FINITE, CONNECTED, SEMISIMPLE, RIGID TENSOR CATEGORIES ARE LINEAR

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ABSTRACT. Fusion categories are fundamental objects in quantum algebra, but their definition is narrow in some respects. By definition a fusion category must be k-linear for some field k, and every simple object V is strongly simple, meaning that $\text{End}(V) = k$. We prove that linearity follows automatically from semisimplicity: Every connected, finite, semisimple, rigid, monoidal category $\mathbb{C}$ is k-linear and finite-dimensional for some field k. Barring inseparable extensions, such a category becomes a multifusion category after passing to an algebraic extension of k.

The proof depends on a result in Galois theory of independent interest, namely a finiteness theorem for abstract composita.

1. Introduction

We take as prerequisites to this article the first two chapters of a survey of Bakalov and Kirillov [1] and the introductions to articles by Etingof, Nikshych, and Ostrik [2] and Müger [4]. Following these three works, a *fusion category* is a k-linear, finite, strongly semisimple rigid tensor category. (Precise definitions of these terms are given below.) The previous works also present the structure theory and applications of fusion categories. But although fusion categories are an important a fairly general class of objects, their definition is narrow in some respects.

If $\mathcal{C}$ is any abelian, k-linear category for some field k, we say that an object $V \in \mathcal{C}$ is *strongly simple* if $\text{End}(V) = k$. For example, if k is algebraically closed, then Schur's Lemma says that every simple object is strongly simple. If $\mathcal{C}$ is semisimple and every simple object is strongly simple, then we say that $\mathcal{C}$ is *strongly semisimple*.

A fusion category $\mathcal{C}$ is endowed with a field k over which it must be linear and strongly semisimple. It is also assumed that the identity object I is simple. But as noted previously [2], it is reasonable to drop the condition that I is simple. In this case $\mathcal{C}$ is a *multifusion category*, or a folded form of a fusion 2-category in which each identity 1-morphism is simple. (To "fold" an abelian 2-category means to combine finitely many objects into one object whose 1-identity is non-simple; see Section 2.)

The aim of this article is to show that if $\mathcal{C}$ is semisimple and suitably finite, then suitably finite linearity appears automatically. We adopt the natural generalization to 2-categories in the statement of the main result.

Theorem 1.1. *Let $\mathcal{C}$ be a connected, semisimple, rigid 2-category with finitely many types of simple 1-morphisms. Then there exists a field k over which it is linear and all Hom spaces between 1-morphisms are finite-dimensional. Taking k_s to be the separable closure of k, $k_s \otimes \mathcal{C}$ is a folded form of a semisimple 2-category over k_s with strongly simple 1-identities. If $\mathcal{C}$ has no inseparable extensions of k, then $k_s \otimes \mathcal{C}$ is strongly semisimple.*

To conclude this introduction we offer two related examples, one in which Theorem 1.1 applies, and one in which it does not.

First, consider the category $\mathcal{A} = \mathbb{C}\text{-mod}_{\mathbb{R}}\text{-}\mathbb{C}$ of finite-dimensional real vector spaces with the extra structure of bimodules over the complex numbers. The category $\mathcal{A}$ is semisimple, and it is monoidal with respect to tensoring in the middle. It has two simple objects I and A, both of which have real dimension 2. But in I, left and right complex multiplication agree, while in A, they are conjugate. The reader can check that $A \otimes A \cong I$, from which it follows that $\mathcal{A}$ is rigid, and that $\mathrm{End}(I) = \mathrm{End}(A) = \mathbb{C}$. Nonetheless, $\mathcal{A}$ is not $\mathbb{C}$-linear. It is $\mathbb{R}$-linear, while $\mathbb{C} \otimes_{\mathbb{R}} \mathcal{A}$ is a multifusion category which unfolds to a 2-category $\widetilde{\mathbb{C} \otimes_{\mathbb{R}} \mathcal{A}}$ with two objects. Each endocategory of $\widetilde{\mathbb{C} \otimes_{\mathbb{R}} \mathcal{A}}$ is the category of complex representations of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$.

