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ON THE POLYNOMIAL MOMENT PROBLEM

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1. Introduction

In this paper we treat the following "polynomial moment problem": *for complex polynomials $P(z)$, $Q(z) = \int q(z)\mathrm{d}\, z$ and distinct $a, b \in \mathbb{C}$ such that $P(a) = P(b)$, $Q(a) = Q(b)$ to find conditions under which*

$$\int_a^b P^i(z) q(z) \mathrm{d}\, z = 0 \qquad (*)$$

for all integer non-negative i.

The polynomial moment problem was proposed in the series of papers of M. Briskin, J.-P. Francoise and Y. Yomdin [1]-[5] as an infinitesimal version of the center problem for the polynomial Abel equation in the complex domain in the frame of a programme concerning the classical Poincaré center-focus problem for the polynomial vector field on the plane. It was suggested that the following "composition condition" imposed on $P(z)$ and $Q(z) = \int q(z)\mathrm{d}\, z$ is necessary and sufficient for the pair $P(z)$, $q(z)$ to satisfy (*): *there exist polynomials $\tilde{P}(z), \tilde{Q}(z), W(z)$ such that*

$$(**) \qquad P(z) = \tilde{P}(W(z)), \quad Q(z) = \tilde{Q}(W(z)), \quad \textit{and} \quad W(a) = W(b).$$

It is easy to see that the composition condition is sufficient: since after the change of variable $z \to W(z)$ the way of integration becomes closed, the sufficientness follows from the Cauchy theorem. The necessity of the composition condition in the case when a, b are not critical points of $P(z)$ was proved by C. Christopher in [6] (see also the paper of N. Roytvarf [12] for a similar result) and in some other special cases by M. Briskin, J.-P. Francoise and Y. Yomdin in the papers cited above.

Nevertheless, in general the composition conjecture fails to be true. Namely, in the paper [9] a class of counterexamples to the composition conjecture was constructed. These counterexamples exploit polynomials $P(z)$ which admit double decompositions: $P(z) = A(B(z)) = C(D(z))$, where $A(z)$, $B(z)$, $C(z)$,

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$D(z)$ are non-linear polynomials. If $P(z)$ is such a polynomial and, in addition, $B(a) = B(b)$, $D(a) = D(b)$ then for any polynomial $Q(z)$ which can be represented as $Q(z) = E(B(z)) + F(D(z))$ for some polynomials $E(z), F(z)$ condition (*) is satisfied with $q(z) = Q'(z)$. On the other hand, it was shown in [9] that if $\deg B(z)$ and $\deg D(z)$ are coprime then condition (**) is not satisfied already for $Q(z) = B(z) + D(z)$.

Note that double decompositions with $\deg A(z) = \deg D(z)$, $\deg B(z) = \deg C(z)$ and $\deg B(z), \deg D(z)$ coprime are described explicitly by Ritt's theory of factorization of polynomials. They are equivalent either to decompositions with $A(z) = z^n R^m(z)$, $B(z) = z^m$, $C(z) = z^m$, $D(z) = z^n R(z^m)$ for a polynomial $R(z)$ and $\mathrm{GCD}(n, m) = 1$ or to decompositions with $A(z) = T_m(z)$, $B(z) = T_n(z)$, $C(z) = T_n(z)$, $D(z) = T_m(z)$ for Chebyshev polynomials $T_n(z)$, $T_m(z)$ and $\mathrm{GCD}(n, m) = 1$ (see [11], [13]).

The counterexamples above suggest to weaken the composition conjecture as follows: *polynomials $P(z)$, $q(z)$ satisfy condition (*) if and only if $\int q(z) \mathrm{d}\, z$ can be represented as a sum of polynomials Q_j such that*

$$(***) \qquad P(z) = \tilde{P}_j(W_j(z)), \quad Q_j(z) = \tilde{Q}_j(W_j(z)), \quad \text{and} \quad W_j(a) = W_j(b)$$

for some $\tilde{P}_j(z), \tilde{Q}_j(z), W_j(z) \in \mathbb{C}[z]$. For the case when $P(z) = T_n(z)$ this statement was verified in [10]. Moreover, it was shown that for $P(z) = T_n(z)$ the number of terms in the representation $\int q(z) \mathrm{d}\, z = \sum_j Q_j(z)$ can be reduced to two.

