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ON THE AVERAGE NUMBER OF OCTAHEDRAL MODULAR FORMS

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Let $N > 0$ be an integer, χ a Dirichlet character modulo N, and k either 0 or 1. Let f be a primitive eigenform, not necessarily holomorphic, of level N and Nebentypus χ, and let $\lambda_f(n)$ be the eigenvalue of T_p on f. We say f is associated to a Galois representation $\rho : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathbb{C})$ if

$$\begin{aligned} \lambda_f(p) &= \mathrm{Tr}(\rho(\mathrm{Frob}_p)) \\ \chi(p) &= \det(\rho(\mathrm{Frob}_p)) \end{aligned}$$

for all p not dividing N. Following [4], we define $S^{\mathrm{Artin}}_{1/4,k}(N,\chi)$ to be the finite set of primitive weight k cupsidal eigenforms which admit an associated Galois representation. If $\rho : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathbb{C})$ is a Galois representation, we define $\mathbb{P}\rho$ to be the composition of ρ with the natural projection $GL_2(\mathbb{C}) \to PGL_2(\mathbb{C})$.

Two-dimensional complex Galois representations fall naturally into four types; we call ρ dihedral, tetrahedral, octahedral, or icosahedral according as the projectivized image $\mathbb{P}\rho(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}))$ is isomorphic to a dihedral group, A_4, S_4, or A_5. Cusp forms associated to Galois representations are classified likewise.

The latter three types are called "exotic"; it is widely believed that the number of exotic cusp forms of level N is at most N^ϵ. The first results in this direction are due to Duke [3]. These results were later sharpened by Wong [6] and Michel and Venkatesh [4]. The latter authors proved that

$$n^{\mathrm{tetr}}(N,\chi,k) \ll_\epsilon N^{2/3+\epsilon}, n^{\mathrm{oct}}(N,\chi,k) \ll_\epsilon N^{4/5+\epsilon}, n^{\mathrm{icos}}(N,\chi,k) \ll_\epsilon N^{6/7+\epsilon}$$

where $n^T(N,\chi,k)$ is the number of weight k cusp forms of level N, Dirichlet character χ, and type T. (Note that $n^T(N,\chi,k) = 0$ unless χ sends complex conjugation to $(-1)^k$.)

The goal of this paper is to show that the Michel-Venkatesh bound on octahedral forms can be sharpened *on average* over square-free levels N.

We begin by showing that, in the case of square-free level, one does not need to consider very many different Dirichlet characters χ when counting exotic cusp forms.

Lemma 1. *Let N range over square-free integers. Then the number of Dirichlet characters χ of conductor N such that there exists an exotic cusp form of level N and Nebentypus χ is $O(N^\epsilon)$.*

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Proof. Let ρ be the Galois representation attached to an exotic cusp form of level N and conductor χ. Let p be a prime dividing N. Then, since $p \| N$, the restriction $\rho : I_p \to GL_2(\mathbb{C})$ must decompose as $\chi \oplus 1$. In particular, the projection from $\rho(I_p)$ to $\mathbb{P}\rho(I_p)$ is an isomorphism. So $\chi(I_p)$ is a cyclic subgroup of either A_4, S_4, or S_5; in particular, χ^{60} is unramified everywhere, whence trivial. Now the number of characters χ of level N such that $\chi^{60} = 1$ is $O(N^\epsilon)$, which proves the lemma. □

Suppose from now on that N is square-free. Let $n^{\mathrm{oct}}(N)$ be the number of octahedral cusp forms of level N. It follows from Lemma 1 and the theorem of Michel and Venkatesh that

$$n^{\mathrm{oct}}(N) \ll_\epsilon N^{4/5+\epsilon}.$$

Let $p : S_4 \to S_3$ be the natural surjection. Then every homomorphism $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to S_4$ can be composed with p to yield a homomorphism $\psi : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to S_3$. By combining arguments from [6] and [4], we obtain the following sharpening of Theorem 10 of [6]:

Proposition 2. *Let k be 0 or 1, and N a positive square-free integer. Let $\psi : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to S_3$ be a homomorphism, and let $n_\psi^{oct}(N)$ be the number of octahedral weight k cusp forms associated to Galois representations ρ such that $p \circ \mathbb{P}\rho = \psi$.*

Then $n_\psi^{oct}(N) \ll_\epsilon N^{2/3+\epsilon}$.

