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FAMILIES OF $K3$ SURFACES OVER CURVES REACHING THE ARAKELOV-YAU TYPE UPPER BOUNDS AND MODULARITY

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Let $f : X \to C$ be a family of semi-stable curves of genus g over a smooth projective C of genus q, and $S \subset C$ the degeneration locus of f. The so-called Arakelov inequality states that

$$\deg f_*\omega_{X/C} \leq \frac{g}{2} \deg \Omega^1_C(\log S) = \frac{g}{2}(2q - 2 + \#S).$$

When $g \geq 2$ and $\#S = 0$, the Miyaoka-Yau inequality for surfaces implies a much stronger inequality

$$\deg f_*\omega_{X/C} \leq \frac{g-1}{6} \deg \Omega^1_C.$$

In general, Tan [28] proved that the Arakelov inequality for a family $f : X \to C$ of semi-stable curves of genus ≥ 2 holds strictly.

If $g = 1$, then $\deg f_*\omega_{X/C}$ can reach the upper bound in the inequality. Beauville has classified such families over $C = \mathbb{P}^1$ with $\#S = 4$. More precisely, there are exactly 6 non-isotrivial families of semi-stable elliptic curves over $\mathbb{P}^1$ with 4 singular fibres. All of them are modular families of elliptic curves [2].

In this paper, we will consider the similar question for families of higher dimensional varieties. The Arakelov inequality is a special case of some more general inequalities for Hodge bundles. To state them, let $\mathbb{V}$ denote a polarized real variation of Hodge structure on a smooth projective curve $C \setminus S$ such that the local monodromies around S are all unipotent, let

$$(\oplus_{p+q=k} E^{p,q}, \theta)$$

denote the corresponding Hodge bundles. In [8] the following Arakelov-Yau type inequality was proven (also see [18] for a similar inequality):

If $k = 2l + 1$, then

$$\deg E^{k,0} \leq \left(\frac{1}{2}(h^{k-l,l} - h_0^{k-l,l}) + \sum_{j=0}^{l-1} (h^{k-j,j} - h_0^{k-j,j}) \right) \cdot \deg(\Omega^1_C(\log S)).$$

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If $k = 2l$,

$$\deg E^{k,0} \le \sum_{j=0}^{l-1} (h^{j,k-j} - h_0^{j,k-j}) \cdot \deg(\Omega^1_C(\log S)).$$

These inequalities generalize the original Arakelov inequality for a family $f : A \to C$ of semi-stable abelian varieties due to Deligne. In general, Yau [31] proved the so-called Yau's Schwarz type inequality, which can be formulated as follows. Let (M, ds) be a Hermitian manifold with holomorphic sectional curvature bounded above by a negative constant K, and let $(C \setminus S, ds_\mu)$ be a Poincare type metric. Then there exists a positive constant c, such that for any holomorphic map $\phi : C \setminus S \to M$, one has $\phi^* ds \le c ds_\mu$. It is the reason why we call such inequalities are of Arakelov-Yau type.

We now consider a family $f : X \to C$ of semi-stable algebraic K3 surfaces. Let X^0 denote the largest subscheme where f is smooth and projective, and assume that $R^2 f_* \mathbb{Z}_{X^0}$ extends to a local system $\mathbb{V}$ on $C \setminus S$. We call S the singularities of $R^2 f_* \mathbb{Z}_X$ and write $\Delta := f^*(S)$, which is a normal crossing *divisor*. Then the corresponding Hodge bundles read

$$(f_* \omega_{X/C} \oplus R^1 f_* \Omega^1_{X/C}(\log \Delta) \oplus R^2 f_*(\mathcal{O}_X), \theta).$$

If f is non-isotrivial, it is known that $f_* \omega_{X/C}$ is ample on C by Fujita [6] (also see [9] [29] for higher dimensional base). Applying the above Arakelov-Yau type inequality for Hodge bundles of weight-2 one obtains

$$\deg f_* \omega_{X/C} \le \deg \Omega^1_C(\log S). \tag{0.0.1}$$

If the iterated Kodaira-Spencer map of this family is zero, one shows a stronger inequality

$$\deg f_* \omega_{X/C} \le \frac{1}{2} \deg \Omega^1_C(\log S). \tag{0.0.2}$$

In this note we shall study non-isotrivial algebraic families of semi-stable K3 surfaces over curves when the inequality (0.0.1), or (0.0.2) becomes an equality. One shows in Theorem 0.1 below that such a numerical equality has a strong consequence for the geometry of the generic fibre. The corresponding question has been considered in [30] for families of abelian varieties. The final presentation of this note has been influenced by [30]. It has also been motivated by Mok's work on rigidity theorems of locally Hermitian symmetric spaces [13] and [14], where he use the Gaussian curvature of the induced metric on a holomorphic curves in a locally Hermitian symmetric space to characterize when this curve will be a totally geodesic embedding.

To state the main result, we recall some notation. Let $a : A^0 \to C^0$ be a family of abelian surfaces with a section, then the desingularization $Z^0 \to C^0$ of the quotient $A^0/\{\pm 1\} \to C^0$ is a family of Kummer surfaces (the so called Kummer construction). The rational map $A^0 \to Z^0$ is called a rational quotient of A^0. The family $a : A^0 \to C^0$ is called the associated family of abelian surfaces of $Z^0 \to C^0$. In general, it is not true that every family of Kummer surfaces has an associated family of abelian surfaces.

An involution $\imath$ on a $K3$ surface X is called a Nikulin involution if $\imath^*\omega = \omega$ for every $\omega \in H^0(X, \Omega^2_X)$. It is known (Nikulin [17]) that every Nikulin involution $\imath$ has eight isolated fixed points, and the rational quotient $X \to Z$ by $\imath$ is a $K3$ surface.

Theorem 0.1. *Let $f : X \to C$ be a family of semi-stable $K3$ surfaces over C, and $S \subset$ the singular locus of $\mathbb{V} =: R^2 f_*(\mathbb{Z}_{X^0})$. If $S \neq \emptyset$ and if $\deg f_*\omega_{X/C}$ reaches the Arakelov bound in* (0.0.1)*, then the following properties hold true:*

- **a)** *The general fibre of $f : X \to C$ has Picard number* 19.
- **b)** *There exist a finite étale cover $\sigma : C' \to C$, a Zariski open set $C'^0 \subset C'$ and a global Nikulin involution $\imath$ on $f : X^0 = f^{-1}(C'^0) \to C'^0$ such that the rational quotient $X^0 \to Z^0$ by $\imath$ is a family of Kummer surfaces over C'^0, which has an associated family of abelian surfaces that is isogenous to the square product of a family of elliptic curves $g : E \to C'$.*
- **c)** *The projective monodromy representation of the local system $R^1 g_*(\mathbb{Z}_{E^0})$ extends to*
$$\tau : \pi_1(C' \setminus \sigma^{-1}S, *) \to PSL_2(\mathbb{Z})$$
such that
$$C' \setminus \sigma^{-1}S \simeq \mathcal{H}/\tau\pi_1(C' \setminus \sigma^{-1}S, *).$$

A family of $K3$ surfaces satisfying Property b) will be called a family coming from Nikulin-Kummer construction of the square product of a family of elliptic curves.

Theorem 0.2. *If the second iterated Kodaira-Spencer map of the family $f : X \to C$ is zero and if the family reaches the Arakelov bound in* (0.0.2)*. Then then the following properties hold true:*

- **a)** *The general fibres of $f : X \to C$ have the Picard number at least* 18.
- **b)** *After passing to a finite étale cover $\sigma : C' \to C$, the monodromy representation ρ of $R^2 f_*(\mathbb{Z}_{X^0})$ is of the form*
$$\rho = \textit{trivial rank-2 representation} \otimes (\tau : \pi_1(C' \setminus \sigma^{-1}S, *) \to SL_2(\mathbb{Z})),$$
$$\textit{and} \qquad C' \setminus \sigma^{-1}S \simeq \mathcal{H}/\tau\pi_1(C' \setminus \sigma^{-1}S, *).$$

Remark 0.3.

i) Theorem 0.1 can be used to explain the observation of B. Lian and S.-T. Yau ([10], [11]) that the weight-2 VHS attached to a certain one dimensional families of $K3$ surfaces coming from the Mirror of $K3$ surfaces of Picard number ≥ 1 can be expressed as the square products of the weight 1 VHS attached to a certain one dimensional families of elliptic curves (also see [5]). Note that such a family must reach the Arakelov bound in (0.0.1). We thank A. Todorov for pointing that out to us. Note that, if $S = \emptyset$ then there is another type families of $K3$ surfaces reaching the Arakelov bound (0.0.1). Namely, let $a : A \to C$ be a modular family of false elliptic curves, i.e. abelian surface whose endomorphism ring is isomorphic to an order of an indefinite quaternion algebra over $\mathbb{Q}$ ([26]). Then the Kummer construction gives rise to a family $f : X \to C$ of smooth $K3$ surfaces reaching the Arakelov bound (0.0.1), and C is a Shimura curve. One likes to know what is the mirror pair of this family.

ii) For a family $f : X \to C$ as in Theorem 0.2 one can find a family $f' : X' \to C$, which comes from the Nikulin-Kummer construction of a product of a modular family of elliptic curves $g : E_1 \to C$ with an elliptic curve E_2 over $\mathbb{C}$, and such that sub VHSs of transcendental lattices of f and f' are Hodge isometric to each other. Are there closer geometric relations among these families?

