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BOUNDED SMOOTH STRICTLY PLURISUBHARMONIC EXHAUSTION FUNCTIONS ON TEICHMÜLLER SPACES

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Let $\mathcal{T}_{g,n}$ be the Teichmüller space of finite Riemann surfaces of genus g with n punctures. The main purpose of this note is to construct a smooth bounded strictly plurisubharmonic exhaustion function on $\mathcal{T}_{g,n}$. Some geometric consequences will also be studied.

The problem about the existence of bounded plurisubharmonic exhaustion function on $\mathcal{T}_{g,n}$ was originally proposed by Gromov (cf. [Kr]). An example of weakly plurisubharmonic exhaustion function with log poles but bounded from above is produced by Krushkal [Kr]. The construction is complex analytic, based on Bers embedding of $\mathcal{T}_{g,n}$ as a bounded domain in C^{3g-3+n} and constructed from the Grunsky coefficients of the Schwarzian derivative representing a point on the bounded domain. One may pose a more precise problem and ask if smooth bounded strictly plurisubharmonic exhaustion functions exists on $\mathcal{T}_{g,n}$. The motivation behind the specification is that in complex geometry, existence of strictly instead of weakly plurisubharmonic function is very useful in the construction of holomorphic sections of certain line bundles using L^2-estimates. The question is answered affirmatively in this article by a geometric construction which is very different from the constructions in [Kr].

In contrast to the complex analytic method used in [Kr], which does not seem to yield strictly plurisubharmonicity, the approach here is more geometric and explicit, depending heavily on the computation of Wolpert [W] on length functions. In [W], a strictly convex exhaustion function is constructed. The function goes to ∞ as we approach the boundary of the Teichmüller space. Our construction is a modification of the construction in [W]. It does not lead to a bounded convex exhaustion function, but somewhat surprisingly, leads to a bounded plurisubharmonic exhaustion function. The real Hessian form of the bounded exhaustion function is not definite, but the complex Hessian or the Levi form is positive definite.

We consider only moduli space of hyperbolic Riemann surface. In particular, the dimension of the Teichmüller space is $3g-3+n > 0$. Since we are considering a fixed Teichmüller space, we are going to suppress the subscript (g, n) and denote the Teichmüller and moduli spaces by $\mathcal{T}$ and $\mathcal{M}$ respectively. $\mathcal{M}$ can be considered as the quotient of $\mathcal{T}$ by the mapping class group.

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In a fixed homotopy class of curves on a negatively curved Riemann surface X, there is a unique geodesic representative minimizing the length in its class. A family of geodesic curves $\{\gamma_j\}_{n=1}^m$ is said to fill up X if $X - \cup_{i=1}^m \gamma_j$ is topologically a two cell or a cylinder with a boundary contained in ∂X. Denote by ℓ_γ the geodesic length function along a geodesic in the class of γ. Obviously for a finite Riemann surface S, there exist a finite number of geodesics which fill up S.

Lemma 1 (Ke). *Let $\{\gamma_j\}_{j\in A}$ be a finite number of geodesic curves filling up a Riemann surface S. Then $L_A = \sum_{j\in A} \ell_{\gamma_j}$ is a proper exhaustion function on $\mathcal{T} = \mathcal{T}(S)$.*

It is furthermore proved in [W] that L_A is a convex exhaustion function on $\mathcal{T}$. Here is our main result.

Theorem 1. *Suppose that $\{\gamma_j\}_{j\in A}$ represents a finite number of geodesic curves filling up a Riemann surface S. Then $-L_A^{-\alpha}$ is a bounded non-positive strictly plurisubharmonic exhaustion function on $\mathcal{T}(S)$ for all $0 < \alpha < 1$. $-L_A^{-\alpha}(x)$ approaches 0 as x tends to the boundary of $\mathcal{T}$. The Levi form has lower bound given by*

$$\mathrm{Levi}(-L_A^{-\alpha})(\nu, \overline{\nu}) \geq \alpha(1-\alpha)[\frac{\mathrm{Levi}(L_A)(\nu,\overline{\nu})}{L_A^{\alpha+1}}],$$

which is positive definite.

The followings are some immediate corollaries from Theorem 1. Some of the statements can however be obtained by other methods.