Second, let B be the field $\mathbb{C}(y)$ with a $\mathbb{C}(x)$-bimodule structure defined as follows: Left multiplication by x is defined as multiplication by y, while right multiplication is defined as multiplication by y^2. Let B^* be B with left and right switched, and let $\mathcal{B}$ be the abelian monoidal category of bimodules over $\mathbb{C}(x)$ generated by B and B^*. It is not hard to show that $\mathcal{B}$ is semisimple and that the simple objects are $I = \mathbb{C}(x)$, $B^{\otimes n}$, and $(B^*)^{\otimes n}$. In this case, the largest field over which $\mathcal{B}$ is linear is $\mathbb{C}$, which is already algebraically closed. No change of base field of $\mathcal{B}$ renders it strongly semisimple.

2. Semisimple categories

We assume various relevant definitions from Mac Lane [3] and Müger [4]: additive, abelian, monoidal, k-linear, etc. An object in an abelian category is *simple* if it has no subobjects. An abelian category is *semisimple* if every object is a direct sum of finitely many simple objects. As mentioned in the introduction, if V is a simple object in an abelian, k-linear category, then it is *strongly simple* if $\mathrm{End}(V) = k$. A semisimple, k-linear category is *strongly semisimple* if all simple objects are strongly simple.

A (strict) monoidal category $\mathcal{C}$ can be reinterpreted as a 2-category $\mathcal{C}'$ with one object. This phenomenon is known as *dimension shifting*, because n-morphisms in $\mathcal{C}$ become $(n + 1)$-morphisms in $\mathcal{C}'$. In light of this relationship, we will use $V \otimes W$ to denote the composition of 1-morphisms $V \in \mathrm{Hom}_1(A, B)$ and $W \in \mathrm{Hom}_1(B, C)$, and $f \otimes g$ for the attendant "horizontal" composition of

2-morphisms. We use fg or $f \circ g$ to denote "vertical" composition of two 2-morphisms f and g in the same category $\mathrm{Hom}_1(A, B)$. A 2-category is also additive, abelian, k-linear, semisimple, or strongly semisimple if each category $\mathrm{Hom}_1(A, B)$ has the same property. If additivity or linearity is part of the structure, we also assume that $f \otimes g$ is biadditive or bilinear in f and g.

Remark. It is common to assume, at least intuitively, that categories are *skeletal* (there is only one object of each isomorphism type), because every category can be made skeletal. It is also common to assume that monoidal categories are *strict* ($\otimes$ is strictly associative), because every monoidal category can be made strict. But most monoidal categories cannot be made simultaneously strict and skeletal! This is the origin of the non-trivial structure of associators. On balance, we prefer strictness and we will not assume that categories are skeletal. However, we can assume that 2-categories are skeletal at the level of objects.

Let $\mathcal{C}$ be a semisimple monoidal category and suppose that the identity object I is not simple. Then

$$I = \bigoplus_{A \in S} I_A,$$

where S is some indexing set and each I_A is simple. As previously noted [2], I is necessarily multiplicity-free, *i.e.*, $I_A \not\cong I_B$ when $A \neq B$, and $\mathcal{C}$ can be reorganized as a 2-category $\widetilde{\mathcal{C}}$ whose objects are the elements of S. In $\widetilde{\mathcal{C}}$, the identity of A is I_A, and the Hom space $\mathrm{Hom}_1(A, B)$ consists of those objects V of $\mathcal{C}$ such that

$$I_A \otimes V \otimes I_B = V \in \mathcal{C}.$$

(Note that $I_A \otimes V \otimes I_B$ is always a subobject of V.) Thus $\widetilde{\mathcal{C}}$ has simple 1-identities. We call it the *unfolded* form of $\mathcal{C}$.