Let $f : X \to \mathbb{P}^1$ be a Calabi-Yau 3-fold fibred by non-constant families of semi-stable $K3$ surfaces. The triviality of ω_X implies that $\deg f_*\omega_{X/\mathbb{P}^1} = 2$.

Corollary 0.4. *Let $f : X \to \mathbb{P}^1$ be a Calabi-Yau 3-fold fibred by non-constant semi-stable $K3$ surfaces. Then the followings hold true:*

i) *If the iterated Kodaira-Spencer map of f is non-zero, then f has at least 4 singular fibres. If f has 4 singular fibres, then X is rigid and birational to the Nikulin-Kummer construction of a square product of a family of elliptic curves $g : E \to \mathbb{P}^1$. After passing to (if necessary) a double cover $E' \to E$, the family $g' : E' \to \mathbb{P}^1$ is one of the 6 modular families of elliptic curves constructed by Beauville.*

ii) *If the iterated Kodaira-Spencer map of f is zero, then f has at least 6 singular fibres. If f has 6 singular fibres over $S \subset \mathbb{P}^1$, then X is non-rigid, the general fibres have Picard number 18, and $\mathbb{P}^1 \setminus S \simeq \mathcal{H}/\Gamma$, where Γ is a subgroup of $SL_2(\mathbb{Z})$ of index 24.*

Remark 0.5.

i) Any $K3$-fibred Calabi-Yau 3-fold $f : X \to \mathbb{P}^1$ in 0.4, i) is rigid because of the modular construction for X. Since all 6 examples of Beauville are defined over $\mathbb{Z}$, we may assume that X has a suitable integral model. The $L-$series of X is defined to be the $L-$series of the Galois representation on $H^3_{et}(\bar{X}, \mathbb{Q})$. One should be able to verify the so-called modularity conjecture for X. M.-H. Saito and N. Yui [20] checked for one example that up to a finite Euler factor, $L(X, s) = L(f, s)$ for $f \in S_4(\Gamma_0(N))$.

ii) Does any rigid Calabi-Yau 3-fold fibred by semi-stable $K3$ surfaces come from the modular construction in 0.4, i)?

iii) One can construct an example for the case ii) of Corollary 0.4. Let

$$g : E(4) \to X(4)$$

be the modular family of elliptic curves corresponding to the congruence group $\Gamma(4)$. Then $X(4) \simeq \mathbb{P}^1$ with six cusps, and $\deg g_*\omega_{E(4)/X(4)} = 2$. The Nikulin-Kummer construction applied to the product of $g : E(4) \to X(4)$

with a constant family of elliptic curves gives an $K3$ fibred Calabi-Yau 3-fold reaching the upper bound in (0.0.2), which is non-rigid.

1. Weight-2 VHS and $\mathbb{R}-$Splitting

Let $f : X \to C$ be a family of semi-stable K3 surfaces. Consider its weight-2 variation of Hodge structure (VHS for simplicity)

$$\mathbb{V}_0 = R^2 f_*(\mathbb{Z}_{X^0}).$$

Let $S \subset C$ denote the subset, where the local monodromies of $\mathbb{V}_0$ are non-trivial, hence of infinite order and $\Delta := f^*(S)$. We will write $\mathbb{V}$ for the extension of $\mathbb{V}_0$ to $C \setminus S$. One has the canonical extension of Hodge bundles

$$E^{p,q} = R^q f_*(\Omega^p_{X/C}(\log \Delta), \quad p + q = 2,$$

together with the cup product of Kodaira-Spencer map

$$\theta^{p,q} : E^{p,q} \to E^{p-1,q+1} \otimes \Omega^1_C(\log S).$$

$\theta = \theta^{2,0} + \theta^{1,1}$ is called the Higgs field of $\mathbb{V}$.

Lemma 1.1. *We have* $\deg E^{2,0} \leq \deg \Omega^1_C(\log S)$*, and if the equality*

$$\deg E^{2,0} = \deg \Omega^1_C(\log S)$$

holds, then there is a real splitting $\mathbb{V} \otimes \mathbb{R} = \mathbb{W}_\mathbb{R} \oplus \mathbb{U}_\mathbb{R}$*, which is orthogonal w.r.t. the polarization, and* $\mathbb{U}$ *is unitary. The corresponding Higgs bundle splitting is*

$$(E^{2,0} \oplus E^{1,1}_1 \oplus E^{0,2}, \theta) \oplus (E^{1,1}_2, 0)$$

where $E^{1,1} = E^{1,1}_1 \oplus E^{1,1}_2$ *and* $E^{1,1}_1$ *is a line bundle of degree zero such that*

$$\theta : E^{2,0} \to E^{1,1}_1 \otimes \Omega^1_C(\log S), \quad \theta : E^{1,1}_1 \to E^{0,2} \otimes \Omega^1_C(\log S)$$

are isomorphisms.

Proof. We consider the map $\theta^{1,1} : E^{1,1} \to E^{0,2} \otimes \Omega^1_C(\log S)$, and let $E^{1,1}_2 \subset E^{1,1}$ denote the kernel of $\theta^{1,1}$. Then $(E^{1,1}_2, 0)$ is a Higgs sub-bundle.

Claim. $\deg E^{1,1}_2 \leq 0$, and if the equality holds then the Higgs subbundle

$$(E^{1,1}_2, 0) \subset (E, \theta)$$

induces a splitting $(E, \theta) = (E^{2,0} \oplus E^{1,1}_1 \oplus E^{0,2}, \theta) \oplus (E^{1,1}_2, 0)$, which corresponds to a $\mathbb{C}-$splitting of the local system $\mathbb{V} \otimes \mathbb{C} = \mathbb{W}_\mathbb{C} \oplus \mathbb{U}_\mathbb{C}$.

Proof of the claim. Let h denote the Hodge metric on $E|_{C\setminus S}$, and let $\Theta(E|_{C\setminus S}, h)$ be its curvature form. Then we have ([7], Chapter II)

$$\Theta(E|_{C\setminus S}) + \theta \wedge \bar{\theta} + \bar{\theta} \wedge \theta = 0,$$

where $\bar{\theta}$ is the complex conjugation of θ with respect to h. Consider the $\mathcal{C}^\infty$ $h-$orthogonal decomposition $E|_{C\setminus S} = E^{1,1}_2|_{C\setminus S} \oplus E^{1,1}_2|_{C\setminus S}{}^{\perp}$. One has

$$\Theta(E^{1,1}_2|_{C\setminus S}, h) = \Theta(E|_{C\setminus S}, h)|_{E^{1,1}_2} + \bar{A} \wedge A = -(\theta \wedge \bar{\theta})|_{E^{1,1}_2} - (\bar{\theta} \wedge \theta)|_{E^{1,1}_2} + \bar{A} \wedge A,$$

where $A \in A^{1,0}(\mathrm{Hom}(E_2^{1,1}, E_2^{1,1\perp}))$ is the second fundamental form of the sub-bundle $E_2^{1,1} \subset E$ and $\bar{A}$ is the complex conjugation with respect to h. Since $\theta(E_2^{1,1}) = 0$, we have $(\bar{\theta} \wedge \theta)|_{E_2^{1,1}} = 0$. Hence

$$\Theta(E_2^{1,1}|_{C \setminus S'}, h) = -(\theta \wedge \bar{\theta})_{E_2^{1,1}} + \bar{A} \wedge A.$$

$\Theta(E_2^{1,1}|_{C\setminus s'}, h)$ is negative semidefinite since $\theta \wedge \bar{\theta}_{E_2^{1,1}}$ is positive semidefinite and $\bar{A} \wedge A$ is negative semidefinite. Since the local monodromies around points in S are unipotent, $\mathrm{Tr}\,\Theta(E_2^{1,1}|_{C\setminus S'}, h)$ represents (by [22]) the Chern class $c_1(E_2^{1,1})$ as a current. Thus

$$\deg E_2^{1,1} = \int_{C\setminus S} \mathrm{Tr}\Theta(E_2^{1,1}|_{C\setminus S}, h) \leq 0,$$

and $\Theta(E_2^{1,1}|_{C\setminus S}, h) = 0$ if $\deg E_2^{1,1} = 0$. This implies that $\bar{\theta}(E_2^{1,1}) = 0$ and $A = 0$. Altogether this shows that the sub-Higgs bundle $(E_2^{1,1}, 0)$ of (E, θ) induces a splitting of the Higgs bundle

$$(E, \theta) = (E^{2,0} \oplus E_1^{1,1} \oplus E^{0,2}, \theta) \oplus (E_2^{1,1}, 0)$$

and the corresponding splitting $\mathbb{V} \otimes \mathbb{C} = \mathbb{W}_{\mathbb{C}} \oplus \mathbb{U}_{\mathbb{C}}$ of the complex local system. Thus the claim is proved. $\square$