In this paper we give a solution of the polynomial moment problem in the case when $P(z)$ is indecomposable that is when $P(z)$ can not be represented as a composition $P(z) = P_1(P_2(z))$ with non-linear polynomials $P_1(z)$, $P_2(z)$. In this case conditions (**) and (***) are equivalent and the composition conjecture reduces to the following statement.

Theorem 1. *Let $P(z)$, $Q(z) = \int q(z) \mathrm{d}\, z$ be complex polynomials and let a, b be distinct complex numbers such that $P(a) = P(b)$, $Q(a) = Q(b)$, and*

$$\int_a^b P^i(z) q(z) \mathrm{d}\, z = 0$$

for $i \geq 0$. Suppose that $P(z)$ is indecomposable. Then there exists a polynomial $\tilde{Q}(z)$ such that $Q(z) = \tilde{Q}(P(z))$.

We also examine the following condition which is stronger than (*):

$$\int_a^b P^i(z) Q^j(z) Q'(z) \mathrm{d}\, z = 0$$

for $i \geq 0, j \geq 0$. If γ is a curve which is the image of the segment $[a, b]$ in $\mathbb{C}^2$ under the map $z \to (P(z), Q(z))$ then this condition is equivalent to the condition that $\int_\gamma \omega = 0$ for all global holomorphic 1-forms ω in $\mathbb{C}^2$ ("the moment condition"). For an oriented simple closed curve δ of class C^2 in $\mathbb{C}^2$ the moment condition is necessary and sufficient to be a boundary of a bounded analytic variety Σ

in $\mathbb{C}^2$; it is a special case of the result of R. Harwey and B. Lawson [7]. The case when δ is an image of S^1 under the map $z \to (f(z), g(z))$, where $f(z), g(z)$ are functions analytic in an annulus containing S^1 was investigated earlier by J. Wermer [14]: in this case the moment condition is equivalent to the condition that there exists a finite Riemann surface Σ with border S^1 such that $f(z)$, $g(z)$ have an analytic extension to Σ.

Unlike condition (*) the more restrictive moment condition imposed on polynomials $P(z)$, $Q(z)$ turns out to be equivalent to composition condition (**). We show that actually even a weaker condition is needed.

Theorem 2. *Let $P(z), Q(z)$ be complex polynomials and let a, b be distinct complex numbers such that $P(a) = P(b), Q(a) = Q(b)$, and*

$$\int_a^b P^i(z) Q^j(z) Q'(z) \mathrm{d}\, z = 0$$

for $0 \leq i \leq \infty$, $0 \leq j \leq d_a + d_b - 2$, where d_a (resp. d_b) is the multiplicity of the point a (resp. b) with respect to $P(z)$. Then there exist polynomials $\tilde{P}(z)$, $\tilde{Q}(z)$, $W(z)$ such that $P(z) = \tilde{P}(W(z))$, $Q(z) = \tilde{Q}(W(z))$, and $W(a) = W(b)$.

Note that if a, b are not critical points of $P(z)$ that is if $d_a = d_b = 1$ then conditions of the theorem reduce to condition (*) and therefore Theorem 2 includes as a particular case the result of C. Christopher.

2. Proofs

2.1. Lemmata about branches of $Q(P^{-1}(z))$. Let $P(z)$ and $Q(z)$ be rational functions and let $U \subset \mathbb{C}$ be a domain in which there exists a single-valued branch $p^{-1}(z)$ of the algebraic function $P^{-1}(z)$. Denote by $Q(P^{-1}(z))$ the complete algebraic function obtained by the analytic continuation of the functional element $\{U, Q(p^{-1}(z))\}$. Since the monodromy group $G(P^{-1})$ of the algebraic function $P^{-1}(z)$ is transitive this definition does not depend of the choice of $p^{-1}(z)$. Denote by $d(Q(P^{-1}(z)))$ the degree of the algebraic function $Q(P^{-1}(z))$ that is the number of its branches.