The same construction applies if $\mathcal{C}$ is a semisimple 2-category such that not all 1-identities are simple. If each 1-identity I_A decomposes as

$$I_A = \bigoplus_{B \in S_A} I_B,$$

then $S = \bigcup_A S_A$ is the set of objects of the unfolded category $\widetilde{\mathcal{C}}$.

Lemma 2.1. *If $V \in \mathrm{Hom}_1(A, B)$ is a simple 1-morphism in an abelian 2-category $\mathcal{C}$, then $\mathrm{End}_2(V)$ is a division ring.*

Proof. The lemma is a form of Schur's Lemma. Suppose that V is simple and that $f \in \mathrm{End}_2(V)$. Then both $\ker f$ and $\mathrm{im}\, f$ are either 0 or V. If $\ker f = V$ or if $\mathrm{im}\, f = 0$, then $f = 0$. Otherwise, if $\ker f = 0$ and $\mathrm{im}\, f = V$, then f has an inverse on each side and therefore a two-sided inverse. Thus every non-zero f has a reciprocal and $\mathrm{End}_2(V)$ is a division ring. $\square$

Lemma 2.2. *The division ring $\mathrm{End}_2(I_A)$ is a field. If $V \in \mathrm{Hom}_1(A, B)$ is any 1-morphism, then $\mathrm{End}_2(I_A)$ and $\mathrm{End}_2(I_B)$ embed in the center $Z(\mathrm{End}_2(V))$.*

Proof. The identity $V = I_A \otimes V$ induces a unital ring homomorphism $\mathrm{End}_2(I_A) \to \mathrm{End}_2(V)$, which must be an inclusion since the domain is a division ring. To show that $\mathrm{End}_2(I_A)$ lies in the center of $\mathrm{End}_2(V)$, let $f \in \mathrm{End}_2(I_A)$ and $g \in \mathrm{End}_2(V)$. Then

$$fg = (f \otimes 1_V)(1_{I_A} \otimes g) = (1_{I_A} \otimes g)(f \otimes 1_V) = gf.$$

In particular, if $V = I_A$, then $\mathrm{End}_2(I_A)$ lies in the center of itself, so it is a field. $\qquad\square$

Remark. If A is an object in any 2-category $\mathcal{C}$, then $\mathrm{End}_2(I_A)$ is commutative. This fact is familiar in topology as the commutativity of the second homotopy group $\pi_2(X)$ of a topological space X. Our proof is the usual one.

Note that $\mathrm{End}_2(V)$ may not be commutative even if V is simple. For example, if V is the defining representation of $\mathfrak{sl}(2, \mathbb{C})$ viewed as a 4-dimensional real representation, then $\mathrm{End}_2(V) = \mathbb{H}$, the quaternions.

In light of Lemma 2.2, let $k_A = \mathrm{End}_2(I_A)$, let k_V be the compositum of the inclusions k_A and k_B in $Z(\mathrm{End}(V))$, and denote the restrictions of these inclusions to k_V by

$$k_A \xrightarrow{\alpha_{A,V}} k_V \xleftarrow{\alpha_{V,B}} k_B$$

Thus k_V is an *abstract compositum* of the fields k_A and k_B.

Now let $\mathcal{C}$ be a semisimple 2-category with simple 1-identities. Then the fields k_A are entirely unrelated on different connected components of $\mathcal{C}$. Even when $\mathcal{C}$ is connected, the fields k_A may differ for different $A \in \mathcal{C}$, although they must have the same characteristic since they are connected by abstract composita.

If $A, B \in \mathcal{C}$ are objects and $V, W \in \mathrm{Hom}_1(A, B)$ are two 1-morphisms connecting them, then $\mathrm{Hom}_2(V, W)$ is a bimodule over the two fields k_A and k_B. If $k_A \not\cong k_B$, then the left and right module structures certainly differ. But even if $\mathcal{C}$ only has one object A, so that it is a dimension-shifted monoidal category, the left and right k_A-module structures may differ; an example was given in Section 1.