Let $I \subset E^{0,2} \otimes \Omega^1_C(\log S)$ denote the image of $\theta^{1,1}$. Then the exact sequence

$$0 \to E_2^{1,1} \to E^{1,1} \to I \to 0,$$

together with $\deg E^{1,1} = 0$ implies that

$$-\deg E_2^{1,1} = \deg I.$$

Hence,

$$-\deg E^{2,0} + \deg \Omega^1_C(\log S) = \deg(E^{0,2} \otimes \Omega^1_C(\log S)) \geq \deg I = -\deg E_2^{1,1} \geq 0.$$

Thus the inequality $\deg E^{2,0} \leq \deg \Omega^1_C(\log S)$ becomes an equality if and only if $\deg E_2^{1,1} = 0$ and $I = E^{0,2} \otimes \Omega^1_C(\log S)$, which is our $E_1^{1,1}$. It is easy to see that the Higgs field of $\mathbb{W}_{\mathbb{C}}$ is an isomorphism, thus $\mathbb{W}_{\mathbb{C}}$ is irreducible over $\mathbb{C}$. Now we only need to show that the decomposition $\mathbb{V} \otimes \mathbb{C} = \mathbb{W}_{\mathbb{C}} \oplus \mathbb{U}_{\mathbb{R}}$ can be, in fact, defined over $\mathbb{R}$. Taking the complex conjugation on $\mathbb{W}_{\mathbb{C}}$ one has

$$\overline{\mathbb{W}_{\mathbb{C}}} \subset \overline{\mathbb{V} \otimes \mathbb{C}} = \mathbb{V} \otimes \mathbb{C}.$$

$\bar{\mathbb{W}}_{\mathbb{C}}$ is again of the Hodge type $(2,0) + (1,1) + (0,2)$, irreducible and with non-zero Higgs field. The projection $p : \overline{\mathbb{W}_{\mathbb{C}}} \subset \mathbb{V} \otimes \mathbb{C} \to \mathbb{U}_{\mathbb{C}}$ can not be injective since $\mathbb{U}_{\mathbb{C}}$ is unitary. Moreover, since $\bar{\mathbb{W}}_{\mathbb{C}}$ can not have a proper sub local system, this projection must be zero. Thus $\overline{\mathbb{W}_{\mathbb{C}}} = \mathbb{W}_{\mathbb{C}}$ and we obtain a real sub local system $\mathbb{W}_{\mathbb{R}} \subset \mathbb{V} \otimes \mathbb{R}$. The intersection form restricted to $\mathbb{W}_{\mathbb{R}}$ is non-degenerated. Thus the orthogonal complement of $\mathbb{W}_{\mathbb{R}}$ with respect to the intersection form gives the desired real decomposition $\mathbb{V} \otimes \mathbb{R} = \mathbb{W}_{\mathbb{R}} \oplus \mathbb{U}_{\mathbb{R}}$. $\square$

Lemma 1.2. *If the iterated Kodaira-Spencer map* $\theta^{1,1}\theta^{2,0} = 0$, *then*

$$\deg E^{2,0} \leq \frac{1}{2} \deg \Omega^1_C(\log S).$$

When the equality $\deg E^{2,0} = \frac{1}{2} \deg \Omega^1_C(\log S)$ *holds, then there is a real splitting*

$$\mathbb{V} \otimes \mathbb{R} = \mathbb{W} \oplus \mathbb{U},$$

which is orthogonal w.r.t. the polarization, and $\mathbb{U}$ *is unitary. The corresponding Higgs bundle splitting is*

$$(E^{2,0} \oplus (E^{1,1}_1 \oplus E^{1,1^*}_1) \oplus E^{0,2}, \theta) \oplus (E^{1,1}_2, 0)$$

where $E^{1,1}_1$ *and* $E^{1,1^*}_1$ *are sub line bundles of* $E^{1,1}$ *with*

$$\deg E^{1,1}_1 = -\deg E^{2,0} = -\frac{1}{2} \deg \Omega^1_C(\log S),$$

and $E^{1,1} = E^{1,1}_1 \oplus E^{1,1^*}_1 \oplus E^{1,1}_2$. *The Higgs field*

$$\theta : (E^{2,0} \oplus (E^{1,1}_1 \oplus E^{1,1^*}_1) \oplus E^{0,2}) \to (E^{2,0} \oplus (E^{1,1}_1 \oplus E^{1,1^*}_1) \oplus E^{0,2}) \otimes \Omega^1_C(\log S)$$

is defined by $\theta = \tau \oplus -\tau^*$, *where* $\tau : E^{2,0} \simeq E^{1,1}_1 \otimes \Omega^1_C(\log S), \quad E^{1,1}_1 \to 0.$

Proof. Since $\theta^{1,1}\theta^{2,0} = 0$, the map $\theta^{2,0}$ factors through

$$\theta^{2,0} : E^{2,0} \to E^{1,1}_1 \otimes \Omega^1_C(\log S),$$

where $E^{1,1}_1 \subset E^{1,1}$ is a sub-line bundle such that $\theta^{1,1}(E^{1,1}_1) = 0$. Thus

$$(E^{2,0} \oplus E^{1,1}_1, \theta^{2,0}) \subset (E, \theta)$$

is a rank-2 Higgs sub bundle. By the same arguments as in the proof of Lemma 1.1, one has $\deg E^{2,0} \oplus E^{1,1}_1 \leq 0$, thus

$$\deg E^{2,0} \leq \frac{1}{2} \Omega^1_C(\log S).$$

If the equality holds, then $\theta^{2,0} =: \tau : E^{2,0} \to E^{1,1}_1 \otimes \Omega^1_C(\log S)$ is an isomorphism with $\deg E^{1,1}_1 = -\deg E^{2,0} = -\frac{1}{2} \deg \Omega^1_C(\log S)$, and the Higgs sub bundle $(E^{2,0} \oplus E^{1,1}_1, \theta^{2,0}) \subset (E, \theta)$ gives rise to a complex sub local system $\mathbb{W}_1 \subset \mathbb{V} \otimes \mathbb{C}$. The dual $\bar{\mathbb{W}}_1 \subset \mathbb{V} \otimes \mathbb{C}$ corresponds to Higgs subbundle

$$(E^{2,0} \oplus E^{1,1}_1)^* = E^{1,1^*}_1 \oplus E^{0,2}$$

together with the Higgs field $-\tau^* : E^{1,1^*}_1 \to E^{0,2} \otimes \Omega^1_C(\log S)$. The sub-local system $\mathbb{W} := \mathbb{W}_1 \oplus \bar{\mathbb{W}}_1$ is real, and the intersection form restricted to $\mathbb{W}$ is non-degenerated. Hence, the orthogonal complement defines the desired decomposition. $\square$

2. Splitting over $\bar{\mathbb{Q}}$

We start with a very simple observation. Suppose that $\mathbb{V}$ is a local system defined over $\bar{\mathbb{Q}}$. Fixing a positive integer r, let $\mathcal{G}(r, \mathbb{V})$ denote the set of all rank-r sub-local systems of $\mathbb{V}$. Then $\mathcal{G}(r, \mathbb{V})$ is a projective variety defined over $\bar{\mathbb{Q}}$. The following property is well known.

Lemma 2.1. *If $[W] \in \mathcal{G}(r, \mathbb{V})$ is an isolated point, then W is defined over $\bar{\mathbb{Q}}$.*

Lemma 2.2. *The $\mathbb{R}$-splittings $\mathbb{V} \otimes \mathbb{R} = \mathbb{W} \oplus \mathbb{U}$ in Lemma 1.1 and Lemma 1.2 can be defined over $\bar{\mathbb{Q}}$.*

Proof. By Lemma 2.1, one only needs to show that $\mathbb{W}$ is a rigid sub-local system of $\mathbb{V} \otimes \mathbb{C}$. Suppose that there is a family of sub-local systems

$$\{\mathbb{W}_t\}, \quad \mathbb{W}_0 = \mathbb{W}.$$

By semi-continuity, the Higgs fields $\theta^{p,q}$ of $\mathbb{W}_t$ are again isomorphisms for t being sufficiently closed to 0. Then the projection $\mathbb{W}_t \to \mathbb{V} \otimes \mathbb{C} \to \mathbb{U}$ must be zero, otherwise, $\mathbb{W}_t$ would contain a non-trivial unitary component, which contradicts that $\theta^{p,q}$ are isomorphisms. Hence $\mathbb{W}_t = \mathbb{W}$.

Similarly, we show that the sub-local system $\mathbb{W} = \mathbb{W}_1 \oplus \bar{\mathbb{W}}_1 \subset \mathbb{V} = \mathbb{W} \oplus \mathbb{U}$ is rigid. Suppose that there is a family of sub local systems $\{\mathbb{W}_t\}$ with $\mathbb{W}_0 = \mathbb{W}$, we decompose $\mathbb{W}_t$ into the direct sum of irreducible components over $\mathbb{C}$, which has only following possible types up to isomorphism

$$\mathbb{W}_1 \oplus \bar{\mathbb{W}}_1; \quad \mathbb{W}_1 \oplus \mathbb{U}'; \quad \bar{\mathbb{W}}_1 \oplus \mathbb{U}''; \quad \mathbb{U}''',$$

where $\mathbb{U}'$, $\mathbb{U}''$, $\mathbb{U}'''$ are unitary. By semicontinuity, the last three cases are impossible if t is sufficiently closed to 0 (otherwise $\theta^{1,1}$ would be zero). Thus

$$\mathbb{W}_t \simeq \mathbb{W}_1 \oplus \bar{\mathbb{W}}_1,$$

which implies that the projection $\mathbb{W}_t \to \mathbb{V} \otimes \mathbb{C} \to \mathbb{U}$ must be zero. Otherwise, $\mathbb{W}_1$ would contain a non-trivial unitary component, which contradicts that the Higgs fields of $\mathbb{W}$ are isomorphisms. $\square$

3. Splitting over $\mathbb{Q}$ and $\mathbb{Z}$-structures

We call the splitting in Lemma 1.1 of type (0.0.1) and the splitting in Lemma 1.2 of type (0.0.2).