Lemma 1. *Let $P(z), Q(z)$ be rational functions. Then*

$$d(Q(P^{-1}(z))) = \deg P(z)/[\mathbb{C}(z) : \mathbb{C}(P, Q)].$$

Proof. Since any algebraic relation over $\mathbb{C}$ between $Q(p^{-1}(z))$ and z supplies an algebraic relation between $Q(z)$ and $P(z)$ and vice versa we see that $d(Q(P^{-1}(z))) = [\mathbb{C}(P, Q) : \mathbb{C}(P)]$. As $[\mathbb{C}(P, Q) : \mathbb{C}(P)] = [\mathbb{C}(z) : \mathbb{C}(P)]/ [\mathbb{C}(z) : \mathbb{C}(P, Q)]$ the lemma follows now from the observation that $[\mathbb{C}(z) : \mathbb{C}(P)] = \deg P(z)$. □

Recall that by Lüroth theorem each field k such that $\mathbb{C} \subset k \subset \mathbb{C}(z)$ and $k \neq \mathbb{C}$ is of the form $k = \mathbb{C}(R)$, $R \in \mathbb{C}(z) \setminus \mathbb{C}$. Therefore, the field $\mathbb{C}(P, Q)$ is a proper subfield of $\mathbb{C}(z)$ if and only if $P(z) = \tilde{P}(W(z))$, $Q(z) = \tilde{Q}(W(z))$ for some rational functions $\tilde{P}(z)$, $\tilde{Q}(z)$, $W(z)$ with $\deg W(z) > 1$; in this case we say that $P(z)$ and $Q(z)$ have a common right divisor in the composition algebra. The Lemma 1 implies the following explicit criterion which is essentially due to Ritt [11] (cf. also [6], [12]).

Corollary 1. *Let $P(z), Q(z)$ be rational functions. Then $P(z)$ and $Q(z)$ have a common right divisor in the composition algebra if and only if*

$$Q(p^{-1}(z)) = Q(\tilde{p}^{-1}(z)) \tag{1}$$

for two different branches $p^{-1}(z)$, $\tilde{p}^{-1}(z)$ of $P^{-1}(z)$.

Proof. Indeed, by Lemma 1, the field $\mathbb{C}(P,Q)$ is a proper subfield of $\mathbb{C}(z)$ if and only if $d(Q(P^{-1}(z))) < \deg P(z)$. On the other hand, the last inequality is clearly equivalent to condition (1). □

Lemma 2. *Let $P(z)$, $Q(z)$ be rational functions,* $\deg P(z) = n$. *Suppose that there exist $a_i \in \mathbb{C}$, $1 \leq i \leq n$, not all equal between themselves such that*

$$\sum_{i=1}^{n} a_i Q(p_i^{-1}(z)) = 0. \tag{2}$$

If, in addition, the group $G(P^{-1})$ is doubly transitive then $Q(z) = \tilde{Q}(P(z))$ for some rational function $\tilde{Q}(z)$.

Proof. Let $G \subset S_n$ be a permutation group and let $\rho_G : G \to GL(\mathbb{C}^n)$ be the permutation representation of G that is $\rho_G(g)$, $g \in G$ is the linear map which sends a vector $\vec{a} = (a_1, a_2, ..., a_n)$ to the vector $\vec{a_g} = (a_{g(1)}, a_{g(2)}, ..., a_{g(n)})$. It is well known (see e.g. [15], Th. 29.9) that G is doubly transitive if and only if ρ_G is the sum of the identical representation and an absolutely irreducible representation. Clearly, the one-dimensional ρ_G-invariant subspace $E \subset \mathbb{C}^n$ corresponding to the identity representation is generated by the vector $(1, 1, ..., 1)$. Therefore, since the Hermitian inner product $(\vec{a}, \vec{b}) = a_1\bar{b}_1 + a_2\bar{b}_2 + ... + a_n\bar{b}_n$ is invariant with respect to ρ_G, the group G is doubly transitive if and only if the subspace E and its orthogonal complement $E^{\perp}$ are the only ρ_G-invariant subspaces of $\mathbb{C}^n$.