Example. Let $\mathcal{A}$ be a 2-category with two objects, $\mathbb{R}$ and $\mathbb{C}$, and define its Hom categories by

$$\mathrm{Hom}_1(k_1, k_2) = k_1\text{-mod}_{\mathbb{R}}\text{-}k_2$$

for every $k_1, k_2 \in \{\mathbb{R}, \mathbb{C}\}$. Then $\mathrm{End}_2(I_k) = k$, so the endomorphism fields of the 1-identities differ.

3. Rigidity

A monoidal category is *rigid* if every object V has both a left dual $^*V$ and a right dual V^* together with morphisms

$$a_V : I \to V \otimes V^* \qquad\qquad b_V : V^* \otimes V \to I$$
$$c_V : I \to {}^*V \otimes V \qquad\qquad d_V : V \otimes {}^*V \to I$$

that satisfy the compatibility conditions

$$(1_V \otimes b_V)(a_V \otimes 1_V) = 1_V$$
$$(d_V \otimes 1_V)(1_V \otimes c_V) = 1_V.$$

these definitions generalize readily to 2-categories. To be explicit, if $V \in \mathrm{Hom}_1(A, B)$, then $V^*, {}^*V \in \mathrm{Hom}_1(B, A)$, and

$$a_V : I_A \to V \otimes V^* \qquad\qquad b_V : V^* \otimes V \to I_B$$
$$c_V : I_B \to {}^*V \otimes V \qquad\qquad d_V : V \otimes {}^*V \to I_A,$$

and the compatibility conditions are the same. As explained by Müger [4], duals can also be called *adjoints*, with the significant consequence that $V \mapsto V^*$ and $V \mapsto {}^*V$ are contravariant endofunctors of the 2-category $\mathcal{C}$. In fact they can be made inverse to each other, so that $V = {}^*V^*$.

Note also that if $\mathcal{C}$ is semisimple, then $V \cong V^{**}$ [2], although unless $\mathcal{C}$ is *pivotal*, these isomorphisms are not functorial. We will not need pivotal structure in this article.

Henceforth let $\mathcal{C}$ be a semisimple, rigid 2-category with simple 1-identities.

Lemma 3.1. *If $V \in \mathrm{Hom}_1(A, B)$, then there is an isomorphism $\sigma_V : k_V \to k_{V^*}$ that makes the following diagram commute:*

Proof. Since $V \mapsto V^*$ extends to an anti-automorphism of $\mathcal{C}$, it produces σ_V. It is only necessary to check that the anti-automorphism is the identity on $k_A = \mathrm{End}_2(I_A)$. $\square$

Lemma 3.2. *If $V \in \mathrm{Hom}_1(A, B)$ is a 1-morphism in $\mathcal{C}$, then $\mathrm{End}_2(V)$ is finite-dimensional as a left k_A-module and as a right k_B-module.*

Proof. If $W \in \mathrm{Hom}_1(A, A)$, then it has an *invariant space* defined as

$$\mathrm{Inv}(W) = \mathrm{Hom}_2(I_A, W).$$

Since $\mathcal{C}$ is semisimple, $\mathrm{Inv}(W)$ is finite-dimensional as a vector space over k_A. If $W = V \otimes {}^*V$, then

$$\mathrm{Inv}(V \otimes {}^*V) \cong k_A^n$$

with $n > 0$ by the existence of d_V. Let κ_V be the composition of the maps

$$\mathrm{End}_2(V) \to \mathrm{End}_2(V \otimes {}^*V) \to \mathrm{End}(\mathrm{Inv}(V \otimes {}^*V)) \cong \mathrm{Mat}_n(k_A),$$

where the first term is given by $f \mapsto f \otimes 1_{*V}$ and the second by isotypic decomposition of $V \otimes {}^*V$. The map κ_V is both a unital ring homomorphism and morphism of k_A-linear spaces. Since the domain of α is a division ring and $1 \neq 0$ in the target, κ_V is injective. Since the target is a finite-dimensional vector space over k_A, the domain $\mathrm{End}_2(V)$ is finite-dimensional as well.