Lemma 3.1. *If $S \neq \emptyset$, the splittings in Lemma 2.2 can be defined over $\mathbb{Q}$.*

Proof. Let $\mathbb{V} \otimes K = \mathbb{W} \oplus \mathbb{U}$ be the splitting of type (0.0.1) in Lemma 2.2, where K is a Galois extension of $\mathbb{Q}$. For any $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$, we claim that $\sigma\mathbb{W} = \mathbb{W}$. Otherwise, the projection $p : \sigma\mathbb{W} \to \mathbb{V} \otimes K \to \mathbb{U}$ must be non-zero and $\sigma\mathbb{W}$ is isomorphic to a unitary sub local system $\mathbb{U}' \subset \mathbb{U}$ under p since $\mathbb{W}$ is irreducible (thus $\sigma\mathbb{W}$ is also irreducible). Let γ be a short loop around $s \in S$. Then the monodromy matrix $\rho_{\mathbb{W}}(\gamma)$ has infinite order, hence $\rho_{\sigma\mathbb{W}}(\gamma)$ has also infinite order, which contradicts that $\rho_{\mathbb{U}'}(\gamma)$ is identity. We proved that $\mathbb{W}$

is invariant under $\mathrm{Gal}(K/\mathbb{Q})$. Hence $\mathbb{W}$ is defined over $\mathbb{Q}$ and the orthogonal complement of $\mathbb{W} \subset \mathbb{V} \otimes \mathbb{Q}$ w.r.t. the intersection form defines an $\mathbb{Q}-$splitting

$$\mathbb{V} \otimes \mathbb{Q} = \mathbb{W} \oplus \mathbb{U}.$$

By the same argument, we show that the splitting of type (0.0.2) in Lemma 2.2 is also defined over $\mathbb{Q}$. □

Lemma 3.2. *After passing to a finite etale cover of C the splittings in Lemma 3.1 induce $\mathbb{Z}-$sub lattices*

$$\mathbb{V} \supset \mathbb{W}_{\mathbb{Z}} \oplus \mathbb{Z}^{\nu},$$

where $\nu = 19$ under the assumptions of 1.1 and $\nu = 18$ under those in 1.2 such that

$$\mathbb{V} \otimes \mathbb{Q} = (\mathbb{W}_{\mathbb{Z}} \oplus \mathbb{Z}^{\nu}) \otimes \mathbb{Q},$$

where $\mathbb{Z}^{\nu}$ is respectively a rank-ν constant $\mathbb{Z}$-lattice of type-(1,1).

Proof. Let $\mathbb{W}_{\mathbb{Z}} = \mathbb{V} \cap \mathbb{W}, \quad \mathbb{U}_{\mathbb{Z}} = \mathbb{V} \cap \mathbb{U}$. It is easy to check that

$$\mathbb{W}_{\mathbb{Z}} \otimes \mathbb{Q} = \mathbb{W}, \quad \mathbb{U}_{\mathbb{Z}} \otimes \mathbb{Q} = \mathbb{U},$$

thus $\mathbb{W}_{Z}$ and $\mathbb{U}_{\mathbb{Z}}$ are lattices in $\mathbb{W}$ and $\mathbb{U}$. Since $\mathbb{U}$ is unitary and carries an $\mathbb{Z}-$structure, the monodromy group of $\mathbb{U}$ is finite. Since the local monodromies of $\mathbb{U}$ around S are trivial, $\mathbb{U}$ extends to a local system on C. Therefore, after passing to the cover corresponding to this monodromy group, $\mathbb{U}$ becomes a constant local system $\mathbb{Z}^{19}$, $\mathbb{Z}^{18}$ respectively. □

Corollary 3.3. *Let $f : X \to C$ be a family of semi-stable $K3$ surfaces over a curve C. When it reaches the upper bound $\deg f_*\omega_{X/C} = \deg \Omega^1_C(\log S)$, then the Picard number of the general fibres is at least 19. If $\theta^{1,1}\theta^{2,0} = 0$ and f reaches the upper bound $\deg f_*\omega_{X/C} = \frac{1}{2}\deg \Omega^1_C(\log S)$, then the Picard number of the general fibres is at least 18.*

4. Nikulin and Kummer construction

Let $f : X \to C$ be a family of semi-stable $K3$ surfaces, which reaches the upper bound $\deg f_*\omega_{X/C} = \deg \Omega^1_C(\log S)$. By Lemma 3.2, after passing to a finite étale cover of C, one has

$$\mathbb{V} \otimes \mathbb{Q} = \mathbb{W} \oplus \mathbb{Q}^{19},$$

where $\mathbb{W}$ is an $\mathbb{C}$-irreducible representation of $\pi_1(C \setminus S, *)$ and $\mathbb{Q}^{19}$ is a constant local system of rank 19 such that $\mathbb{Q}^{19}_t \subset NS(X_t) \otimes \mathbb{Q}$ for any $t \in C \setminus S$. We obtain therefore,

Lemma 4.1. *For any $t \in C \setminus S$, the Picard number $\rho(X_t) \geq 19$ and for any class $s_t \in \mathbb{Q}^{19}_t \subset \mathrm{Pic}(X_t) \otimes \mathbb{Q}$ there is a $\mathbb{Q}$-divisor $D \in \mathrm{Div}(X) \otimes \mathbb{Q}$ such that $D|_{X_t} = s_t$.*

Let Y be an algebraic K3 surface and $H^2(Y,\ Z) = T_Y \oplus NS(Y)$ be the orthogonal decomposition. T_Y is the so called transcendental lattice of Y, which is even and has signature $(2,\, 20 - \rho(Y))$. It is well-known that as lattices

$$H^2(Y,\mathbb{Z}) \cong U^3 \oplus E_8(-1)^2.$$

We recall some results about embeddings of lattices (see [15] and references given there).

Lemma 4.2 (Theorem 2.4 of [12], or Corollary 2.6 of [15]). *Let T be a non-degenerate even lattice of rank r. Then there is a primitive embedding*

$$T \hookrightarrow U^r$$

In particular, if $\rho(X) \geq 19$, then there is a primitive embedding

$$T_X \hookrightarrow U^3.$$

Lemma 4.3. *If $12 < \rho \leq 20$, then every even lattice T of signature $(2, 20-\rho)$ occurs as the transcendental lattice of some algebraic K3 surface and the primitive embedding $T \hookrightarrow U^3 \oplus E_8(-1)^2$ is unique.*

Theorem 4.4 ([15]). *If $\rho(Y) \geq 19$, then there exists a primitive embedding*

$$\varphi : E_8(-1)^2 \hookrightarrow NS(Y) \subset H^2(Y,\mathbb{Z})$$

and a Nikulin involution $\tau : Y \to Y$ such that $\tau^* : H^2(Y,\mathbb{Z}) \to H^2(Y,\mathbb{Z})$ is identity on $(\varphi(E_8(-1)^2)^\perp$.

Proof. By Lemma 4.2, there is a primitive embedding $\phi : T_Y \hookrightarrow U^3$, thus a primitive embedding $\phi \oplus 0 : T_Y \hookrightarrow U^3 \oplus E_8(-1)^2$. By Lemma 4.3 (uniqueness), the above embedding is isomorphic to

$$T_Y = NS(X)^\perp \subset H^2(Y,\mathbb{Z}) \cong U^3 \oplus E_8(-1)^2.$$

Thus, there is a primitive embedding

$$\psi : E_8(-1)^2 \hookrightarrow T_Y^\perp = NS(Y) \subset H^2(Y,\mathbb{Z}).$$

Let $\{c_j^1\}_{1\leq j\leq 8}$ and $\{c_j^2\}_{1\leq j\leq 8}$ be the bases of $E_8(-1)\oplus 0$ and $0 \oplus E_8(-1)$ and

$$g : H^2(Y,\mathbb{Z}) \to H^2(Y,\mathbb{Z})$$

be defined as: $g(\psi(c_j^1)) = \psi(c_j^2)$, $g(\psi(c_j^2)) = \psi(c_j^1)$ and $g(e) = e$ for any $e \in (\psi(E_8(-1)^2))^\perp$. Then, by theorems of Nikulin (see Theorem 5.6 of [Mo]), there is a Nikulin involution $\tau : Y \to Y$ and $w \in W(Y)$ (the group of Picard-Lefschetz reflections) such that $\tau^* = w \cdot g \cdot w^{-1}$. Let

$$\varphi : E_8(-1)^2 \xrightarrow{\psi} H^2(Y,\mathbb{Z}) \xrightarrow{w} H^2(Y,\mathbb{Z}),$$

then $\varphi : E_8(-1)^2 \hookrightarrow NS(Y) \subset H^2(Y,\mathbb{Z})$ is another primitive embedding, and

$$\tau^*(\varphi(c_j^1)) = \varphi(c_j^2), \quad \tau^*(\varphi(c_j^2)) = \varphi(c_j^1), \quad \tau^*(e) = e, \quad \forall e \in (\varphi(E_8(-1)^2))^\perp.$$