Suppose that (2) holds. In this case also

$$\sum_{i=1}^{n} a_i Q(p_{\sigma(i)}^{-1}(z)) = 0 \tag{3}$$

for all $\sigma \in G(P^{-1})$ by the analytic continuation. To prove the lemma it is enough to show that $Q(p_i^{-1}(z)) = Q(p_j^{-1}(z))$ for all i, j, $1 \leq i, j \leq n$; then by Lemma 1 $[\mathbb{C}(z) : \mathbb{C}(P,Q)] = \deg P(z) = [\mathbb{C}(z) : \mathbb{C}(P)]$ and therefore $Q(z) = \tilde{Q}(P(z))$ for some rational function $\tilde{Q}(z)$. Assume the converse i.e. that there exists $z_0 \in U$ such that not all $Q(p_i^{-1}(z_0))$, $1 \leq i \leq n$, are equal between themselves. Without loss of generality we can suppose that all $Q(p_i^{-1}(z_0))$, $1 \leq i \leq n$, are finite. Consider the subspace $V \subset \mathbb{C}^n$ generated by the vectors $\vec{v}_\sigma$, $\sigma \in G(P^{-1})$, where $\vec{v}_\sigma = (Q(p^{-1}_{\sigma(1)}(z_0)), Q(p^{-1}_{\sigma(2)}(z_0)), ..., Q(p^{-1}_{\sigma(n)}(z_0))$. Clearly, V is $\rho_{G(P^{-1})}$-invariant and $V \neq E$. Moreover, it follows from (3) that V is contained in the orthogonal complement $A^\perp$ of the subspace $A \subset \mathbb{C}^n$ generated by the vector $(\bar{a}_1, \bar{a}_2, ..., \bar{a}_n)$. Since $A \neq E$ we see that V is a proper ρ_G-invariant subspace of $\mathbb{C}^n$ distinct from E and $E^\perp$ that contradicts the assumption that the group $G(P^{-1})$ is doubly transitive. □

2.2. Lemma about preimages of domains. For a polynomial $P(z)$ denote by $c(P)$ the set of finite critical values of $P(z)$.

Lemma 3. *Let $P(z)$ be a polynomial and let $V \subset \mathbb{CP}^1$ be a simply connected domain containing infinity such that $c(P) \cap V = \emptyset$. Then $P^{-1}\{V\}$ is conformally equivalent to the unit disk and $P^{-1}\{\partial V\}$ is connected.*

Proof. Indeed, by the Riemann theorem V is conformally equivalent to the unit disk $\mathbb{D}$ whenever ∂V contains more than one point. It follows from $c(P) \cap V = \emptyset$ that ∂V contains a unique point if and only if $P(z)$ has a unique finite critical value c and $\partial V = c$; in this case there exist linear functions σ_1, σ_2 such that $\sigma_1(P(\sigma_2(z))) = z^n$, $n \in \mathbb{N}$ and the lemma is obvious. Therefore, we can suppose that $V \cong \mathbb{D}$. Since $c(P) \cap V = \emptyset$ the restriction of the map $P(z) : \mathbb{CP}^1 \to \mathbb{CP}^1$ on $P^{-1}\{V\} \setminus P^{-1}\{\infty\}$ is a covering map. As $V \setminus \infty$ is conformally equivalent to the punctured unit disc $\mathbb{D}^*$ it follows from covering spaces theory that $P^{-1}\{V\} \setminus P^{-1}\{\infty\}$ is a disjoint union of domains $\cup V_i$ conformally equivalent to $\mathbb{D}^*$ such that all induced maps $f_i : \mathbb{D}^* \to \mathbb{D}^*$ are of the form $z \to z^{l_i}$, $l_i \in \mathbb{N}$. But, as $P^{-1}\{\infty\} = \{\infty\}$, there may be only one such a domain. Therefore, the preimage $P^{-1}\{V\}$ is conformally equivalent to the unit disk. In particular, since $P^{-1}\{\partial V\} = \partial P^{-1}\{V\}$ we see that $P^{-1}\{\partial V\}$ is connected. □

2.3. Proof of Theorem 2: the case of a regular value. In this section we investigate the case when $t_0 = P(a) = P(b)$ is not a critical value of the polynomial $P(z)$. For a simple closed curve $M \subset \mathbb{C}$ denote by D^+_M (resp. by D^-_M) the domain that is interior (resp. exterior) with respect to M.