Corollary 1.

a. *Let ω be a complete Kähler metric on $\mathcal{T}$. Let L be a holomorphic line bundle on $\mathcal{T}$ equipped with a Hermitian metric with non-negative curvature. Then there exist non-trivial L^2 holomorphic sections of $K_{\mathcal{T}}+L$ on $\mathcal{T}$, where $K_{\mathcal{T}}$ is the canonical line bundle of the Teichmüller space $\mathcal{T}$. Moreover, the L^2-sections separate points on $\mathcal{T}$ and generate any jet of the line bundle $K_{\mathcal{T}} + L$.*
b. *Let ω be a complete Kähler metric on the moduli space $\mathcal{M}$. Let L be a holomorphic line on $\mathcal{M}$ endowed with a Hermitian metric of bounded positive curvature on $\mathcal{M}$. Then the orbifold Euler characteristic $\chi(\mathcal{M}, K_{\mathcal{M}}+L) > 0$. Furthermore, there exists a non-trivial L^2-holomorphic section of $K_{\mathcal{M}}+L$ on $\mathcal{M}$, where $K_{\mathcal{M}}$ is the canonical line bundle of the moduli space $\mathcal{M}$.*

For simplicity of notations, we will suppress the subscript and denote by K the canonical line bundle in the appropriate spaces or manifolds. We note that for L trivial, the L^2-norm of K is conformal invariant and independent of the metric. In such a case, completeness is not required.

Corollary 2. *The Bergman metric on $\mathcal{T}$ is complete. In fact, it is greater than the Carathéodory metric on $\mathcal{T}$.*

Before we proceed to the proof of Theorem 1, we recall briefly the notion of quasi-Fuchsian uniformization of Riemann surfaces used in standard theory of Teichmüller spaces. General reference for the formulation of variation formulae in our discussions can be found in [W] or [G]. Consider a Riemann surface $X \in \mathcal{T} = \mathrm{Teich}(S)$, the Teichmüller space of a Riemann surface S. As a consequence of Riemann Uniformization Theorem, we may represent $X = H/\Gamma$, where H is the upper half plane and Γ is a lattice on H. Tangent vectors in $\mathcal{T}_X$ are represented by harmonic Beltrami differentials. For a harmonic Beltrami differential μ, we denote by f^{μ} the solution of the Beltrami equation

$$\begin{aligned} f_{\overline{z}} &= \mu(z) f_z, z \in H \\ f_{\overline{z}} &= \overline{\mu(\overline{z})} f_z, z \in L \\ f(0) &= 0, f(1) = 1, f(\infty) = \infty, \end{aligned}$$

where L is the lower half plane in C. Let μ be a fixed Beltrami differential of norm 1. Let ϵ be a small complex number. We consider the Beltrami equation associated to $\epsilon\mu$ and let $f^{\epsilon\mu}$ be the solution of the equation. For ϵ small, $f^{\epsilon\mu}$ defines a Fuchsian group $\Gamma^{\epsilon} = f^{\epsilon\mu}\Gamma(f^{\epsilon\mu})^{-1}$. The Riemann surfaces $X^{\epsilon} = H/\Gamma^{\epsilon}$ defines a curve in $\mathcal{T}$ with $X^0 = X$ for ϵ small.

We consider now the geodesic length function ℓ_γ for the closed geodesic curve γ in a fixed homotopy class of curves. As in [W], we lift γ to the universal covering H and normalize the coordinate if necessary so that the lift lies in the positive imaginary axis with the deck transformation corresponding to γ given by $z \mapsto \lambda z$ with $\lambda > 0$. Consider now the length function $\ell_\gamma(X^{\epsilon})$ on X^{ϵ} determined by the homotopy class of γ. We need to estimate the derivatives $\frac{d\ell_\gamma(X^{\epsilon})}{d\epsilon}$.

On H, the Beltrami differential μ on X defining X^{ϵ} can be considered as a Γ invariant tensor of the form $\mu = b(z)\frac{\partial}{\partial z} \otimes d\overline{z}$. Since μ is harmonic, it follows that $\phi = \overline{\frac{1}{(z-\overline{z})^2} b(z)} dz \otimes dz$ is a Γ-invariant holomorphic quadratic differential. Let $a(z) = \overline{\frac{1}{(z-\overline{z})^2} b(z)}$. We have a series expansion of $a(z)$ in terms of powers of z^{σ}, where $\sigma = \frac{2\pi i}{\log \lambda}$. Hence

$$a(z) = \frac{1}{z^2} \sum_{n=-\infty}^{\infty} a_n z^{\sigma n}.$$

The following computations are essentially contained in [W]. We go through the calculations briefly to present it in a way convenient for our later use. It is necessary to go into some details since the precise constants are important for our proof.