Proof. Let χ be a character of level N. In the proof of [6, Thm. 10], Wong constructs an amplifier–that is, a set of complex numbers $\{c_n\}_{n\in\mathbb{N}}$ such that, for some absolute constants C and C',

- $\sum_{n\leq B} |c_n| \leq CB^{1/4}$.
- $\sum_{n\leq B} |c_n|^2 \leq CB^{1/4}$.
- If ρ is an octahedral Galois representation such that $p \circ \mathbb{P}\rho = \psi$, and f is a cusp form in $S^{\mathrm{Artin}}_{1/4,k}(N, \chi)$ associated to ρ, then $|\sum_{n\leq B} c_n\lambda_f(n)| \geq C'B^{1/4}/\log B$.

In [4, §3], Michel and Venkatesh use the Petersson-Kuznetzov formula and standard bounds on Kloosterman sums to obtain the following inequality:

$$(1) \quad \sum_{f\in\Sigma} (f,f)^{-1} \left| \sum_{\substack{n\leq B \\ (n,N)=1}} c_n\lambda_f(n) \right|^2 \ll_\epsilon \sum_{\substack{n\leq B \\ (n,N)=1}} |c_n|^2 + (BN)^\epsilon B^{1/2} N^{-1} \left(\sum_{\substack{n\leq B \\ (n,N)=1}} |c_n| \right)^2$$

where

- Σ is a set of eigenforms in $S^{\text{Artin}}_{1/4,k}(N,\chi)$;
- (f,f) is the Petersson self-product of f;
- $\{c_n\}$ is an arbitrary sequence of complex numbers.

Speaking loosely, the idea of [4] and [3] is that, by the Petersson-Kuznetzov formula, the vectors $\{\lambda_f(n)\}_{f\in\Sigma}$ and $\{\lambda_f(m)\}_{f\in\Sigma}$ are "approximately orthogonal" when m and n are distinct integers. On the other hand, Wong shows that there exists a large set of n such that the Fourier coefficients $\lambda_f(n)$ are real numbers of a fixed sign as f ranges over Σ. The desired bound on Σ will follow from the tension between these two constraints.

In the above inequality, take $\{c_n\}$ to be Wong's amplifier and Σ to be the set of octahedral forms in $S^{\text{Artin}}_{1/4,k}(N,\chi)$ associated to Galois representations ρ such that $p\circ\mathbb{P}\rho=\psi$.

Note that $(f,f)=O(N\log^3 N)$ by [6, Lemma 6]. So the left hand side of (1) is bounded below by a constant multiple of

$$N^{-1}\log^{-3}N\sum_{f\in\Sigma}\left|\sum_{\substack{n\le B\\(n,N)=1}}c_n\lambda_f(n)\right|^2\ge N^{-1}\log^{-3}N|\Sigma|(C')^2B^{1/2}\log^{-2}(B)$$

while the right hand side is bounded above by a constant multiple of

$$B^{1/4}+(BN)^{\epsilon}B^{1/2}N^{-1}B^{1/2}.$$

Combining these bounds, one has

$$|\Sigma|\ll_{\epsilon}N\log^3(N)B^{-1/2}\log^2(B)(B^{1/4}+(BN)^{\epsilon}BN^{-1})$$

The bound is optimized when we take $B\sim N^{4/3}$, which yields

$$|\Sigma|\ll_{\epsilon}N^{2/3+\epsilon}.$$

Combined with the fact that the number of χ under consideration is $O(N^{\epsilon})$, this yields the desired result. □

Proposition 2, in combination with the theorem of Davenport and Heilbronn on cubic fields, allows us to improve Michel and Venkatesh's bound on $n^{\text{oct}}(N)$ in the average.

Theorem 3. *For all $\epsilon>0$ there exists a constant C_ϵ such that*

$$(1/X)\sum_{\substack{N<X\\N\,sq.free}}n^{oct}(N)<C_{\epsilon}X^{2/3+\epsilon}$$

for all $X>1$.

Proof. Let f be an octahedral form of level N associated to a representation ρ. For each prime $p|N$, the group $\mathbb{P}\rho(I_p)$ is a cyclic subgroup of S_4. (Recall that N is square-free.) Define

- N_1 to be the product of primes p such that $\mathbb{P}\rho(I_p)$ is a nontrivial subgroup of the Klein four-group;
- N_2 to be the product of primes p such that $\mathbb{P}\rho(I_p)$ is contained in A_4 but not in the Klein four-group;
- N_3 to be the product of primes p such that $\mathbb{P}\rho(I_p)$ is not contained in A_4.