$\square$

Let $t_0 \in C\setminus S$ be a point such that the fibre X_{t_0} satisfying $\rho(X_{t_0}) = 19$. Thus,

$$\mathbb{Q}^{19}_{t_0} = NS(X_{t_0}) \otimes \mathbb{Q}.$$

Since the monodromy action of $\pi_1(C\setminus S, t_0)$ on $\mathbb{Q}^{19}_{t_0}$ is trivial, $\varphi(c^1_j)$ and $\varphi(c^2_j)$, $1 \leq j \leq 8$ can be lifted to divisors D^1_j and D^2_j, $1 \leq j \leq 8$ on X. Then we have

Lemma 4.5. *For any $t \in C\setminus S$, let $d^i_{j\,t} = D^i_j|_{X_t} \in H^2(X_t, \mathbb{Z})$. Then $\{d^i_{j\,t}\}_{1\leq j\leq 8}$ $(i = 1, 2)$ generate a sublattice of $H^2(X_t, \mathbb{Z})$, which is isomorphic to $E_8(-1)^2$ such that $E_8(-1)^2 \hookrightarrow H^2(X_t, \mathbb{Z})$ is a primitive embedding, $E_8(-1) \oplus 0$ and $0 \oplus E_8(-1)$ are isomorphic to $\mathbb{Z}\{d^1_{j\,t}, j = 1, ..., 8\}$ and $\mathbb{Z}\{d^2_{j\,t}, j = 1, ..., 8\}$*

Proof. The proof is straightforward. For example, to prove that $\{d^1_{j\,t}\}_{1\leq j\leq 8}$ are $\mathbb{Z}$-linearly independent: if $\sum n_j d^1_{j\,t} = 0$ in $H^2(X_t, \mathbb{Z})$, we claim that $\sum n_j\varphi(c^1_j) = 0$, which will imply the $\mathbb{Z}$-linearly independence of $\{d^1_{j\,t}\}_{1\leq j\leq 8}$. The claim is clear, otherwise there is a $A \in NS(X_{t_0})$ such that $(\sum n_j\varphi(c^1_j),\, A) \neq 0$. Let $\tilde{A}$ be a lifting of A, then

$$\begin{aligned}\left(\sum n_j d^1_{j\,t},\, \tilde{A}|_{X_t}\right) &= \left(\sum n_j D^1_j|_{X_t},\, \tilde{A}|_{X_t}\right) = \left(\sum n_j D^1_j|_{X_{t_0}},\, \tilde{A}|_{X_{t_0}}\right)\\ &= \left(\sum n_j\varphi(c^1_j),\, A\right) \neq 0.\end{aligned}$$

To see that the embedding $E_8(-1)^2 \hookrightarrow H^2(X_t, \mathbb{Z})$ is primitive, let $B \in H^2(X_t, \mathbb{Z})$ be a class with $mB \in \mathbb{Z}\{d^i_{j\,t}, i = 1, 2, j = 1, ..., 8\}$. Since B is invariant under the monodromy, one finds a lifting $\tilde{B}$ of B. Since $\varphi : E_8(-1)^2 \hookrightarrow H^2(X_{t_0}, \mathbb{Z})$ is primitive and $m\tilde{B}|_{X_{t_0}} \in \varphi(E_8(-1)^2)$, $\tilde{B}|_{X_{t_0}} = \sum n^i_j\varphi(c^i_j)$. Then

$$\left(m\left(\tilde{B} - \sum n^i_j D^i_j\right)|_{X_t},\, m\left(\tilde{B} - \sum n^i_j D^i_j\right)|_{X_t}\right) = 0$$

and $(m(\tilde{B} - \sum n^i_j D^i_j)|_{X_t}, H|_{X_t}) = 0$. By Hodge index theorem one obtains $m(\tilde{B} - \sum n^i_j D^i_j)|_{X_t} = 0$, hence, $(\tilde{B} - \sum n^i_j D^i_j)|_{X_t} = 0$. □

Let $E = \bigoplus_{p+q=2} E^{p,q}$ denote the canonical extension of the Hodge bundle associated to the local system $R^2f_*(\mathbb{Z}_{X^0})$, and $\mathrm{End}(E) \to C$ denote the sheaf of endomorphisms of the vector bundle $m(\tilde{B} - \sum n^i_j D^i_j)|_{X_t} = 0E \to C$, which represents the functor

$$\mathrm{End}(E)^\sharp : \{\text{schemes over } C\} \to \{\text{sets}\}$$

where $\mathrm{End}(E)^\sharp(T) = \{\text{bundle morphism } E_T \to \mathbb{E}_T \text{ over } T\}$. For $t \in C \setminus S$, by Lemma 4.5, we can define an isometric involution

$$g_t : H^2(X_t, \mathbb{Z}) \to H^2(\mathbb{X}_t, \mathbb{Z})$$

by $g_t(d^1_{j\,t}) = d^2_{j\,t}$, $g_t(d^2_{j\,t}) = d^1_{j\,t}$, $g_t(e) = e$ for all $e \in \mathbb{Z}\{d^i_{j\,t}\}^\perp$ and $1 \leq j \leq 8$. It is easy to see that $g_t : H^2(X_t, \mathbb{Z}) \to H^2(X_t, \mathbb{Z})$ is a morphism of $\pi_1(C\setminus S)$-modules. Thus, they give rise an involution

$$g : R^2f_*(\mathbb{Z}_{X^0}) \to R^2f_*(\mathbb{Z}_{X^0})$$

of local system, which corresponds to a section $g \in H^0(C \setminus S, \mathrm{End}(E))$.

Lemma 4.6. *The section $g \in H^0(C \setminus S, \mathrm{End}(E))$ defined above can be extended to a section in $H^0(C, \mathrm{End}(E))$, and thus g is an algebraic section.*

Proof. Recall that $R^2 f_*(\mathbb{Z}_{X^0}) \otimes \mathbb{Q} = \mathbb{W} \oplus \mathbb{Q}^{19}$ and the canonical extension of the Hodge bundle corresponding to $R^2 f_*(\mathbb{Z}_{X^0})$ can be written into

$$(E, \theta) = (E_{\mathbb{W}}, \theta) \oplus (\mathcal{O}_C^{19}, 0),$$

where $(E_{\mathbb{W}}, \theta)$ and $(\mathcal{O}_C^{19}, 0)$ are the canonical extension of the Hodge bundles corresponding to $\mathbb{W}$ and $\mathbb{Q}^{19}$ respectively. By the construction of g, it is identity on $\mathbb{W}$ (thus extended to $E_{\mathbb{W}}$), and is well-defined on the constant lattice $\mathbb{Z}^{19}$. Thus it is clear that g can be extended on C. $\square$

Lemma 4.7. *Let H be an ample divisor on X and $g_t : H^2(X_t, \mathbb{Z}) \to H^2(X_t, \mathbb{Z})$ be the Hodge isometry involutions defined above. Then there exists a non-empty Zariski open set $C^0 \subset C \setminus S$ such that $g_t(H|_{X_t})$ is an ample divisor for any $t \in C^0$, In particular, g_t is an effective Hodge isometry for any $t \in C^0$.*

Proof. We may write $H|_{X_{t_0}} = \sum n_j^1 \varphi(c_j^1) + \sum n_j^2 \varphi(c_j^2) + e$, where $e \in \varphi(E_8(-1)^2)^{\perp}$. Let E be a lifting of e and

$$D = \sum_{j=1}^{8} n_j^1 D_j^1 + \sum_{j=1}^{8} n_j^2 D_j^2 + E, \quad \tilde{D} = \sum_{j=1}^{8} n_j^1 D_j^2 + \sum_{j=1}^{8} n_j^2 D_j^1 + E.$$

Then, for any $t \in C \setminus S$, $H|_{X_t} = D|_{X_t}$ and $g_t(D|_{X_t}) = \tilde{D}|_{X_t}$. Thus D is a relative ample divisor on $f^{-1}(C \setminus S)$ and $\tilde{D}|_{X_{t_0}}$ is ample (here we have chosen t_0 such that g_{t_0} is effective). Thus there exists a Zariski open set $C^0 \subset C \setminus S$ such that $\tilde{D}$ is relative ample on $f^{-1}(C^0)$. $\square$

Lemma 4.8. *The g induces an involution $\tau : f^{-1}(C^0) \to f^{-1}(C^0)$ over C^0 such that $\tau_t : X_t \to X_t$ (for $t \in C^0$) are Nikulin involutions with $\tau_t^* = g_t$.*

Proof. Let $\mathcal{L} = D + \tilde{D}$, where D and $\tilde{D}$ are the divisors defined in the proof of Lemma 4.7. Then we know that $\mathcal{L}$ is relative ample on $f^{-1}(C^0)$ and $\mathcal{L}_t = \mathcal{L}|_{X_t}$ is invariant under the involution g_t. Let $\pi : \mathrm{Aut}^{\mathcal{L}}(f^{-1}(C^0)/C^0) \to C^0$ denote the automorphism group scheme, which represents the functor

$$\mathrm{Aut}^{\mathcal{L}}_{f^{-1}(C^0)/C^0}(T) = \left\{ \begin{array}{l} \text{Isomorphisms } h : f^{-1}(C^0) \times_{C^0} T \to f^{-1}(C^0) \times_{C^0} T \\ \text{over } T \text{ such that } h^*(p_T^* \mathcal{L}) = p_T^*(\mathcal{L}) \end{array} \right\}.$$