The same proof works on the other side, replacing κ_V with

$$\mathrm{End}_2(V) \to \mathrm{End}_2(V^* \otimes V) \to \mathrm{End}(\mathrm{Inv}(V^* \otimes V)) \cong \mathrm{Mat}_n(k_A),$$

which for later use we call λ_V.   □

Remark. In fact, $\mathrm{Inv}(V \otimes {}^*V)$ and $\mathrm{End}_2(V)$ are isomorphic as k_A-vector spaces.

Combining Lemmas 2.1 and 3.2, the abstract compositum k_V associated to a simple 1-morphism $V \in \mathrm{Hom}_1(A, B)$ is bifinite: a finite extension of both k_A and k_B.

Remark. Theorem 1.1 postulates a common finite-index subfield of all k_A and all k_V on which every $\alpha_{A,V}$ is the identity. Given that there are only finitely many simple V up to isomorphism, the fact that k_V is a bifinite compositum of k_A and k_B suggests looking at their intersection in k_V. Unfortunately, if k_1 and k_2 are two finite-index subfields of a field k_3, it does not follow that $k_1 \cap k_3$ is finite index in k_3. For example, let

$$k_3 = \mathbb{C}(q) \qquad k_1 = \mathbb{C}(q^2) \qquad k_2 = \mathbb{C}((q-1)^2).$$

The reader can check that $k_1 \cap k_2 = \mathbb{C}$. So it is not enough to know that bifinite abstract composita connect every pair k_A and k_B.

Lemma 3.3. *Let $V \in \mathrm{Hom}_1(A, B)$ and $W \in \mathrm{Hom}_1(B, C)$. Then the map*

$$\sigma_{V,W} : \mathrm{End}_2(V) \otimes_{k_B} \mathrm{End}_2(W) \to \mathrm{End}_2(V \otimes W)$$

is injective.

Proof. Consider the commutative diagram

Since the maps κ_W and λ_V from Lemma 3.2 are both injective, the left arrow is injective. The bottom arrow is trivially injective. Therefore the top arrow, $\sigma_{V,W}$, is also injective. $\square$

Lemma 3.4. *If $V \in \mathrm{Hom}_1(A,B)$, then the abstract compositum k_V is a separable extension of both k_A and k_B.*

Proof. By abuse of notation, we omit the embeddings $\alpha_{A,V}$ and $\alpha_{V,B}$. (This already arises in the statement of the lemma.) Applying Lemma 3.3 to $V \otimes {}^*V$, we know that $k_V \otimes_{k_B} k_V$ embeds in $\mathrm{End}_2(V \otimes {}^*V)$. Moreover, the subalgebra R generated by both copies of k_A in $k_V \otimes_{k_B} k_V$ lies in the center of $\mathrm{End}_2(V \otimes {}^*V)$, since the latter is an algebra over k_A on both the left and the right. It suffices to show that R has nilpotent elements when k_V is inseparable over k_B, because this would violate the semisimplicity of $V \otimes {}^*V$.

Let $p \neq 0$ be the common characteristic of k_A, k_B, and k_V. Suppose as a special case that k_V is a non-trivial purely inseparable extension of k_B with exponent e. Since k_V is the compositum of k_A and k_B, k_A contains an element x which is not in k_B. In this case

$$x \otimes 1 - 1 \otimes x \neq 0 \in k_V \otimes_{k_B} k_V,$$

while

$$(x \otimes 1 - 1 \otimes x)^{p^e} = 0.$$

Thus $x \otimes 1 - 1 \otimes x$ is the desired nilpotent element in R.