Let $L \subset \mathbb{C}$ be a simple closed curve such that $t_0 \in L$ and $c(P) \subset D^+_L$. Denote by $\vec{L}$ the same curve considered as an oriented graph embedded into the complex plane. By definition, the graph $\vec{L}$ has one vertex t_0 and one counter-clockwise oriented edge l. Let $\vec{\Omega} = P^{-1}\{\vec{L}\}$ be an oriented graph which is the preimage of the graph $\vec{L}$ under the mapping $P(z) : \mathbb{C} \to \mathbb{C}$, i.e. vertices of $\vec{\Omega}$ are preimages of t_0 and oriented edges of $\vec{\Omega}$ are preimages of l. As $L \cap c(P) = \emptyset$ the graph $\vec{\Omega}$ has $n = \deg P(z)$ vertices and n edges. Furthermore, by Lemma 3 the graph $\vec{\Omega} = P^{-1}\{\partial D^-_L\}$ is connected. Therefore, as a point set in $\mathbb{C}$ the graph $\vec{\Omega}$ is a simple closed curve. Let l_j, $1 \leq j \leq n$, be oriented edges of $\vec{\Omega}$ and let a_j (resp. b_j) be the starting (resp. ending) point of l_j. We will suppose that edges of $\vec{\Omega}$ are numerated by such a way that $a_1 = a$ and that under a moving around the domain $P^{-1}\{D^-_L\}$ along its boundary $\vec{\Omega}$ the edge l_i, $1 \leq i \leq n-1$, is followed by the edge l_{i+1} (see fig. 1).

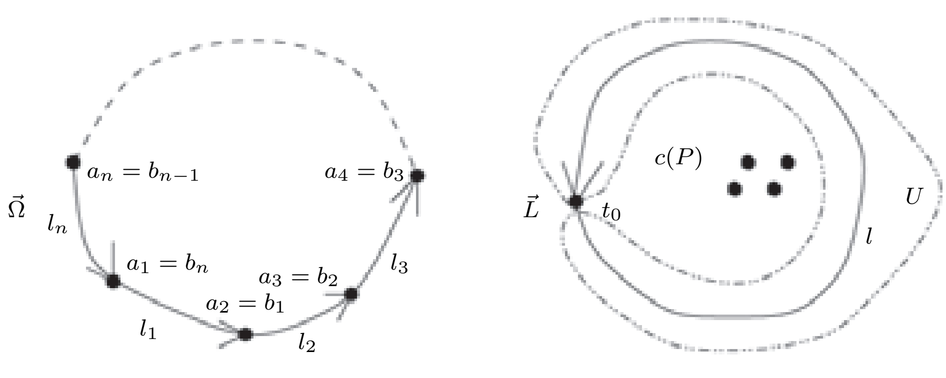


FIGURE 1

Let $U \subset \mathbb{C}$ be a simply connected domain such that $U \cap c(P) = \emptyset$ and $L \setminus \{t_0\} \subset U$. By the monodromy theorem, in such a domain there exist n single-valued branches of $P^{-1}(t)$. Denote by $p_j^{-1}(t)$, $1 \le j \le n$, the single-valued branch of $P^{-1}(t)$ defined in U by the condition $p_j^{-1}\{l \setminus t_0\} = l_j \setminus \{a_j, b_j\}$; such a numeration of branches of $P^{-1}(t)$ means that the analytic continuation of the functional element $\{U, p_j^{-1}(t)\}$, $1 \le j \le n-1$, along L is the functional element $\{U, p_{j+1}^{-1}(t)\}$. Let l_k, $k < n$, be the edge of Ω such that $b_k = b$ and let $\Gamma = \{l_1, l_2, ..., l_k\}$ be the oriented path in the graph Ω joining the vertices $a_1 = a$ to $b_k = b$. For $t \in U$ set $\varphi(t) = \sum_{j=1}^{k} Q(p_j^{-1}(t))$.

Consider an analytic function on $\mathbb{CP}^1 \setminus L$

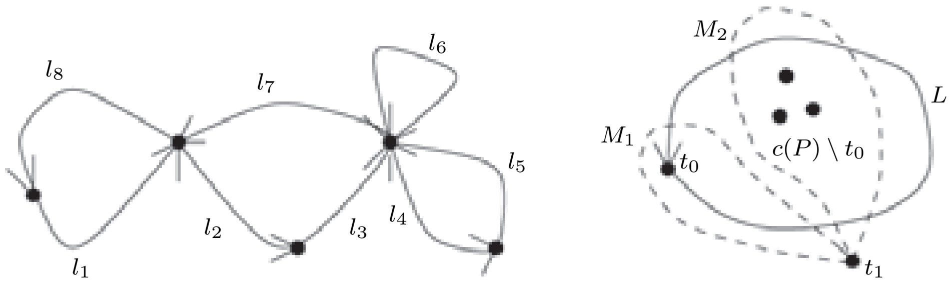
$$I(\lambda) = \oint_L \frac{\varphi(t)}{t-\lambda} \mathrm{d}\, t = \int_\Gamma \frac{Q(z)P'(z)\mathrm{d}\, z}{P(z)-\lambda}.$$