Thus there exists a universal automorphism

$$\begin{array}{ccc} f^{-1}(C^0) \times_{C^0} \mathrm{Aut}^{\mathcal{L}}(f^{-1}(C^0)/C^0) & \xrightarrow{h} & f^{-1}(C^0) \times_{C^0} \mathrm{Aut}^{\mathcal{L}}(f^{-1}(C^0)/C^0) \\ \tilde{f} \downarrow & & \tilde{f} \downarrow \\ \mathrm{Aut}^{\mathcal{L}}(f^{-1}(C^0)/C^0) & = & \mathrm{Aut}^{\mathcal{L}}(f^{-1}(C^0)/C^0) \end{array}$$

and h^* induces an endomorphism $\pi^* E \to \pi^* E$, which gives a homomorphism

$$\begin{array}{ccc} \mathrm{Aut}^{\mathcal{L}}(f^{-1}(C^0)/C^0) & \xrightarrow{\alpha} & \mathrm{End}(E) \\ \pi \downarrow & & \downarrow \\ C^0 & = & C^0. \end{array}$$

By Torelli theorem of K3 surfaces, α is injective. On the other hand, the fibres of α are isomorphic to group schemes, which are smooth. Thus α is an embedding. By Lemma 4.6 and Lemma 4.7, $g(C^0)$ is algebraic and contained in the image of α, which gives a section of $\pi : \mathrm{Aut}^{\mathcal{L}}(f^{-1}(C^0)/C^0) \to C^0$. That is an automorphism

$$\begin{array}{ccc} f^{-1}(C^0) & \xrightarrow{\tau} & f^{-1}(C^0) \\ f\downarrow & & f\downarrow \\ C^0 & = & C^0 \end{array}$$

such that $\tau_t^* = g_t$ for any $t \in C^0$. Thus τ_t are Nikulin involutions, i.e. $\tau_t^*\omega = \omega$ for any $\omega \in H^{2,0}(X_t)$. □

Since all fibres X_t are algebraic K3 surfaces, the τ_t gives rise a Shioda-Inose structure on X_t by theorems of Morrison (see Theorem 6.3 of [15]). Let $g : Z^0 \to C^0$ be the desingularization of $f^{-1}(C^0)/\tau \to C^0$. Then $g : Z^0 \to C^0$ is a family of Kummer surfaces and there exist divisors $N_1, ..., N_8$ on Z^0 such that their restrictions $(N_1)_t, ..., (N_8)_t$ on Z^0_t are the exceptional (-2)-curves of the double points of X_t/τ_t (produced by the eight isolated fixed points of τ_t). By Lemma 3.2, we write $R^2 f_*(\mathbb{Z}_{f^{-1}(C^0)}) = \mathbb{W} \oplus \mathbb{Z}^{19}$. Then we have (see Lemma 3.1 of [15])

$$R^2 g_*(\mathbb{Z}_{Z^0}) \simeq (\mathbb{W} \oplus \mathbb{Z}^{19^\tau})(2) \oplus \mathbb{Z}[N_1, ..., N_8],$$

where $\mathbb{Z}^{19^\tau}$ is the invariant sub local system of $\mathbb{Z}^{19}$ under τ, $(\mathbb{W} \oplus \mathbb{Z}^{19^\tau})(2)$ has the same underlying local system as $(\mathbb{W} \oplus \mathbb{Z}^{19^\tau})$, and with the intersection form defined by multiplication by 2 of the the intersection form on $(\mathbb{W} \oplus \mathbb{Z}^{19^\tau})$.

Lemma 4.9. *By making C^0 smaller, there exists a family of abelian surfaces*

$$a : A^0 \to C^0$$

with $\rho(A^0_t) \geq 3$ such that $g : Z^0 \to C^0$ is its Kummer construction.

Proof. It is easy to see that, for any $t \in C^0$, $NS(Z^0_t)$ contains a sub-lattice, which is isomorphic to $\mathbb{Z}^{19^\tau}(2) \oplus \mathbb{Z}[N_1, ..., N_8]$ as a trivial $\pi_1(C \setminus S)$-modules. Thus $g : Z^0 \to C^0$ is a family of Kummer surfaces with $\rho(Z_t) \geq 19$. Let $t_0 \in C^0$ with $\rho(Z^0_{t_0}) = 19$. Then $NS(Z^0_{t_0}) \supset \mathbb{Z}^{19^\tau} \oplus \mathbb{Z}[N_1, ..., N_8]$ and

$$NS(Z^0_{t_0}) \otimes \mathbb{Q} = (\mathbb{Z}^{19^\tau} \oplus \mathbb{Z}[N_1, ..., N_8]) \otimes \mathbb{Q}.$$

Let $F_1, ..., F_{16}$ be the liftings of the sixteen pairwise-disjoint (-2)-curves on $Z^0_{t_0}$ to Z^0. It is not difficult to see that we can choose F_i $(i = 1, ..., 16)$ to be effective divisors on Z^0. In fact, since $g_*\mathcal{O}_{Z^0}(F_i) \neq 0$ (because $H^0(F_i|_{Z^0{}_t}) \neq 0$ for any $t \in C^0$ by Riemann-Roch theorem), we have, for m large enough and a point $p \in C^0$, $H^0(\mathcal{O}_{Z^0}(F_i + mg^{-1}(p))) = H^0(\mathcal{O}_{C^0}(mp) \otimes g_*\mathcal{O}_{Z^0}(F_i)) \neq 0$. Thus there is an effective divisor D on Z^0 such that $D|_{Z^0_{t_0}}$ is numerical equivalent to $F_i|_{Z^0_{t_0}}$, which implies that $D|_{Z^0_{t_0}} = F_i|_{Z^0_{t_0}}$ since a nodal class is represented by only one effective divisor. We can choose F_i $(i = 1, ..., 16)$ to be irreducible further. In fact, we will show that $F_i|_{Z^0_t}$ is irreducible if $\rho(Z^0_t) = 19$. Otherwise, let $F_i|_{Z^0_t} = D_1 + D_2$, where D_1 is irreducible with $D_1^2 = -2$ and D_2 is effective.

Note that for any lifting of an irreducible curve, whose restriction to any other fibre is equivalent to an effective divisor. Thus if $\tilde{D}_1$ and $\tilde{D}_2$ are the liftings of D_1 and D_2 ($\tilde{D}_2$ obtained by lifting the irreducible components of D_2), we see that $\tilde{D}_1|_{Z^0_{t_0}}$ and $\tilde{D}_2|_{Z^0_{t_0}}$ are equivalent to effective divisors. On the other hand, $F_i|_{Z^0_{t_0}} - \tilde{D}_1|_{Z^0_{t_0}}$ is numerically equivalent to $\tilde{D}_2|_{Z^0_{t_0}}$ since it is so on Z^0_t. But this is impossible since $F_i|_{Z^0_{t_0}}$ is a nodal class. Let $g : Z \to C$ be a compactification of $g : Z^0 \to C^0$ with Z smooth and $F_1, ..., F_{16}$ be extended to Z. It is known that $F_1|_{Z_{t_0}} + \cdots + F_{16}|_{Z_{t_0}} \equiv 2\delta$. Let Δ be a divisor on Z such that $\Delta|_{Z_{t_0}} = \delta$. Then $F_1 + \cdots + F_{16} - 2\Delta$ is numerically equivalent to zero on the general fibres, thus

$$F_1 + \cdots + F_{16} - 2\Delta \equiv g^* D_a, \quad D_a \in \mathrm{Div}(C).$$

Choose C^0 smaller so that $F_i|_{Z_t}$ $(i = 1, ..., 16)$ are irreducible for $t \in C^0$ and

$$F_1 + \cdots + F_{16} \equiv 2\Delta \quad \text{on } Z^0.$$

Let $A^{0'} \to Z^0$ be the double covering with branch locus $F_1 + \cdots + F_{16}$, and let $\varpi : A^{0'} \to A^0$ be the uniform blowing down of the sixteen (-1)-curves on the fibres A'_{0t}. Then $a : A^0 \to C^0$ is the family of abelian surfaces with $\rho(A^0{}_t) \geq 3$. □

5. Splitting on families of abelian surfaces

Let $a : A^0 \to C^0$ be the family of abelian surfaces constructed in Lemma 4.9. We take a compactification $a : A \to C$, (which may not be semi-stable). We consider the decomposition

$$R^2 a_*(\mathbb{Z}_{A^0}) \otimes \mathbb{Q} = \mathbb{Q}^\rho \oplus \mathbb{T}_a,$$

where $\mathbb{Q}^\rho$ is the maximal constant sub local system and its complement $\mathbb{T}_a$ is the so-called the sub VHS of the transcendental part of $R^2 a_*(\mathbb{Z}_{A^0})$. It is known that $\mathbb{T}_a$ is Hodge isometric to $\mathbb{T}_g(2)$, where $\mathbb{T}_g$ is the sub VHS of the transcendental part of the weight-2 VHS $R^2 g_*(\mathbb{Z}_{Z^0})$attached to the family of Kummer surfaces $g : Z^0 \to C^0$ arisen from $a : A \to C$. Furthermore, $\mathbb{T}_g$ is Hodge isometric to $\mathbb{T}_f(2)$, where $\mathbb{T}_f = \mathbb{W}$ is the sub VHS of the transcendental part of the weight-2 VHS $R^2 f_*(\mathbb{Z}_{f^{-1}(C^0)})$ attached to one original family $f : f^{-1}(C^0) \to C^0$. Since $\mathbb{W}$ is, in fact, defined on $C \setminus S$, $\mathbb{T}_a$ can be extended to $C \setminus S$ as an VHS.