In this general case, k_B has a maximal separable extension s_V in k_V. Then $k_V \otimes_{k_B} k_V$ surjects onto $k_V \otimes_{s_V} k_V$. Replacing k_B by s_V in the previous paragraph, the image R' of R in $k_V \otimes_{s_V} k_V$ has a nilpotent element. At the same time, $k_V \otimes_{k_B} k_V$ is a finite-dimensional algebra with respect to its left k_A structure;

$$\dim_{k_A} k_V \otimes_{k_B} k_V = (\dim_{k_A} k_V)(\dim_{k_B} k_V) < \infty$$

by Lemma 3.2. Thus R is also a finite-dimensional (commutative) algebra over k_A. Since its quotient R' has a nilpotent element, R must have a nilpotent element as well. $\square$

Lemma 3.5. *Let $V \in \mathrm{Hom}_1(A,B)$ and $W \in \mathrm{Hom}_1(B,C)$. Then any compositum of k_A and k_C which occurs as a subring of $k_V \otimes_{k_B} k_W$ is k_X for some summand $X \subseteq V \otimes W$.*

Proof. Since the extensions k_V and k_W are separable over k_B, $k_V \otimes_{k_B} k_W$ is semisimple and decomposes as a direct sum of fields:

$$k_V \otimes_{k_B} k_W = \bigoplus_{i=1}^{n} k_i.$$

Let $P_i \in k_V \otimes_{k_B} k_W$ be the projection onto the summand k_i. Since $k_V \otimes_{k_B} k_W$ embeds in $\mathrm{End}_2(V \otimes W)$, we can view P_i as a non-zero idempotent in $\mathrm{End}_2(V \otimes W)$ as well. Then $\mathrm{im}\, P_i \in \mathrm{Hom}_1(A,C)$ and $k_i \subseteq \mathrm{End}_2(\mathrm{im}\, P_i)$. It follows that the subfield of k_i generated by $\alpha_{A,V}(k_A)$ and $\alpha_{W,C}(k_C)$ is also the field k_X for any simple summand X of $\mathrm{im}\, P_i$. $\square$

4. Galois theory

In this section we complete the proof of Theorem 1.1 using a result in Galois theory which is of separate interest.

Section 3 constructs, from a suitable 2-category $\mathcal{C}$, a finite collection of fields $\{k_A\}$ and a finite collection of bifinite, biseparable abstract composita

$$k_A \xrightarrow{\alpha_{A,V}} k_V \xleftarrow{\alpha_{V,B}} k_B$$

Moreover, $k_{I_A} = k_A$ and α_{A,I_A} is the identity. By Lemma 3.1, $\alpha_{A,V^*} = \alpha_{V,A}$. And by Lemma 3.5, given two abstract composita

$$k_A \xrightarrow{\alpha_{A,V}} k_V \xleftarrow{\alpha_{V,B}} k_B \xrightarrow{\alpha_{B,W}} k_W \xleftarrow{\alpha_{W,C}} k_C,$$

every compositum of k_A and k_C that appears in $k_V \otimes_{k_B} k_W$ is k_X for some summand $X \subseteq V \otimes W$. We call this method of producing k_X from k_V and k_W *amalgamation* of composita.

Theorem 4.1. *Let $K = \{k_A\}$ be a finite set of fields, and let $E = \{k_V\}$ be a finite set of biseparable, bifinite composita with embeddings*

$$k_A \xrightarrow{\alpha_{A,V}} k_V \xleftarrow{\alpha_{V,B}} k_B.$$

Suppose that E contains identities, is closed with respect to duality and amalgamation, and connects every pair of elements of K. Then there is a field k and finite-index embeddings $\{\beta_A\}$ and $\{\beta_V\}$ that form commutative triangles:
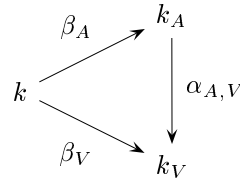
Theorem 4.1 can also be reformulated as the following corollary.