Proposition 1. *Let $\mu = b(z)\frac{\partial}{\partial z} \otimes d\overline{z}$ be a harmonic Beltrami differential representing a tangent vector in $T\mathcal{T}_X$ and $X^{\epsilon} = H/f^{\epsilon\mu}\Gamma(f^{\epsilon\mu})^{-1}$, $|\epsilon| < \delta$, be a local $1-$parameter family of Riemann surfaces determined by μ. In terms of series* *expansion* $\overline{\frac{1}{(z-\overline{z})^2}b(z)} = \frac{1}{z^2}\sum_{n=-\infty}^{\infty} a_n z^{\sigma n}$, *the length function satisfies*

$$\begin{aligned}
\ell_\gamma(X) &= \log\lambda, \\
\frac{d\ell_\gamma(X^\epsilon)}{d\epsilon}|_{\epsilon=0} &= -4\log\lambda \mathrm{Re}(a_o), \\
\frac{d^2\ell_\gamma(X^\epsilon)}{d\epsilon^2}|_{\epsilon=0} &= 8\log\lambda(2|a_0|^2 - \mathrm{Re}(a_0^2)) \\
&\quad + \sum_{n=1}^{\infty} \frac{2}{1+|\sigma n|^2}\int_0^{2\pi} (|a_n e^{i\theta(\sigma n-1)} + \overline{a_{-n}}e^{-i\theta(\sigma n-1)}|^2 \\
&\quad + |a_n e^{i\theta(\sigma n+1)} + \overline{a_{-n}}e^{-i\theta(\sigma n+1)}|^2)\sin^2\theta d\theta
\end{aligned}$$

Proof. On the upper half plane H, the hyperbolic metric is given by $\frac{|dz|^2}{y^2}$. γ lies along the imaginary axis. Hence the length $\ell_\gamma(X) = \int_1^\lambda \frac{dy}{y} = \log\lambda$. This proves (1).

For (2), denote by Ω_λ the region $\{z \in C | 1 < |z| < \lambda\} \cap H$. We used as in [W], Theorem 3.1 or [G] the classical fact that

$$\begin{aligned}
\frac{d\ell_\gamma(X^\epsilon)}{d\epsilon}|_{\epsilon=0} &= \frac{2}{\pi}\mathrm{Re}\int_{\Omega_\lambda} \frac{b(z)}{z^2} dE \\
&= \frac{2}{\pi}\mathrm{Re}\int_{\Omega_\lambda} [\frac{(z-\overline{z})^2}{|z|^4}\sum_{n=-\infty}^{\infty}\overline{a_n z^{\sigma n}}] dE,
\end{aligned}$$

where dE is the Euclidean area element. In terms of polar coordinates $z = re^{i\theta}$ and $\rho = \frac{\log r}{\log\lambda}$, $z^\sigma = e^{\frac{-2\pi\theta}{\log\lambda}}(\cos(2\pi\rho) + i\sin(2\pi\rho))$. Upon simplification,

$$\begin{aligned}
&\frac{d\ell_\gamma(X^\epsilon)}{d\epsilon}|_{\epsilon=0} \\
&= \frac{2}{\pi}\mathrm{Re}\{\sum_{n=-\infty}^{\infty}\overline{a_n}[\int_1^\lambda \frac{1}{r^2}(\cos(2\pi n\rho) + i\sin(2\pi n\rho)) r dr][\int_0^\pi (e^{2i\theta} - 2 + e^{-2i\theta})e^{\frac{2\pi\theta n}{\log\lambda}} d\theta]\}
\end{aligned}$$

The second integral is trivial unless $n = 0$. After a change of variable in the integral,

$$\begin{aligned}
&\frac{d\ell_\gamma(X^\epsilon)}{d\epsilon}|_{\epsilon=0} \\
&= \frac{2}{\pi}\mathrm{Re}\{a_0\log\lambda\int_0^1 d\rho\int_0^\pi (e^{2i\theta} - 2 + e^{-2i\theta})d\theta\} \\
&= \frac{2}{\pi}(-2\log\lambda \mathrm{Re}(a_0)(\pi)) \\
&= -4\log\lambda \mathrm{Re}(a_0),
\end{aligned}$$

concluding the proof of (2).