Lemma 5.1. *The $\mathbb{Q}-$vector space of endomorphisms of*

$$R^1 a_*(\mathbb{Z}_{A^0}) \otimes \mathbb{Q}$$

has dimension 4*, and is of* $(0, 0)$*-type.*

Proof. By the construction of $a : A^0 \to C^0$, we see $R^2 a_*(\mathbb{Z}_{A^0}) \otimes \mathbb{Q}$ contains a constant local system of dimension 3 of (1,1)-type (this corresponds to a sub-lattice of Picard lattice of A^0). Hence, it corresponds to a 3-dimensional subspace of

$\mathrm{End}(R^1a_*(\mathbb{Z}_{A^0}))$ of (0,0)-type. Using a non-scalar endomorphism of this space, we can split $R^1a_*(\mathbb{Z}_{A^0})\otimes\mathbb{C}$ into the following type

$$R^1a_*(\mathbb{Z}_{A^0})\otimes\mathbb{C}\simeq\mathbb{W}_1\oplus\mathbb{W}_2,$$

where both $\mathbb{W}_i$ are of rank-2 and irreducible over $\mathbb{C}$. Otherwise $R^1a_*(\mathbb{Z}_{A^0})\otimes\mathbb{C}$ would contain a rank-1 sub-local system with zero Higgs field. This implies that the Higgs field of (p,g)-type on $\wedge^2R^1a_*(\mathbb{Z}_{A^0})\otimes\mathbb{C}$ can not be isomorphism, a contradiction. We claim that $\mathbb{W}_1\simeq\mathbb{W}_2$. Otherwise, the space $\mathrm{End}(R^1a_*(\mathbb{Z}_{A^0}))\otimes\mathbb{C}$ of global sections of $\mathrm{End}(R^1a_*(\mathbb{Z}_{A^0}))\otimes\mathbb{C}$ has at most dimension 2, a contradiction. Let $\mathbb{W}_1\simeq\mathbb{W}_2\simeq\mathbb{W}$. We have then

$$\mathrm{End}(R^1a_*(\mathbb{Z}_{A^0}))\otimes\mathbb{C}\simeq\mathrm{End}(\mathbb{W})^{\oplus4}=\mathrm{End}_0(\mathbb{W})^{\oplus4}\oplus\mathbb{C}^4=\mathbb{W}'\oplus\mathbb{C}^4.$$

Since $\mathbb{W}$ is irreducible, one shows that $\mathbb{W}'$ does not contain any constant sub-local system and the last splitting can be defined over$\mathbb{Q}$. Hence,

$$\dim\mathrm{End}(R^1a_*(\mathbb{Z}_{A^0}))\otimes\mathbb{Q}=4.$$

□

Lemma 5.2. *The family $a:A^0\to C^0$ is isogenous to the square product of a family of elliptic curves $e:E^0\to C^0$.*

Proof. Case 1). Suppose that there is a subset $T\subset C^0$ of non-countable many points such that A_t is isogenous to $E_t\times E_t$, $t\in T$. Since there are only countable many isomorphic classes of elliptic curves having complex multiplication, we find an $t_0\in T$ such that $\mathrm{End}(E_{t_0})\otimes Q=\mathbb{Q}$. Hence, the endomorphism algebra

$$\mathrm{End}(A_{t_0})\otimes Q\simeq M_2(\mathbb{Q}).$$

In the other words, we have $\mathrm{End}(R^1a_*(\mathbb{Z}_{A^0})\otimes\mathbb{Q})|_{t_0}\simeq M_2(\mathbb{Q})$. Since

$$\mathrm{End}(R^1a_*(\mathbb{Z}_{A^0})\otimes\mathbb{Q})$$

is constant local system, we have $\mathrm{End}(R^1a_*(\mathbb{Z}_{A^0})\otimes\mathbb{Q})\simeq M_2(\mathbb{Q})$. The element

$$\begin{bmatrix}1&0\\0&0\end{bmatrix}\in\mathrm{End}(R^1a_*(\mathbb{Z}_{A^0})\otimes\mathbb{Q})$$

gives a $\mathbb{Q}-$splitting $R^1a_*(\mathbb{Z}_{A^0})\otimes\mathbb{Q}=\mathbb{W}_{\mathbb{Q}}\oplus\mathbb{W}_{\mathbb{Q}}$, thus isogeny splitting of $f:A^0\to C^0$ into the square product of a family of elliptic curves $e:E^0\to C^0$.

Case 2). Suppose that there are non-countable many points $\{t\}\subset C^0$ such that A_t is simple. Since the Picard number $\rho(A_t)\ge3$, one checks easily that $\rho(A_t)=3$ and $\mathrm{End}(A_t)\otimes\mathbb{Q}$ is the totally indefinite quaternion algebra over $\mathbb{Q}$. An abelian surface with this type endomorphism algebra is called a false elliptic curve. There are countable many projective curves $\{C_i\}_{i\in\mathbb{N}}$ in the moduli space of polarized abelian surfaces, which are Shimura curves of certain type and parameterize all false elliptic curves. So, the family $a:A^0\to C^0$ induces a morphism $\phi:C^0\to C_i$ for some $i\in\mathbb{N}$, which extends to a morphism $\phi:C\to C_i$. This implies that the local monodromies of $R^2a_*(\mathbb{Z}_{A^0})$ around the singularity has finite order. It contradicts to $S\neq\emptyset$. □

6. Proof of Theorems 0.1, 0.2 and Corollary 0.4

Proof of Theorem 0.1. Only the modularity of $C' \setminus \sigma^{-1}S$ needs to be checked. The isogeny $a : A^0 \to C^0 \sim e^2 : E^0 \times_{C^0} E^0 \to C^0$ induces an isomorphism $S^2(R^1e_*(\mathbb{Z}_{E^0})) \simeq \mathbb{W}|_{C^0}$. There are natural group homomorphisms

$$1 \to \{\pm 1\} \to SL_2(\mathbb{R}) \to SO(1,2),$$

which induce an isomorphism between $\mathcal{H}$ and a connected component of the symmetric space $SO(1,2)/SO(2) \times O(1)$, say

$$i : \mathcal{H} \simeq SO^+(1,2)/SO(2) \times O(1).$$

Since $\mathbb{W}|_{C^0}$ is the restriction of $\mathbb{W}$ on $C \setminus S$ to C^0, the local monodromies of $R^1e_*(\mathbb{Z}_{E^0})$ around $(C \setminus S) \setminus C^0$ are either $+1$, or -1. Thus the projective monodromy representation of $R^1e_*(\mathbb{Z}_{E^0})$ is actually defined on $C \setminus S$, say

$$\rho_{R^1e_*\mathbb{Z}_{E^0}} : \pi_1(C \setminus S, *) \to PSL_2(\mathbb{Z}).$$

Let $\widetilde{\phi}_{R^1e_*(\mathbb{Z}_{E^0})} : \widetilde{C \setminus S} \to \mathcal{H}$ be the period map corresponding to $R^1e_*\mathbb{Z}_{E^0}$ and

$$\widetilde{\phi}_{\mathbb{W}} : \widetilde{C \setminus S} \to SO^+(1,2)/SO(2) \times O(1))$$

denote the period map corresponding to $\mathbb{W}$. Then $\widetilde{\phi}_{\mathbb{W}} = i \cdot \widetilde{\phi}_{R^1e_*(\mathbb{Z}_{E^0})}$ is an isomorphism. In fact, the tangent map of $\widetilde{\phi}_{\mathbb{W}}$ is precisely the Kodaira-Spencer map of $\mathbb{W}$: $\theta^{2,0} : E^{2,0} \to E_1^{1,1} \otimes \Omega^1_C(\log S)$, which is isomorphic at each point by Lemma 1.1. Thus $\widetilde{\phi}_{\mathbb{W}}$ is a local diffeomorphism. Since the Hodge metric on the Higgs bundle corresponding to $\mathbb{W}$ has logarithmic growth at S and bounded curvature by Schmid [22], together with the remarks after Proposition 9.1 and Proposition 9.8 in [25], $\widetilde{\phi}_{\mathbb{W}}$ is a covering map, hence an isomorphism. This implies that $\widetilde{\phi}_{R^1e_*(\mathbb{Z}_{E^0})}$ is an isomorphism. Thus

$$\phi_{R^1e_*(\mathbb{Z}_{E^0})} : C \setminus S \simeq \mathcal{H}/\rho_{R^1e_*(\mathbb{Z}_{E^0})}$$

is an isomorphism. □

In order to prove Theorem 0.2, we need the following lemma.

