2. Let F be a hyperholomorphic bundle on M. Then $F \oplus F^*$ is $\mathbb{H}$-hyperholomorphic.

Example 5.3. Consider the tangent bundle TM on M. Assume that M is hyperkähler. Then TM is $\mathbb{H}$-hyperholomorphic.

The main property of $\mathbb{H}$-hyperholomorphic bundles is the following.

Claim 5.4. Let M be a hypercomplex manifold, and B an $\mathbb{H}$-hyperholomorphic bundle. Denote by $\operatorname{Tot} B$ the total space of B. Then $\operatorname{Tot} B$ is equipped with a natural hypercomplex structure. In particular, the total space of TM is hypercomplex.

Proof. Since the holonomy of B is contained in $Sp(n)$, there is a natural parallel action of $\mathbb{H}$ on B. Given a quaternion $L \in \mathbb{H}$, $L^2 = -1$, consider B as a complex vector bundle with the complex structure defined by L. Denote this complex vector bundle as (B, L). Since the curvature of B is $SU(2)$-invariant, the bundle (B, L) is hyperholomorphic. Consider (B, L) as a holomorphic vector bundle on (M, L). Denote the corresponding complex structure on $\operatorname{Tot} B$ by L. We obtained an integrable complex structure on $\operatorname{Tot} B$ for each quaternion $L \in \mathbb{H}$, $L^2 = -1$. It is easy to check that these complex structures satisfy quaternionic relations, inducing a hypercomplex structure on $\operatorname{Tot} B$. □

6. The Obata connection on $\operatorname{Tot} B$.

Let M be a hyperkähler manifold, and B an $\mathbb{H}$-hyperholomorphic bundle. By Claim 5.4, the total space $\operatorname{Tot} B$ is hypercomplex. One can ask whether this hypercomplex structure is hyperkähler. The answer is - never (unless B is flat).

Given a hypercomplex manifold, one can easily establish whether M admits a hyperkähler structure. This is done most easily using the so-called Obata connection.

Theorem 6.1 (Obata)**.** *Let M be a hypercomplex manifold. Then M admits a unique torsion-free connection which preserves the hypercomplex structure.*[1]

Proof. Well known (see [Ob]). □

If M is hyperkähler, then the Levi-Civita connection preserves the hypercomplex structure. In this case, the Levi-Civita connection coincides with the Obata connection.

To determine whether a hypercomplex manifold M admits a hyperkähler structure, one needs to compute the holonomy of the Obata connection. The manifold is hyperkähler if and only if the holonomy Hol preserves a metric; that is, M is hyperkähler if and only if Hol is contained in $Sp(n)$.

Proposition 6.2. *Let M be a hyperkähler manifold, B an $\mathbb{H}$-hyperholomorphic bundle, and* $\operatorname{Tot} B$ *its total space considered as a hypercomplex manifold (see*

[1]This connection is called the Obata connection.

Claim 5.4). Assume that the curvature of B is non-zero. Then $\operatorname{Tot} B$ *does not admit a hyperkähler structure.*

Proof. One could compute the holonomy group of the Obata connection of $\operatorname{Tot} B$, and show that it is non-compact. To avoid excessive computations, we use a less straightforward argument.

Suppose that $\operatorname{Tot} B$ is hyperkähler. Given $m \in M$, let $B_m \subset \operatorname{Tot} B$ be the fiber of B in m. By construction, B_m is a hypercomplex submanifold in $\operatorname{Tot} B$. Such submanifolds are called *trianalytic* (see [V2], [V3] for a study of trianalytic cycles on hyperkähler manifolds). In [V3], it was shown that trianalytic submanifolds are completely geodesic. In other words, for any trianalytic submanifold $Z \subset X$, the Levi-Civita connection on $TX\big|_Z$ preserves the orthogonal decomposition

$$TX\big|_Z = TZ \oplus TZ^{\perp} \tag{6.1}$$

If we have a hypercomplex fibration $X \longrightarrow Y$, the decomposition (6.1) gives a connection for this fibration. In [V3] it was shown that this connection is flat, for any hyperkähler fibration.