More precisely, the integral above defines two analytic functions: one of them $I^+(\lambda)$ is analytic in D_L^+ and the other one $I^-(\lambda)$ is analytic in D_L^-. Furthermore, calculating the Taylor expansion of $I^-(\lambda)$ at infinity and using integration by part we see that condition (*) reduces to the condition that $I^-(\lambda) \equiv 0$ in D_L^-. By a well-known result about integrals of the Cauchy type (see e.g. [8]) the last condition implies that $\varphi(t)$ is the boundary value on L of the analytic function $I^+(\lambda)$ in D_L^+. It follows from the uniqueness theorem for boundary values of analytic functions that the functional element $\{U, \varphi(t)\}$ can be analytically continued along any curve $M \subset D_L^+$. As $c(P) \subset D_L^+$ this fact implies that $\{U, \varphi(t)\}$ can be analytically continued along any curve $M \subset \mathbb{C}$. Therefore, by the monodromy theorem, the element $\{U, \varphi(t)\}$ extends to a single-valued analytic function in the whole complex plane. In particular, the analytic continuation of $\{U, \varphi(t)\}$ along any closed curve coincides with $\{U, \varphi(t)\}$. On the other hand, by construction the analytic continuation of $\{U, \varphi(t)\}$ along the curve L is $\{U, \varphi_L(t)\}$, where $\varphi_L(t) = \sum_{j=2}^{k+1} Q(p_j^{-1}(t))$. It follows from $\varphi(t) = \varphi_L(t)$ that $Q(p_1^{-1}(t)) = Q(p_{k+1}^{-1}(t))$ and by Corollary 1 we conclude that $P(z)$ and $Q(z)$ have a common right divisor in the composition algebra.

As the field $\mathbb{C}(P,Q)$ is a proper subfield of $\mathbb{C}(z)$ and $P(z)$, $Q(z)$ are polynomials it is easy to prove that $\mathbb{C}(P,Q) = \mathbb{C}(W)$ for some polynomial $W(z)$, $\deg W(z) > 1$. It means that $P(z) = \tilde{P}(W(z))$, $Q(z) = \tilde{Q}(W(z))$ for some polynomials $\tilde{P}(z)$, $\tilde{Q}(z)$ such that $\tilde{P}(z)$ and $\tilde{Q}(z)$ have no a common right divisor in the composition algebra. Let us show that $W(a) = W(b)$. Since t_0 is not a critical value of the polynomial $P(z) = \tilde{P}(W(z))$ the chain rule implies that t_0 is not a critical value of the polynomial $\tilde{P}(z)$. Therefore, if $W(a) \neq W(b)$ then after the change of variable $z \to W(z)$ in the same way as above we find that $\tilde{P}(z) = \bar{P}(U(z))$, $\tilde{Q}(z) = \bar{Q}(U(z))$ for some polynomials $\bar{P}(z), \bar{Q}(z), U(z)$ with $\deg U(z) > 1$ that contradicts the fact that $\tilde{P}(z)$, $\tilde{Q}(z)$ have no a common right divisor in the composition algebra. This completes the proof in the case when z_0 is not a critical value of $P(z)$.