(3) is just a slightly more detailed version of the computations in [W]. Differentiating the expression of $\frac{d\ell_\gamma(X^\epsilon)}{d\epsilon}$ with respect to ϵ as explained in Theorem

3.2 and Lemma 4.1 of [W],

$$
\begin{aligned}
&\frac{d^2\ell_\gamma(X^\epsilon)}{d\epsilon^2}|_{\epsilon=0}\\
&= \frac{4}{\pi}\mathrm{Re}\int_{\Omega_\lambda}[\frac{\mu}{z^3}(z\frac{df_z}{d\epsilon}|_{\epsilon=0} - \frac{df}{d\epsilon}|_{\epsilon=0})]dE\\
&= \frac{4}{\pi}\mathrm{Re}\int_{\Omega_\lambda}\frac{\mu}{z^3}[z^2 2\mathrm{Re}(\sum_{n=-\infty}^{\infty}\frac{a_n z^{\sigma n-1}}{\sigma n-1}) - 2\mathrm{Re}(\sum_{n=-\infty}^{\infty}\frac{a_n z^{\sigma n+1}}{\sigma n+1})]dE\\
&= \frac{4}{\pi}\mathrm{Re}\int_{\Omega_\lambda}\frac{(z-\overline{z})^2}{z^3\overline{z}^2}[\sum_{m=-\infty}^{\infty}\overline{a_m}\overline{z^{\sigma m}}][z^2 2\mathrm{Re}(\sum_{n=-\infty}^{\infty}\frac{a_n z^{\sigma n-1}}{\sigma n-1})\\
&\quad - 2\mathrm{Re}(\sum_{n=-\infty}^{\infty}\frac{a_n z^{\sigma n+1}}{\sigma n+1})]dE\\
&= I + II + III,
\end{aligned}
$$

where

$$
\begin{aligned}
I &= \frac{4}{\pi}\sum_{m\neq\pm n}\mathrm{Re}\int_{\Omega_\lambda}\frac{(z-\overline{z})^2}{z^3\overline{z}^2}[\overline{a_m}\overline{z^{\sigma m}}][z^2 2\mathrm{Re}(\frac{a_n z^{\sigma n-1}}{\sigma n-1}) - 2\mathrm{Re}(\frac{a_n z^{\sigma n+1}}{\sigma n+1})]dE\\
II &= \frac{4}{\pi}\sum_{n=1}^{\infty}\mathrm{Re}\int_{\Omega_\lambda}\frac{(z-\overline{z})^2}{z^3\overline{z}^2}[\overline{a_n z^{\sigma n} + a_{-n}z^{-\sigma n}}]\\
&\quad [z^2 2\mathrm{Re}\{\frac{a_n z^{\sigma n-1}}{\sigma n-1} + \frac{a_{-n}z^{-\sigma n-1}}{-\sigma n-1}\} - 2\mathrm{Re}\{\frac{a_n z^{\sigma n+1}}{\sigma n+1} + \frac{a_{-n}z^{-\sigma n+1}}{-\sigma n+1}\}]dE\\
III &= \frac{4}{\pi}\mathrm{Re}\int_{\Omega_\lambda}\frac{(z-\overline{z})^2}{z^3\overline{z}^2}[\overline{a_0}][-z^2 2\mathrm{Re}(a_0 z^{-1}) - 2\mathrm{Re}(a_0 z)]dE.
\end{aligned}
$$

Recall that in all the expressions, $z^\sigma = e^{-2\pi\theta/\log\lambda}(\cos(2\pi\rho) + i\sin 2\pi\rho)$ with $\rho = \log r/\log\lambda$. Direct computation leads to orthogonality properties that $I = 0$ as verified in Lemma 4.2, Lemma 4.4 of [W]. Similar computations (pp. 289–290 of [W]) shows that