Corollary 4.2. *Let $K = \{k_A\}$ be a finite set of fields, let K-Mod-K be the 2-category of bimodules over elements of K, and let $\mathcal{D}$ be a full, connected, rigid, semisimple sub-2-category of K-Mod-K with finitely many simple 1-morphisms and without inseparable extensions. Then $\mathcal{D}$ admits a forgetful functor to the category k-mod of finite-dimensional vector spaces over a field k which embeds in every k_A.*

By Lemma 3.4, the semisimplicity of the 2-category $\mathcal{D}$ eliminates the possibility of inseparable extensions. Although $\mathcal{D}$ satisfies the hypotheses of Theorem 1.1, and although it is constructed from the more general 2-category $\mathcal{C}$, we do not know a natural functor from $\mathcal{C}$ to $\mathcal{D}$.

Proof. Since the fields $k_A \in K$ are all connected by composita, they all have isomorphic separable closures. Let Ω be a field in this isomorphism class, and realize each k_A arbitrarily as a subfield of Ω. Let $G_A = \mathrm{Gal}(\Omega/k_A)$ be the absolute Galois group of k_A. Finally let $\mathbb{F}$ be the characteristic field of Ω, either $\mathbb{F}_p$ or $\mathbb{C}$.

If $k_V \in E$ is an extension of k_A, we can position k_V so that $k_A \subseteq k_V \subseteq \Omega$; the embedding $\alpha_{A,V}$ is then the inclusion map. But having chosen this position for k_V, we cannot require that the other embedding $\alpha_{V,B}$ is the inclusion map. Rather k_V only contains a subfield isomorphic to k_B and $\alpha_{V,B}$ is the isomorphism. Let $\phi_V \in \mathrm{Gal}(\Omega/\mathbb{F})$ be an extension to all of Ω of the map $\alpha_{V,B}$, so that

$$(1) \qquad G_V \overset{\mathrm{def}}{=} \mathrm{Gal}(\Omega/k_V) = G_A \cap \phi_V G_B \phi_V^{-1}.$$

Note that, having fixed $k_A, k_B \subseteq \Omega$, the connecting automorphism ϕ_V can be replaced by any other element of the double coset $G_A \phi_V G_B$. The double coset determines the mutual extension k_V up to its position in Ω.

Next consider the tensor product $k_V \otimes_{k_B} k_W$, which, as in the proof of Lemma 3.5, is a direct sum of fields:

$$k_V \otimes_{k_B} k_W = \bigoplus_{i=1}^{n} k_i.$$

Any summand k_i contains a copy of the field embeddings

$$k_A \overset{\alpha_{A,V}}{\longrightarrow} k_V \overset{\alpha_{V,B}}{\longleftarrow} k_B \overset{\alpha_{B,W}}{\longrightarrow} k_W \overset{\alpha_{W,C}}{\longleftarrow} k_C.$$

As discussed above, the relative position of k_A and k_B in k_V is described by an element of the double coset $G_A \phi_V G_B$. Likewise the relative position of k_B and k_C is described by an element of the double coset $G_B \phi_W G_C$. Therefore the relative position of k_A and k_C in the summand k is given by an element ϕ of their product

$$G_A \phi_V G_B \phi_W G_C.$$

By hypothesis, the compositum k_X of k_A and k_C in k_i which is in E and is represented by its own double coset $G_A \phi_X G_C$. Thus

$$\phi \in G_A \phi_X G_C \subseteq G_A \phi_V G_B \phi_W G_C.$$

At the same time, universality of tensor products implies that if ϕ is any element of $G_A \phi_V G_B \phi_W G_C$, the corresponding relative position of k_V and k_W is represented by some summand $k \subseteq k_V \otimes_{k_B} k_W$.