2.4. Proof of Theorem 2: the case of a critical value. Assume now that $t_0 = P(a) = P(b)$ is a critical value of $P(z)$. In this case let L be a simple closed curve such that $t_0 \in L$ and $c(P) \setminus t_0 \subset D_L^+$. Consider again a graph $\vec{\Omega} = P^{-1}\{\vec{L}\}$. Since $P^{-1}\{D_L^-\}$ is still conformally equivalent to the unit disk by Lemma 3, we see that the graph $\vec{\Omega}$ topologically is the boundary of a disc although it is not a simple closed curve any more. Let l_j, $1 \leq j \leq n$, be oriented edges of $\vec{\Omega}$ and let a_j (resp. b_j) be the starting (resp. the ending) point of l_j. Let us fix again such a numeration of edges of $\vec{\Omega}$ that $a_1 = a$ and that under a moving around the domain $P^{-1}\{D_L^-\}$ along its boundary $\vec{\Omega}$ the edge l_i, $1 \leq i \leq n-1$, is followed by the edge l_{i+1}. As above denote by U a domain in $\mathbb{C}$ such that $U \cap c(P) = \emptyset$, $L \setminus \{t_0\} \subset U$ and let $p_j^{-1}(t)$, $1 \leq j \leq n$, be the single-valued branch of $P^{-1}(t)$ defined in U by the condition $p_j^{-1}\{l \setminus t_0\} = l_j \setminus \{a_j, b_j\}$. If $k < n$ is a number such that $b_k = b$ then for the same reason as above the function $\varphi(t) = \sum_{j=1}^{k} Q(p_j^{-1}(t))$ extends to an analytic function in $U \cup D_L^+$ but this fact does not imply now that $\varphi(t)$ extends to an analytic function in the whole complex plane since D_L^+ does not contain $t_0 \in c(P)$. Nevertheless, if V is a simply connected domain such that $U \subset V$ and $t_0 \notin V$ then $\varphi(t)$ still extends to a single-valued analytic function in V. In particular, the analytic continuation of $\{U, \varphi(t)\}$ along any simple closed curve M such that $t_0 \subset D_M^-$ coincides with $\{U, \varphi(t)\}$.

Let $t_1 \in U$ be a point and let M_1 (resp. M_2) be a simple closed curve such that $t_1 \in M_1$, $M_1 \cap c(P) = \emptyset$ and $D_{M_1}^+ \cap c(P) = t_0$ (resp. $t_1 \in M_2$, $M_2 \cap c(P) = \emptyset$ and $D_{M_2}^+ \cap c(P) = c(P) \setminus t_0$). Define a permutation $\rho_1 \in S_n$ (resp. $\rho_2 \in S_n$) by the condition that the functional element $\{U, p_{\rho_1(j)}^{-1}(t)\}$ (resp. $\{U, p_{\rho_2(j)}^{-1}(t)\}$) is the result of the analytic continuation of the functional element $\{U, p_j^{-1}(t)\}$, $1 \leq j \leq n$, from t_1 along the curve M_1 (resp. M_2). Having in mind the identification of the set of elements $\{U, p_j^{-1}(t)\}$, $1 \leq j \leq n$, with the set of oriented edges of the graph $\vec{\Omega}$ the permutations ρ_1, ρ_2 can be described as follows: ρ_1 cyclically permutes the edges of $\vec{\Omega}$ around the vertices from which they go



$\rho_1 = (28)(467), \quad \rho_2 = (18)(237)(45)$

FIGURE 2

while cycles $(j_1, j_2, ..., j_k)$ of ρ_2 correspond to simple cycles $(l_{j_1}, l_{j_2}, ..., l_{j_k})$ of the graph $\vec{\Omega}$ and $\rho_1\rho_2 = (12...n)$ (see fig. 2).

To unload notation denote temporarily the element $\{U, Q(p_i^{-1}(t))\}$, $1 \leq i \leq n$, by s_i. Since $t_0 \subset D_{M_2}^-$ we have:

$$0 = \sum_{j=1}^{k} s_{\rho_2(j)} - \sum_{j=1}^{k} s_j = s_{\rho_2(k)} + \sum_{j=1}^{k-1} \left[s_{\rho_2(j)} - s_{j+1} \right] - s_1. \tag{4}$$

Using $\rho_1\rho_2 = (12...n)$ we can rewrite (4) as

$$s_{\rho_1^{-1}(k+1)} - s_1 + \sum_{j=1}^{k-1} \left[s_{\rho_2(j)} - s_{\rho_1\rho_2(j)} \right] = 0.$$