We obtain a flat connection ∇ in the fibration $\operatorname{Tot} B \longrightarrow M$. This connection is clearly compatible with the additive structure on the bundle B. Therefore, ∇ is an affine connection on B. By construction, ∇ is compatible with the hypercomplex structure on $\operatorname{Tot} B$. Therefore, ∇ coincides with the hyperholomorphic connection on B. We proved that B is flat. □

7. HKT-structure on $\operatorname{Tot} B$.

Let M be a smooth manifold. Given a bundle with connection on M, we have a decomposition

$$T \operatorname{Tot} B = T_{\mathrm{ver}} \oplus T_{\mathrm{hor}} \tag{7.1}$$

of the tangent space to $\operatorname{Tot} B$ into horizontal and vertical components. Clearly, the bundle T_{ver} is identified with $\pi^* B$, and T_{hor} with $\pi^* TM$, where $\pi : \operatorname{Tot} B \longrightarrow M$ is the standard projection.

Assume now that M is a Riemannian manifold, and B a vector bundle, equipped with a Euclidean metric. Then $\operatorname{Tot} B$ is equipped with a Riemannian metric g defined by the following conditions.

(i) The decomposition $T \operatorname{Tot} B = T_{\mathrm{ver}} \oplus T_{\mathrm{hor}}$ is orthogonal with respect to g.

(ii) Under the natural identification $T_{\mathrm{ver}} \cong \pi^* B$, the metric g restricted to T_{ver} becomes the metric on B.

(iii) The metric g restricted $T_{\mathrm{hor}} \cong \pi^* TM$ is equal to the metric induced on $\pi^* TM$ from the Riemannian structure on M.

Definition 7.1. In the above assumptions, the metric g is called *the natural metric on* $\operatorname{Tot} B$ *induced by the connection and the metrics on M and B.*

Notice that the metric g depends from the metrics on B and M and from the connection in B. Different connections induce different metrics on $\operatorname{Tot} B$.

Theorem 7.2. *Let M be an HKT-manifold, and B an $\mathbb{H}$-hyperholomorphic vector bundle on M. Consider the metric g on $\operatorname{Tot} B$ defined as in Definition 7.1. Then g is an HKT-metric.*

Proof. Consider the decomposition $g = \pi^* g_M + \pi^* g_B$ of the metric g onto the horizontal and vertical components. Since the decomposition $T \operatorname{Tot} B = T_{\text{ver}} \oplus T_{\text{hor}}$ is compatible with the hypercomplex structure, the 2-forms $g_{\text{hor}} := \pi^* g_M$ and $g_{\text{ver}} := \pi^* g_B$ are $SU(2)$-invariant. Consider the corresponding (2,0)-forms Ω_{hor} and Ω_{ver} obtained as in (1.1);

$$\Omega_{\text{hor}} = \omega_{J\,\text{hor}} + \sqrt{-1}\,\omega_{K\,\text{hor}}$$

where $\omega_{J\,\text{hor}} = g_{\text{hor}}(J\cdot,\cdot)$, $\omega_{K\,\text{hor}} = g_{\text{hor}}(K\cdot,\cdot)$ are differential forms associated with g_{hor} and J, K as in (1.1).

Then Ω_{hor} and Ω_{ver} are horizontal and vertical components of the standard (2,0)-form of $\operatorname{Tot} B$:

$$\Omega = \Omega_{\text{hor}} + \Omega_{\text{ver}} \tag{7.2}$$

The HKT condition can be written as $\partial\Omega = 0$ (1.1). Let Ω_M be the standard $(2,0)$-form of M. Since M is an HKT manifold, (1.1) holds on M and the form Ω_{hor} satisfies

$$\partial\Omega_{\text{hor}} = \partial\pi^*\Omega_M = 0.$$

To prove Theorem 7.2, it remains to show

$$\partial\Omega_{\text{ver}} = 0 \tag{7.3}$$

Consider a function

$$\Psi:\ \operatorname{Tot} B \longrightarrow \mathbb{R}, \quad \Psi(v) = |v|^2, \tag{7.4}$$