Thus the decomposition of $k_V \otimes_{k_B} k_W$ yields a decomposition of double cosets

$$(2) \qquad G_A \phi_V G_B \phi_W G_C = \bigcup_{X \in E_V} G_A \phi_X G_C$$

for some subset $E_V \subseteq E$. In addition, the duality hypothesis implies that we can take we can take $\phi_{V*} = \phi_V^{-1}$ for some $V^* \in E$, while the identity hypothesis implies that we can take $\phi_{I_A} = 1$ for some $I_A \in E$. Combining all of these facts,

if $E_{A,B}$ is the set of all mutual extensions of k_A and k_B in E, then the union of double cosets

$$H_A = \bigcup_{V \in E_{A,A}} G_A \phi_V G_A$$

is a group: It is closed under multiplication and inversion and contains the group G_A (and therefore 1).

We claim that G_A is a finite-index subgroup of H_A. By equation (1), the number of right cosets of G_A in the double coset $G_A \phi_V G_A$ is the same as the index $[G_A : G_V]$ of G_V in G_A, which by hypothesis is finite. Moreover, H_A is a finite union of such double cosets, since E is finite. This establishes the claim. As the remaining arguments indicate, the claim is the heart of the proof of Theorem 4.1.

Let f_A be the fixed field of $H_A \subseteq \mathrm{Gal}(\Omega/\mathbb{F})$. Then k_A is a finite, separable extension of f_A, because its Galois group G_A is a finite-index subgroup of H_A. Moreover, f_A and f_B are canonically isomorphic. Any $\phi_V \in \mathrm{Gal}(\Omega/\mathbb{F})$ is an isomorphism between them. By equation (2), any two choices for ϕ_V (allowing V to vary as well) differ by an element of H_A on the left, and therefore all agree after restriction to f_A. Equation (2) also shows that the isomorphisms between f_A, f_B, and f_C form a commutative triangle. Thus we can let k be a field isomorphic to all of them by maps

$$\beta_A : k \to f_A;$$

the same map β_A can also be taken as an embedding of k in k_A. By construction these field embeddings also extend to commutative triangles
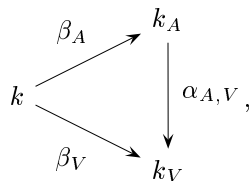
as desired.   □

5. Questions

Question 5.1. *If $\mathcal{C}$ is k-linear over a separably closed field k and every 1-identity is strongly simple, can some $\mathrm{End}_2(V)$ be a non-trivial inseparable extension of k?*

It is noteworthy that if f is an inseparable finite extension of k, then $f \otimes_k f$ is not semisimple; this is a weak form of Lemma 3.4. If a 2-category $\mathcal{C}$ did satisfy Question 5.1, then the result of base change $\overline{k} \otimes_k \mathcal{C}$ would not be multifusion because it would not be semisimple. This could be taken as a loophole in the structure theory of fusion categories in characteristic p: such a category $\mathcal{C}$ would be "morally" but not "technically" fusion.

If $\mathcal{C}$ is a semisimple 2-category with simple 1-identities, then it may be *weakly right-rigid* in the sense that for every $V \in \mathrm{Hom}(A, B)$, there exists $V^* \in \mathrm{Hom}(B, A)$ such that

$$\mathrm{Inv}(V \otimes V^*) \neq 0.$$

If $\mathcal{C}$ is weakly right-rigid, then the structure of $V \otimes V^* \otimes V^{**}$ induces a map

$$s_V : \mathrm{End}_2(V) \otimes_{k_B} \mathrm{Inv}(V^* \otimes V^{**}) \to \mathrm{Inv}(V \otimes V^*) \otimes_{k_A} \mathrm{End}_2(V^{**}).$$

We can always take V^* and V^{**} to be simple. If V is also simple, then either s_V vanishes, or $V^{**} \cong V$ and V^* is (up to isomorphisms) both a left dual and a right dual of V.

Conjecture 5.2. *Every finite, weakly rigid, semisimple 2-category with simple 1-identities is rigid.*

Example. The representation category of $U_q(\mathfrak{sl}(2))$ becomes weakly rigid but not rigid in the crystal limit $q \to 0$. However, it has infinitely many simple objects.

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References

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