$$
\begin{aligned}
II &= \frac{8}{\pi}\sum_{i=1}^{\infty}\frac{2}{1+|\sigma n|^2}[\int_1^\lambda\int_0^\pi \frac{1}{r}|a_n e^{i\theta(\sigma n-1)} + \overline{a_{-n}}e^{i\theta(1-\sigma n)}|^2 sin^2\theta d\theta dr\\
&\quad + \int_1^\lambda\int_0^{2\pi}\frac{1}{r}|a_n e^{i\theta(\sigma n+1)} + \overline{a_{-n}}e^{-i\theta(\sigma n+1)}|^2 sin^2\theta d\theta dr]\\
&= \frac{8\log\lambda}{\pi}\sum_{i=1}^{\infty}\frac{2}{1+|\sigma n|^2}[\int_0^\pi |a_n e^{i\theta(\sigma n-1)} + \overline{a_{-n}}e^{i\theta(1-\sigma n)}|^2\\
&\quad + |a_n e^{i\theta(\sigma n-1)} + \overline{a_{-n}}e^{-i\theta(\sigma n+1)}|^2 sin^2\theta d\theta].
\end{aligned}
$$

III can be evaluated by putting $n = 0$ in the summand of the expression II and dividing the resulting expression by 2, or more directly from definition that

$$\begin{aligned} III &= \frac{4}{\pi}\mathrm{Re}\int_{\Omega_\lambda}\{\frac{(z-\overline{z})^2}{z^3\overline{z}^2}[-z^2(\frac{a_0}{z}+\frac{\overline{a_0}}{z})-(a_0z+\overline{a_0z})]\overline{a_0}\} \\ &= \frac{4}{\pi}\mathrm{Re}\int_1^\lambda\int_0^\pi\frac{1}{r^2}(4a_0-2\overline{a_o})\overline{a_o}d\theta rdr \\ &= 8(2|a_0|^2-\mathrm{Re}(a_0^2))\log\lambda. \end{aligned}$$

Summing up the expressions I, II and III, we get the expression in part (3) of the proposition, thereby concluding the proof of Proposition 1. □

We now proceed to the proof of Theorem 1.

Proof of Theorem 1. Let us label the closed geodesic curves in A by $1, \ldots, m$ so that $L_A = \sum_{j=1}^m \ell_j$. L_A is a function defined on the Teichmüller space $\mathcal{T} = \mathcal{T}(S)$. Let X be a Riemann surface in $\mathcal{T}$. A local coordinates around $X \in \mathcal{T}$ can be described as follows. Let $\mu_i, i = 1, \ldots, d$ be a unitary basis of the space of harmonic Beltrami differentials on $X = H/\Gamma$. The tuple $t = (t_1, ..., t_d) \in C^d$ with $|t_j| < 1$ gives a coordinate neighbourhood of X, so that t represents a Riemann surface $X^t = H/(f^{t\mu}\Gamma(f^{t\mu})^{-1}$, where $t\mu$ represents the harmonic Beltrami differential $\sum_{j=1}^d t_j\mu_j$ and $f^{t\mu}$ is the quasi-Fuchsian uniformization explained earlier. t gives rise to locally geodesic coordinates for the Weil-Peterson metric g_{WP} in the sense that the metric is just the Euclidean metric with respect to t up to an error term of order $|t|^2$ (cf. [A2]). Convexity of a function f on $\mathcal{T}$ is equivalent to the positive definiteness of the Hessian $\mathrm{Hess}(f)(\mu, \mu) > 0$ with respect to any tangent vector $\mu \in T_X\mathcal{T}$, while plurisuharmonicity is given by the positive definiteness of the Levi form. Expressing μ as a linear combination of $\mu_i, i = 1, \ldots, d$ and consider a local $1-$parameter family of Riemann surfaces X^ϵ corresponding to $\epsilon\mu, |\epsilon| < 1$, it is clear that

$$\mathrm{Hess}(f)(\mu, \mu) = \frac{d^2 f(X^\epsilon)}{d\epsilon^2}|_{\epsilon=0}.$$

The Levi form on $\mathcal{T}$ is related to the Hessian by

$$\mathrm{Levi}(f)(\mu + J\mu, \mu - J\mu) = \mathrm{Hess}(f)(\mu, \mu) + \mathrm{Hess}(f)(J\mu, J\mu),$$

where J is the complex structure on $\mathcal{T}$. In our situation, a tangent vector in $\mathcal{T}_X$ is represented by a harmonic differential μ. The complex structure is given by $J\mu = i\mu$, where i is the complex structure on the Riemann surface X.