Lemma 6.1. *Let $f : X \to C$ be a family of semi-stable K3 surfaces, which has zero iterated Kodaira-Spencer map and reaches the Arakelov bound (II)*

$$\deg f_*\omega_{X/C} = \frac{1}{2} \deg \Omega^1_C(\log S).$$

Then, after passing to a finite étale covering $C' \to C$, the VHS $\mathbb{W}$ is non-rigid.

Proof. One needs to show that, after passing through a finite étale covering of C, the local system $R^2f_*(\mathbb{Z}_{X^0}) \otimes \mathbb{C}$ admits a non-zero endomorphism of type $(-1,1)$. By Lemma 3.2, one has splitting

$$R^2f_*(\mathbb{Z}_{X^0}) \supset \mathbb{W}_{\mathbb{Z}} \oplus \mathbb{Z}^{18}, \quad R^2f_*(\mathbb{Z}_{X^0}) \otimes \mathbb{Q} = (\mathbb{W}_{\mathbb{Z}} \oplus \mathbb{Z}^{18}) \otimes \mathbb{Q}.$$

By Lemma 1.2, the Higgs bundle corresponds to $\mathbb{W}$ has the form

$$(E^{2,0} \oplus E_1^{1,1}) \oplus (E_1^{1,1^*} \oplus E^{0,2})$$

such that the Higgs fields

$$\tau : E^{2,0} \to E_1^{1,1} \otimes \Omega^1_C(\log S), \quad \tau^* : E^{1,1^*} \to E^{0,2} \otimes \Omega^1_C(\log S)$$

are isomorphisms. These two Higgs subbundles correspond to two sub-local systems $\mathbb{W}_1$ and $\bar{W}_1$. We claim that, after passing to a finite étale covering of C, one has $\mathbb{W}_1 \simeq \bar{\mathbb{W}}_1$. To prove the claim, consider the sub-local system

$$\mathbb{W}_1 \to \mathbb{W}.$$

If $\mathbb{W}_1$ is not rigid, then there is a small deformation $W_{1,t} \subset \mathbb{W} \otimes \mathbb{C}$ such that both projections $W_{1,t} \subset \mathbb{W} \otimes \mathbb{C} \to \mathbb{W}_1$ and $W_{1,t} \subset \mathbb{W} \otimes \mathbb{C} \to \bar{\mathbb{W}}_1$ are non-zero. Since $\mathbb{W}_1$ is irreducible, one obtains

$$\mathbb{W}_1 \simeq \mathbb{W}_{1,t} \simeq \bar{\mathbb{W}}_1.$$

If $\mathbb{W}_1$ is rigid, then by Lemma 2.1 $\mathbb{W}_1$ is defined over a number field K. Let $\mathcal{O}_K$ denote the ring of algebraic integers in K, and let

$$\mathbb{W}_{1\mathcal{O}_K} = \mathbb{W} \otimes_{\mathbb{Z}} \mathcal{O}_K \cap \mathbb{W}_1.$$

Then $\mathbb{W}_{1\mathcal{O}_K} \otimes K = \mathbb{W}_1$, which means that the corresponding monodromy representation of $\mathbb{W}_1$ can be defined over $\mathcal{O}_K$. The determinant $\det \mathbb{W}_1 = E^{2,0} \otimes E_1^{1,1}$ is a rank-1 unitary local system $\eta \in \mathrm{Pic}^0(C)$ and takes values in $\mathcal{O}_K$. By a theorem of Kronecker, η is a torsion. So, after passing to the finite étale covering corresponding to η, one obtains $E^{2,0} \simeq E_1^{1,1^*}$ and

$$(E^{2,0} \oplus E_1^{1,1}, \tau) \simeq (E_1^{1,1*} \oplus E^{0,2}, \tau^*).$$

Thus, in any case, we obtain a non-zero endomorphism

$$(E^{2,0} \oplus E_1^{1,1}) \oplus (E_1^{1,1^*} \oplus E^{0,2}) \to (E^{2,0} \oplus E_1^{1,1}) \oplus (E_1^{1,1^*} \oplus E^{0,2})$$

of type $(-1,1)$, which corresponds to an endomorphism of $R^2 f_*(\mathbb{Z}_{X^0}) \otimes \mathbb{C}$ of type $(-1,1)$. □

Proof of Theorem 0.2. By Lemma 6.1, after passing to a finite étale covering $C' \to C$, the VHS $\mathbb{W}$ is non-rigid. By Corollary 5.6.3 of [21], one has

$$\mathrm{End}(\mathbb{W}) \otimes \mathbb{Q} \simeq M_2(\mathbb{Q}).$$

Taking an element in $M_2(\mathbb{Q})$ with two distinct rational eigenvalues, we get a $\mathbb{Q}-$splitting $\mathbb{W} \otimes \mathbb{Q} = \mathbb{W}_1 \oplus \mathbb{W}_2$ such that $\mathbb{W}_1$ is isomorphic to $\mathbb{W}_2$ and the Higgs bundle corresponding to $\mathbb{W}_1$ has the form

$$(L \oplus L^{-1}, \theta), \quad \theta : L \simeq L^{-1} \otimes \Omega^1_C(\log S).$$

$\mathbb{W}_1$ has an $\mathbb{Z}-$structure defined by $\mathbb{W}_{1\mathbb{Z}} = \mathbb{W}_{\mathbb{Z}} \cap \mathbb{W}_1$. Again by Proposition 9.1 of [25], the Higgs bundle $\theta : L \simeq L^{-1} \otimes \Omega^1_C(\log S)$ gives rise to the uniformization

$$C \setminus S \simeq \mathcal{H}/\rho_{\mathbb{W}_1}\pi_1(C \setminus S, *),$$

where $\rho_{\mathbb{W}_1}\pi_1(C \setminus S, *) \subset SL_2(\mathbb{Z})$ of finite index. □

Proof of Corollary 0.4. i) By Theorem 0.1 there exists a family of elliptic curves $g : E^0 \to {\mathbb{P}^1}^0 \subset \mathbb{P}^1 \setminus S$ such that the projective representation

$$p\rho_{R^1 g_* \mathbb{Z}_{E^0}} : \pi_1(\mathbb{P}^1 \setminus S, *) \to \Gamma' \subset PSL_2(\mathbb{Z})$$

extends to $\mathbb{P}^1 \setminus S$ and $\mathbb{P}^1 \setminus S \simeq \mathcal{H}/\Gamma'$. By [2], $\Gamma' \subset PSL_2(\mathbb{Z})$ is of index 12 and conjugates to one of the following 6 subgroups of $PSL_2(\mathbb{Z})$, which are images of $\Gamma(3)$, $\Gamma_0^0(4) \cap \Gamma(2)$, $\Gamma_0^0(5)$, $\Gamma_0^0(6)$, $\Gamma_0(8) \cap \Gamma_0^0(4)$ and $\Gamma_0(9) \cap \Gamma_0^0(3)$ in $SL_2(\mathbb{Z})$ of index 24, where

$$\Gamma(n) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) | b \equiv c \equiv 0,\, a \equiv 1 (\mathrm{mod}. n) \right\},$$

$$\Gamma_0^0(n) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) | c \equiv 0,\, a \equiv 1 (\mathrm{mod}. n) \right\},$$

$$\Gamma_0(n) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) | c \equiv 0 (\mathrm{mod}. n) \right\}.$$

In the proof of Theorem 0.1, we have seen already that the monodromy of $R^1 g_* \mathbb{Z}_{E^0}$ of a short loop around a point of $(\mathbb{P}^1 \setminus S) \setminus {\mathbb{P}^1}^0$ is either $+1$, or -1. If all of them equal to $+1$, then the representation $\rho_{R^1 g_* \mathbb{Z}_{E^0}}$ extends to $\mathbb{P}^1 \setminus S$, and the image of $\pi_1(\mathbb{P}^1 \setminus S, *)$ under this representation conjugates to one of the above 6 subgroups. Hence $g : E^0 \to {\mathbb{P}^1}^0$ extends to a modular family of elliptic curves $g : E \to \mathbb{P}^1 \setminus S$ from one of 6 examples in [2]. Suppose that the monodromies of $R^1 g_* \mathbb{Z}_{E^0}$ of short loops around some points of $(\mathbb{P}^1 \setminus S) \setminus {P^1}^0$ equal to -1. Then the image of $\pi_1(\mathbb{P}^1 \setminus S, *)$ conjugates to the preimage $p^{-1} p \Gamma$, where Γ is one of $\Gamma(3)$, $\Gamma_0^0(4) \cap \Gamma(2)$, $\Gamma_0^0(5)$, $\Gamma_0^0(6)$, $\Gamma_0(8) \cap \Gamma_0^0(4)$ and $\Gamma_0(9) \cap \Gamma_0^0(3)$. The inclusion $\Gamma \subset p^{-1} p \Gamma$ of index 2 defines an étale covering $E^{0'} \to E^0$, which is étale along the fibres and the family $g' : E^{0'} \to {\mathbb{P}^1}^0$ extends to the modular family of elliptic curves $g' : E' \to \mathbb{P}^1 \setminus S$ corresponding to Γ.

ii) is straightforward. □

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