mapping a vector $v \in TM$ to the square of its norm. Let

$$0 \longrightarrow \Omega^{1,0} \xrightarrow{\partial,\partial_J} \Omega^{2,0} \xrightarrow{\partial,\partial_J} \Omega^{3,0} \xrightarrow{\partial,\partial_J} \ldots$$

be the bicomplex defined in (2.4). To prove (7.3), and hence Theorem 7.2, it suffices to prove

$$\partial\partial_J\Psi = \Omega_{\text{ver}}. \tag{7.5}$$

By Claim 2.3, we have

$$\partial\partial_J\Psi = \mathcal{R}(\partial\overline{\partial}\Psi),$$

where $R:\ \Lambda^{1,1}(\operatorname{Tot} B) \longrightarrow \Lambda^{2,0}(\operatorname{Tot} B)$ is the operator

$$\mathcal{R} = \frac{\mathcal{J} - \sqrt{-1}\,\mathcal{K}}{2}$$

(see (2.5)). However, the 2-form $\partial\overline{\partial}\Psi$ is quite easy to compute. From [Bes], (15.19), we obtain:

$$\partial\overline{\partial}\Psi = \omega_{\text{ver}} + \xi, \tag{7.6}$$

where $\omega_{\mathrm{ver}} = g_{\mathrm{ver}}(\cdot, I\cdot)$ is the Hermitian form of g_{ver}, and ξ is defined as following. Using the decomposition (7.1), we consider $\Lambda^2 T_{\mathrm{hor}}$ as a subbundle in $\Lambda^2 \operatorname{Tot} B$. Then $\xi \in \Lambda^2 T_{\mathrm{hor}} \subset \Lambda^2 \operatorname{Tot} B$ is a 2-form on T_{hor} mapping a pair of vectors (x, y)

$$x, y \in T_{\mathrm{hor}}\Big|_{(m,b)} \subset T_{(m,b)} \operatorname{Tot} B,$$
$$T_{\mathrm{hor}}\Big|_{(m,b)} = T_m M,$$
$$(m, b) \in \operatorname{Tot} B, m \in M, b \in B\Big|_m$$

to $(R(x, y, b)\overline{b})$, where $R \in \Lambda^2 M \otimes \operatorname{End} B$ is the curvature of B. The form ξ is $SU(2)$-invariant because the curvature of B is $SU(2)$-invariant. Therefore, $\mathcal{R}(\xi) = 0$ (Claim 2.3), and

$$\partial\partial_J \Psi = \mathcal{R}(\partial\overline{\partial}\Psi) = \mathcal{R}(\omega_{\mathrm{ver}}) = \Omega_{\mathrm{ver}} \tag{7.7}$$

(the last equation holds by (2.6)). This proves (7.5). Theorem 7.2 is proven. □

8. New examples of compact HKT-manifolds

Let M be a compact HKT-manifold, e.g. a hyperkähler manifold, and B an $\mathbb{H}$-hyperholomorphic vector bundle on M (for examples of $\mathbb{H}$-hyperholomorphic vector bundles see Examples 5.2 and 5.3). Denote by $\operatorname{Tot}^\circ B$ be the space of non-zero vectors in B. Fix a real number $q > 1$. Consider the map

$$\rho_q : \operatorname{Tot}^\circ B \longrightarrow \operatorname{Tot}^\circ B, \quad \rho_q(b) = qb, \quad b \in \operatorname{Tot}^\circ B,$$

and let $\mathcal{M} = \operatorname{Tot}^\circ B/\rho_q$ be the corresponding quotient space. Since the map $b \longrightarrow qb$ is compatible with the hypercomplex structure, the space $\mathcal{M}$ is hypercomplex. It is fibered over a compact manifold M, with fibers Hopf manifolds which are homeomorphic to $S^1 \times S^{2m-1}$, $m = \dim_{\mathbb{R}} B$, hence it is compact.