Denote by $\nu = \mu + i\mu$ a tangent vector of $(1,0)$ type on $\mathcal{T}$. In terms of the expansion of μ in Proposition 1, we conclude that

$$\begin{aligned}
& \mathrm{Levi}(\ell_\gamma)(\nu,\overline{\nu}) \\
= \;& \mathrm{Hess}(\ell_\gamma)(\mu,\mu) + \mathrm{Hess}(\ell_\gamma)(i\mu,i\mu) \\
\geq \;& 8\log\lambda(2|a_0|^2 - \mathrm{Re}(a_0^2)) + 8\log\lambda(2|a_0|^2 - \mathrm{Re}((ia_0)^2)) \\
= \;& 32\log\lambda|a_0|^2 \\
= \;& 2[(4\log\lambda\mathrm{Re}(a_0))^2 + (4\log\lambda\mathrm{Im}(a_0))^2] \\
= \;& \frac{2}{\ell_\gamma}|\nabla_\nu \ell_\gamma|^2.
\end{aligned}$$

It follows that

$$\begin{aligned}
& L_A\mathrm{Levi}(L_A)(\nu,\overline{\nu}) - 2|\nabla_\nu L_A|^2 \\
= \;& [\sum_{j=1}^m \ell_{\gamma_j}][\sum_{k=1}^m \mathrm{Levi}(\ell_{\gamma_k})(\nu,\overline{\nu})] - 2|\sum_{j=1}^m \nabla_\nu \ell_{\gamma_j}|^2 \\
\geq \;& [\sum_{j=1}^m \ell_{\gamma_j}][\sum_{k=1}^m \frac{2}{\ell_{\gamma_k}}|\nabla_\nu \ell_{\gamma_k}|^2] - 2|\sum_{j=1}^m \nabla_\nu \ell_{\gamma_j}|^2 \\
= \;& 2\sum_{j=1}^m |\frac{\ell_{\gamma_k}\nabla_\nu\ell_{\gamma_j} - \ell_{\gamma_j}\nabla_\nu\ell_{\gamma_k}}{\sqrt{\ell_{\gamma_j}\ell_{\gamma_k}}}|^2 \\
\geq \;& 0.
\end{aligned}$$

Hence

$$\begin{aligned}
& \mathrm{Levi}(-L_A^{-\alpha})(\nu,\overline{\nu}) \\
= \;& \alpha[\frac{L_A\mathrm{Levi}(L_A)(\nu,\overline{\nu}) - (\alpha+1)|\nabla_\nu L_A|^2}{L_A^{\alpha+2}}] \\
\geq \;& \alpha(1-\alpha)[\frac{\mathrm{Levi}(L_A)(\nu,\overline{\nu})}{L_A^{\alpha+1}}]
\end{aligned}$$

from the above estimates. From Proposition 1 again, we conclude that $\mathrm{Levi}(L_A)$ $(\nu,\overline{\nu})$ and hence $\mathrm{Levi}(-L_A^{-\alpha})(\nu,\overline{\nu})$ is positive definite for $0 < \alpha < 1$. From Lemma 1, we know that L_A is an exhaustion function, $L_A(x) \to \infty$ as $x \to \partial\mathcal{T}$, and is proper. It follows that $-L_A^{-\alpha}(x) \to 0$ as $x \to \partial\mathcal{T}$. As L_A is continuous, it is bounded away from 0 on an compact set of $\mathcal{T}$ and hence on $\mathcal{T}$ since it blows up near infinity. Therefore $-L_A^{-\alpha}$ is a bounded strictly plurisubharmonic exhaustion function. This concludes the proof of Theorem 1. $\square$

Proof of Corollary 1. We recall the following standard techniques from L^2-estimates (cf. [Hö]). Let M be a Kähler manifold with a complete Kähler metric ω and canonical line bundle K_M. Let φ be a function on M. Let (L,h) be a hermitian line bundle on M. Assume that

$$c_1(L,h) + \sqrt{-1}\partial\bar{\partial}\varphi - c_1(K_M) > c(x)\omega \tag{1}$$

for some positive function $c(x)$. Let g be a $\bar\partial$-closed $L-$valued $(0,1)-$form on M with $\int_M \frac{1}{c}\|g\|_h^2 e^{-\varphi} < \infty$. Then the equation $\bar\partial f = g$ has a solution satisfying the L^2- estimate

$$\int_M \|f\|_h^2 e^{-\varphi} < \int_M \frac{\|g\|_h^2 e^{-\varphi}}{c}. \tag{2}$$