Therefore, by the analytic continuation

$$s_{\rho_1^{f-1}(k+1)} - s_{\rho_1^f(1)} + \sum_{j=1}^{k-1} \left[s_{\rho_1^f \rho_2(j)} - s_{\rho_1^{f+1}\rho_2(j)} \right] = 0 \tag{5}$$

for $f \geq 0$. Summing equalities (5) from $f = 1$ to $f = o(\rho_1)$, where $o(\rho_1)$ is the order of the permutation ρ_1, changing the order of summing, and observing that

$$\sum_{f=0}^{o(\rho_1)-1} \left[s_{\rho_1^f \rho_2(j)} - s_{\rho_1^{f+1}\rho_2(j)} \right] = s_{\rho_2(j)} - s_{\rho_1^{o(\rho_1)}\rho_2(j)} = 0$$

we conclude that

$$\sum_{s=0}^{o(\rho_1)-1} Q(p^{-1}_{\rho_1^s(k+1)}(t)) = \sum_{s=0}^{o(\rho_1)-1} Q(p^{-1}_{\rho_1^s(1)}(t)) \tag{6}$$

in U. Note that if a, b are regular points of $P(z)$ then $\rho_1(1) = 1$, $\rho_1(k+1) = k+1$ and (6) reduces to the equality $Q(p_{k+1}^{-1}(t)) = Q(p_1^{-1}(t))$.

Since (6) holds for any polynomial $Q(z)$ such that $q(z) = Q'(z)$ satisfies (*), substituting in (6) $Q^j(z)$, $2 \leq j \leq d_a + d_b - 1$, instead of $Q(z)$ we see that

$$\sum_{s=0}^{o(\rho_1)-1} Q^j(p^{-1}_{\rho_1^s(k+1)}(t)) = \sum_{s=0}^{o(\rho_1)-1} Q^j(p^{-1}_{\rho_1^s(1)}(t)) \tag{7}$$

for all j, $1 \leq j \leq \mathrm{d}_b + d_b - 1$. Consider a Vandermonde determinant $D = \parallel d_{j,i} \parallel$, where $d_{j,i} = Q^j(p_i^{-1}(t))$, $0 \leq j \leq d_a + d_b - 1$ and i ranges the set of different indices from the cycles of ρ_1 containing 1 and $k+1$. Since (7) implies that $D = 0$ we conclude again that $Q(p_i^{-1}(t)) = Q(p_j^{-1}(t))$ for some $i \neq j$, $1 \leq i, j \leq n$. Therefore, $P(z)$ and $Q(z)$ have a common right divisor in the composition algebra and we can finish the proof by the same argument as in section 2.3 taking into account that the multiplicity of a point $c \in \mathbb{C}$ with respect to $P(z) = \tilde{P}(W(z))$ is greater or equal than the multiplicity of the point $W(c)$ with respect to $\tilde{P}(z)$. □

2.5. Proof of Theorem 1. Suppose at first that $n = \deg P(z)$ is a prime number. In this case the degree of the algebraic function $Q(P^{-1}(t))$ equals either n or 1 since $d(Q(P^{-1}(t)))$ divides $\deg P(z)$. If $d(Q(p^{-1}(t))) = n$ then Puiseux expansions at infinity

$$Q(p_i^{-1}(t)) = \sum_{k \leq k_0} a_k \varepsilon^{ik} t^{\frac{k}{n}}, \tag{8}$$

$1 \leq i \leq n$, $a_k \in \mathbb{C}$, $\varepsilon = exp(2\pi i/n)$, contain a coefficient $a_k \neq 0$ such that k is not a multiple of n. Substituting (8) in the equality obtained by the analytic continuation of (6) along a curve going to the domain where series (8) converge, we conclude that ε^k is a root of a polynomial with integer coefficients distinct from the n-th cyclotomic polynomial $\Phi_n(z) = 1 + z + ... + z^{n-1}$. Since ε^k is a primitive n-th root of unity it is a contradiction. Therefore, $d(Q(p^{-1}(t))) = 1$ and $Q(z) = \tilde{Q}(P(z))$ for some polynomial $\tilde{Q}(z)$.

Suppose now that n is composite. Since $P(z)$ is indecomposable the group $G(P^{-1})$ is primitive by the Ritt theorem [11]. By the Schur theorem (see e.g. [15], Th. 25.3) a primitive permutation group of composite degree n which contains an n-cycle is doubly transitive. Therefore, by Lemma 2 equality (6) implies that $Q(z) = \tilde{Q}(P(z))$ for some polynomial $\tilde{Q}(z)$. □

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