Theorem 8.1. *In the above assumptions, $\mathcal{M}$ admits a natural HKT-structure.*

Proof. By Corollary 3.3, we need to construct a q-positive ∂-closed $(2, 0)$-form on $\mathcal{M}$. Let $\widetilde{\Omega}$ be a (2,0)-form on $\operatorname{Tot}^\circ B$,

$$\widetilde{\Omega} = \pi^*\Omega_M + \partial\partial_J \log\Psi,$$

where $\pi^*\Omega_M$ is the canonical (2,0)-form on M lifted to $\operatorname{Tot}^\circ B$, and $\Psi : \operatorname{Tot} B \longrightarrow \mathbb{R}$ the square norm function (7.4). The map $v \xrightarrow{\rho_q} qv$ satisfies $\rho_q^* \log\Psi = \log\Psi + \log q^2$, and therefore

$$\rho_q^*\partial\partial_J \log\Psi = \partial\partial_J \log\Psi.$$

This implies that $\widetilde{\Omega} = \pi^*\Omega_M + \partial\partial_J \log\Psi$ is ρ_q-invariant, hence defines a form Ω on $\mathcal{M} = \operatorname{Tot}^\circ B/\rho_q$.

By construction, the form Ω is ∂-closed. To prove Theorem 8.1, it remains to show that $\widetilde{\Omega}$ is strictly q-positive. We use the same argument as used to show that a locally conformal hyperkähler manifold is HKT.

We have

$$\partial\partial_J \log\Psi = \frac{\partial\partial_J\Psi}{\Psi} - \frac{\partial\Psi\wedge\partial_J\Psi}{\Psi^2}. \tag{8.1}$$

In all directions orthogonal to $\partial\Psi, \partial_J\Psi$, the form $\partial\partial_J\log\Psi$ is proportional to $\partial\partial_J\Psi$, hence q-positive by (7.7). Moreover, (8.1) implies that

$$\widetilde{\Omega} = \Omega_{\mathrm{hor}} + \frac{\Omega_{\mathrm{ver}}}{\Psi} - \frac{\partial\Psi\wedge\partial_J\Psi}{\Psi^2},$$

(we use the notation introduced in Section 7). The form $\Omega_{\mathrm{hor}} + \frac{\Omega_{\mathrm{ver}}}{\Psi}$ is strictly q-positive (Theorem 7.2). The vertical and the horizontal tangent vectors are orthogonal with respect to $\widetilde{\Omega}$. Since $\frac{\partial\Psi\wedge\partial_J\Psi}{\Psi^2}$ vanishes on all horizontal tangent vectors, it remains to prove that $\widetilde{\Omega}(x, J\overline{x}) > 0$, where x is vertical.

Let $\xi \in T^{1,0}(\mathrm{Tot}^\circ B)$ be the vertical tangent vector to $\mathrm{Tot}^\circ B$ which is dual to $\frac{d\Psi}{\sqrt{\Psi}}$. Clearly, $\partial\Psi$ is the $(1,0)$-part of ξ. For all $x \in T^{1,0}(\mathrm{Tot}^\circ B)$, we have

$$\frac{\partial\Psi\wedge\partial_J\Psi}{\Psi^2}\Big(x, J(\overline{x})\Big) = (\xi, x)_H^2,$$

where $(\cdot,\cdot)_H$ denotes the Riemannian form. Similarly,

$$\Omega_{\mathrm{ver}}(x, J(\overline{x})) = 2(x,x)_H$$

(this can be checked by writing Ω_{ver} is coordinates as in (3.4)). Using Cauchy inequality and $|\xi| = 1$, we obtain $(x,x)_H \geqslant (\xi,x)_H^2$. Then

$$\begin{aligned}\widetilde{\Omega}(x, J(\overline{x})) &= \frac{\Omega_{\mathrm{ver}}}{\Psi}\Big(x, J(\overline{x})\Big) - \frac{\partial\Psi\wedge\partial_J\Psi}{\Psi^2}\Big(x, J(\overline{x})\Big)\\ &= 2\frac{(x,x)_H}{\Psi} - \frac{(\xi,x)_H^2}{\Psi} \geqslant \frac{(x,x)_H}{\Psi} > 0\end{aligned}$$

for all vertical tangent vectors $x \neq 0$. This proves Theorem 8.1. □

Acknowledgements

I am grateful to G. Grantcharov, D. Kaledin and Y. S. Poon for interesting discussions.

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