To prove Part (a) of Corollary 1, we use h_L to denote the Hermitian metric of L on $\mathcal{T}$. Let $x \in \mathcal{T}$. To construct a holomorphic section of $K_{\mathcal{T}} + L$ non-vanishing at x, we choose a small complex coordinate neighbourhood U_x around x so that both K and L are trivialized by local sections $e_{K,x}$ and $e_{L,x}$ on U_x. Let V_x be a smaller neighbourhood of x so that the closure of V_x is contained in U_x. Let χ_x be a C^∞ cut-off function on $\mathcal{T}$ so that $0 \leq \chi_x \leq 1$ on $\mathcal{T}$, its support is contained in U_x and it is identically 1 on V_x. Let $N = 3g - 3 + n$ be the complex dimension of $\mathcal{T}$ and $\eta_x = (\log \sum_{i=1}^N |z_i|^2)\chi_x$, where z_i are the coordinate functions on U_x and η_x is defined to be identically 0 outside of U_x. $\chi e_{K,x} \otimes e_{L,x}$ is a C^∞ section on U_x and can be regarded as a section on $\mathcal{T}$ after extending by 0 outside of U_x. Choose c to be a positive number large enough so that $\sqrt{-1}c\partial\bar\partial(-L_A^{-\alpha}) + \sqrt{-1}N\partial\bar\partial\eta_x|_{U_x - V_x}$ is positive and moreover greater than $c_1\omega$ for some positive function $c_1(x)$ on $\mathcal{T}$, where α is a fixed number between 0 and 1. This is possible since $\sqrt{-1}N\partial\bar\partial\eta_x$ has compact support. It follows easily that equation (1) is satisfied with $\varphi = -cL_A^{-\alpha} + N\eta_x$, corresponding to

$$\begin{aligned} & c_1(K_{\mathcal{T}}) + c_1(L, h_L) + \sqrt{-1}c\partial\bar\partial(-L_A^{-\alpha}) + \sqrt{-1}N\partial\bar\partial\eta_x - c_1(K_{\mathcal{T}}) \\ \geq \quad & \sqrt{-1}c\partial\bar\partial(-L_A^{-\alpha}) + (N\sqrt{-1}\partial\bar\partial\eta_x)|_{U_x - V_x} + (\sqrt{-1}N\partial\bar\partial(\log \sum_{i=1}^N |z_i|^2))|_{V_x} \\ > \quad & c_1(x)\omega, \end{aligned}$$

where we have used the fact that $c_1(L, h_L) \geq 0$. Letting $g = \overline{\partial}(\chi e_{K,x} \otimes e_{L,x})$ and noting the right hand side of (2) is finite since the section involved is compactly supported, we apply the L^2-estimate above to obtain a section f of $K + L$ satisfying $\overline{\partial} f = g$, $\int_{\mathcal{T}} \|f\|^2 e^{-\varphi} \leq \int_{\mathcal{T}} \frac{1}{c_1}\|g\|^2 e^{-\varphi} < \infty$. It follows from the log-pole of φ that $f(x) = 0$. Since $(\chi e_{K,x} \otimes e_{L,x})(x) = (e_{K,x} \otimes e_L)(x) \neq 0$, $f - \chi e_{K,x} \otimes e_{L,x}$ is a non-trivial holomorphic section of $K + L$. It is furthermore L^2-bounded with respect to the weight $-\varphi = cL_A^{-\alpha} - N\eta_x$ and hence without any weight since both $L_A^{-\alpha}$ and $-\eta$ are bounded from below on $\mathcal{T}$. Hence there exists a L^2-holomorphic section of $K + L$ non-vanishing at x.

The proofs for the separation of points and generation of jets are quite similar and hence only the modification is sketched here. To prove that the L^2-holomorphic sections separate points on $\mathcal{T}$, it suffices to prove that there exists a section non-zero at x but zero at $y \neq x$ for arbitrary points x and y on $\mathcal{T}$. For this purpose, we choose $g = \overline{\partial}[\chi e_{K,x} \otimes e_{L,x} + (1 - \chi)e_{K,y} \otimes e_{L,y}]$, $\eta_{x,y} = \eta_x + \eta_y$ and c large enough so that $\varphi = -cL_A^{-\alpha} + N\eta_x + N\eta_y$ satisfies condition (1). The L^2 estimate yields a L^2 holomorphic section of $K + L$ taking value $-e_{K,x} \otimes e_{L,x}$ at x but vanishes at y.