Then $N_1N_2N_3 = N$. Let $\psi : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to S_3$ be the composition $p \circ \mathbb{P}\rho$. Then the fixed field L of $\ker\psi$ is a cyclic 3-cover of a quadratic number field K, where $K/\mathbb{Q}$ is the unique quadratic field ramified precisely at primes dividing N_3, and L/K is ramified only at primes dividing N_2.

Let $b(N_2, N_3)$ be the number of such S_3-extensions L. (For notational convenience we take $b(N_2, N_3)$ to be 0 when either N_2 or N_3 is not square-free.) From Proposition 2, we have

$$n^{\mathrm{oct}}(N) \ll_\epsilon \sum_{\substack{N_1,N_2,N_3 \\ N_1N_2N_3=N}} b(N_2, N_3)N^{2/3+\epsilon}. \tag{2}$$

Let T be the set of places of K dividing $3N_2\infty$, and let $G_T(K)$ be the Galois group of the maximal extension of K unramified away from T. Each cubic field counted in $b(N_2, N_3)$ is a cyclic 3-extension of K unramified away from T, so

$$b(N_2, N_3) \le |\,\mathrm{Hom}(G_T(K), \mathbb{Z}/3\mathbb{Z})|.$$

The Galois cohomology group above fits in an exact sequence

$$0 \to \mathrm{Hom}(\mathrm{Cl}_T(K), \mathbb{Z}/3\mathbb{Z}) \to \mathrm{Hom}(G_T(K), \mathbb{Z}/3\mathbb{Z}) \to \prod_{v\in T} \mathrm{Hom}(\mathrm{Gal}(\bar{K_v}/K_v), \mathbb{Z}/3\mathbb{Z})$$

where $\mathrm{Cl}_T(K)$ is the quotient of the class group of K by all primes in T. (See [5, (8.6.3)]). Let $h_3(N_3)$ be the order of the 3-torsion subgroup of the class group of K. Since $\dim_{\mathbb{F}_3} \mathrm{Hom}(\mathrm{Gal}(\bar{K_v}/K_v), \mathbb{Z}/3\mathbb{Z})$ is at most 4 (see [5, (7.3.9)]), we have

$$b(N_2, N_3) \le h_3(N_3)3^{4|T|} \ll_\epsilon N_2^\epsilon h_3(N_3). \tag{3}$$

Combining (3) and (2) yields

$$n^{\mathrm{oct}}(N) \ll_\epsilon \sum_{\substack{N_1,N_2,N_3 \\ N_1N_2N_3=N}} h_3(N_3)N^{2/3+\epsilon}.$$

Since the sums over N_1 and N_2 have length at most $d(N) = O(N^\epsilon)$, we have

$$n^{\mathrm{oct}}(N) \ll_\epsilon \sum_{N_3|N} h_3(N_3)N^{2/3+\epsilon}.$$

So

$$\sum_{\substack{N<X \\ N sq.free}} n^{\text{oct}}(N) \ll_\epsilon \sum_{N_3=0}^{X} h_3(N_3) \sum_{k=0}^{X/N_3} (kN_3)^{2/3+\epsilon} \tag{4}$$

$$\leq X^{5/3+\epsilon} \sum_{N_3=0}^{X} h_3(N_3)(1/N_3).$$

The sum $\sum_{d=0}^{X} h_3(d)/d$ can be estimated as follows. Integration by parts yields

$$\sum_{d=0}^{X} h_3(d)/d = (1/X) \sum_{d=0}^{X} h_3(d) + \int_1^X (\sum_{d=1}^{t} h_3(d)) t^{-2} dt.$$

Now by the the theorem of Davenport and Heilbronn [2, Theorem 3] we have $\sum_{d=0}^{t} h_3(d) = O(t)$. It follows that

$$\sum_{d=0}^{X} h_3(d)/d = O(\log X).$$

Substituting this bound into (4) gives

$$\sum_{\substack{N<X \\ N sq.free}} n^{\text{oct}}(N) \ll_\epsilon X^{5/3+\epsilon}$$

which yields the desired result. □

Theorem 3 can be thought of as a bound for the number of quartic extensions of $\mathbb{Q}$ whose Artin conductor, with respect to a certain 2-dimensional projective representation of S_4, is bounded by X. This is quite different from the problem, recently solved by Bhargava [1], of counting the number of quartic extensions of $\mathbb{Q}$ with *discriminant* less than X. For instance, quartic extensions attached to cusp forms of conductor N might have discriminant as large as N^3.

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