Similarly, to prove that the holomorphic sections generate an arbitrary jet $\frac{\partial}{\partial z^{i_1}} \cdots \frac{\partial}{\partial z^{i_k}}$ at $x \in \mathcal{T}$, we replace $\varphi = -cL_A^{-\alpha} + N\eta_x$ by $\varphi = -cL_A^{-\alpha} + (k+N)\eta_x$ for sufficiently large c to make sure that condition (1) still holds. Then we let $g = \overline{\partial}[z^{i_1} \cdots z^{i_k} \chi e_{K,x} \otimes e_{L,x}]$ and apply the the same argument as before to solve $\overline{\partial} f = g$. Letting $f_1 = f - [z^{i_1} \cdots z^{i_k} \chi e_{K,x} \otimes e_{L,x}]$, we get a L^2-holomorphic section with $\frac{\partial}{\partial z^{i_1}} \cdots \frac{\partial}{\partial z^{i_k}} f_1|_x \neq 0$. This concludes the proof of (a).

To prove (b), we assume that $\mathcal{M}$ is equipped with a complete Kähler metric with bounded geometry, such as the ones constructed by McMullen [Mc]. We then apply Atiyah's Covering Index Theorem [A] with modification by Cheeger-Gromov [CG] to relate $\chi_{L^2}(\mathcal{T}, K+L) = \chi(\mathcal{M}, K+L)$. The original version of the Covering Index Theorem of Atiyah is stated for a compact manifold. Here the moduli space is non-compact and we have to use the results of Cheeger-Gromov [CG] to make sure that the covering theorem works for complete non-compact manifolds of bounded geometry. The result of Cheeger-Gromov [CG] also implies that $\chi(\mathcal{M}, K+L)$ is just the usual alternate sum $\sum_{i=0}^m h^i_{L^2}(\mathcal{M}, K+L)$ as in the compact case. The interests in [CG] are mainly for the usual Betti numbers but the arguments work for cohomology groups of differential forms twisted by a line bundle of bounded curvature. Kodaira's Vanishing Theorem or the above L^2-estimates implies that $h^i_{L^2}(\mathcal{T}, K+L) = 0$ for $1 \leq i \leq m$. Considering a finite cover $\mathcal{M}_1$ of $\mathcal{M}$ corresponding to moduli space of curves with a certain level structure (cf. [HM]) so that $\mathcal{M}_1$ is smooth and pulling back differential forms from $\mathcal{M}$ to $\mathcal{M}_1$, we conclude by the same argument that $h^i_{L^2}(\mathcal{M}, K+L) = 0$ for $1 \leq i \leq m$ as well.

Combining the above results and conclusion of part (a), we conclude that

$$h^0_{L^2}(\mathcal{M}, K+L) = \chi(\mathcal{M}, K+L) = \chi_{L^2}(\mathcal{T}, K+L) = h^0_{L^2}(\mathcal{T}, K+L) > 0.$$

This concludes the proof of Corollary 1. □

Proof of Corollary 2. Recall that a domain in $\mathbf{C}^N$ is said to be hyperconvex if there exists a continuous plurisubharmonic exhaustion function from the domain to $(-\infty, 0)$. It is a theorem of Herbort [He] and Biocki-Pflug [BP] that a bounded hyperconvex domain in $\mathbf{C}^N$ is complete with respect to the Bergman metric. Hence completeness of Bergman metric on $\mathcal{T}$ follows directly from Theorem 1 and the observation here.

For the estimate of the Bergman metric, we notice that Corollary 1a implies that the Bergman metric and its infinitesemal form are non-degenerate everywhere. It is also well-known that the Carathéodory metric and its infinitesemal form of $\mathcal{T}$ are non-degenerate and complete as shown in Earle [E]. From the work of Hahn [Ha], the infinitesemal Bergman metric metric is bounded from below by the infinitesemal Carathéodory metric at any point on $\mathcal{T}$. The conclusion follows. □

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