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DWYER'S FILTRATION AND TOPOLOGY OF 4-MANIFOLDS

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Abstract. Topological 4-dimensional surgery is conjectured to fail, in general, for free fundamental groups. M. Freedman and P. Teichner have shown that surgery problems with an arbitrary fundamental group have a solution, provided they satisfy a certain condition on Dwyer's filtration on second homology. We give a new geometric proof of this result, and analyze its relation to the canonical surgery problems.

The lower central series of the fundamental group of a space X is closely related to the Dwyer's [D] filtration $\phi_k(X)$ of the second homology $H_2(X;\mathbb{Z})$. It is well known [FQ] that, for any $k > 1$, the canonical 4-dimensional surgery problems may be arranged to have the kernel represented by a submanifold M which is π_1-null, and satisfies $H_2(M) = \phi_k(M)$. It is conjectured [F] that these canonical problems do not have a solution. On the other hand, Freedman and Teichner showed in [FT] that if the surgery kernel is π_1-null and its second homology lies in the ω-term of the filtration, $H_2(M) = \phi_\omega(M)$, then the surgery problem has a solution.

This theorem was a development of the earlier result ([FQ], Chapter 6) that a surgery problem can be solved if the kernel M is π_1-null and $H_2(M)$ is *spherical*. In the present paper we give a new, geometric, proof of the theorem of Freedman–Teichner. We show that a surgery problem with the kernel M in their setup, i.e. π_1-null and with $H_2(M) = \phi_\omega(M)$, in a 4-manifold N, can be reduced to the π_1-null *spherical* case, in the same manifold N. The proof is based on the idea of splitting of capped gropes. This technique has been useful in solving a number of other problems in 4-manifold topology, see [K], [KQ].

It is interesting to note that the proof goes through if $H_2(M) = \phi_k(M)$ for $k \geq k_0$, where k_0 is a constant depending on the inclusion $M \subset N$. This integer k_0 can be easily read off from the data as the number of group elements in $\pi_1 N$, represented by the double point loops of the Whitney disks for M in N. As is common in the subject, the aforementioned canonical surgery problems may be chosen to satisfy $H_2(M) = \phi_{k_0-1}(M)$, just missing the requirement for the theorem above.

Section 1 gives a brief overview of the splitting operation on gropes. The background material on the lower central series and Dwyer's filtration is presented in Section 2. The main theorem is stated and proved in Section 3.

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1. Gropes and splitting

This is a brief summary of terminology and notations; for a more detailed exposition the reader is referred to [FQ], [FT], [K], [KQ]. Observe that the notion of a *grope of class k* which is needed in this paper is different from the *symmetric gropes* discussed in [FQ], [KQ]. In particular, a symmetric grope of height n has class 2^n. To be precise, we recall the definition:

Definition 1. A *grope* is a special pair (2-complex, circle). A grope has a class $k = 2, 3, \ldots$. A grope of class 2 is a compact oriented surface Σ with a single boundary component. A k-grope is defined inductively as follows: let $\{\alpha_i, \beta_i\}$ be a standard symplectic basis of circles for Σ. For any positive integers p_i, q_i with $p_i + q_i = k$, a k-grope is formed by gluing a p_i-grope to each α_i and a q_i-grope to each β_i.

The *tips* of a grope g is a symplectic basis of circles in its top stage surfaces. They freely generate $\pi_1 g$. A model *capped grope* g^c is obtained from a grope g by attaching disks to its tips. The grope g is then called the *body* of g^c. Finally, a capped grope in a 4-manifold M is an immersion $g^c \longrightarrow M$, where only intersections among the caps are allowed (so the body is embedded, and is disjoint from the interiors of the caps.) Each intersection point between the caps carries an element of $\pi_1 M$. It is determined by the double point loop of the intersection.

Here we use the terminology of [K]. In particular, g denotes a grope (the underlying 2-complex), while the capital letter G indicates the use of its untwisted 4-dimensional thickening. The operations that will be used in the proof are *contraction*, sometimes also referred to as *symmetric surgery*, and *pushoff*, which are described in detail in [FQ, §2.3]. The following lemma (suitably formulated grope splitting) is a central ingredient in the proof of the main theorem in Section 3. For more applications of grope splitting see [K], [KQ].

Lemma 2 (Grope splitting). *Let (g^c, γ) be a capped grope in M^4. Then, given a regular neighborhood N of g^c in M, there is a capped grope $(g^c_{\text{split}}, \gamma) \subset N$, such that each cap of g^c_{split} has double points which represent at most one group element in $\pi_1 M$, and each body surface, above the first stage, of g^c_{split} has genus* 1.

Proof. First assume that N is the untwisted thickening of g^c, $N = G^c$, and moreover let g^c be a model capped grope (without double points). Let C, D be a dual pair of its caps, and let α be an arc in C with endpoints on the boundary of C. (In our applications, α will be chosen to separate intersection points of C corresponding to different group elements.) Recall that the untwisted thickening N of g^c is defined as the thickening in $\mathbb{R}^3$, times the interval I. We consider the 3-dimensional thickening, and surger the top-stage surface of g, which is capped by C and D, along the arc α. The cap C is divided by α into two disks C', C'' which serve as the caps for the new grope; their dual caps D', D'' are formed by parallel copies of D. This operation increases the genus of this top-stage surface by 1. We described this operation for a model capped grope; splitting of a capped grope with double points is defined as an obvious generalization.

Continue the proof of lemma 2 by dividing each cap C by arcs $\{\alpha\}$, so that each component of $C \smallsetminus \cup\alpha$ has double points representing just one group element, and splitting g^c along all these arcs. The crucial observation is that any future application of this technique preserves the progress achieved up to date: the parallel copies D', D'' of D as above inherit the collection of the group elements carried by D. We apply the same operation to the surfaces in the $(h-1)$-st stage of the grope, separating each top stage surface by arcs into genus 1 pieces. This procedure is performed inductively, descending to the first stage of g^c. For example, if originally each body surface of a k-grope g had genus one, and each cap carried n group elements, then after this complete splitting procedure the first stage surface will have genus n^k. □

2. Dwyer's filtration

In this section we recall basic facts about the lower central series and Dwyer's filtration, and their geometric reformulation in terms of gropes. For proofs of the propositions, see [FT]. Recall that the lower central series of a group H is defined by $H^1 = H$, $H^k = [H, H^{k-1}]$ for $k \geq 1$, and $H^\omega = \cap_{k\in\mathbb{N}} H^k$. The following proposition provides a geometric reformulation:

Proposition 3. *A loop γ in a space X lies in $\pi_1(X)^k$ if and only if γ bounds a map of some k-grope in X.*

Clearly, a loop γ is in $\pi_1(X)^\omega$ iff for each finite k, γ bounds a map of a k-grope in X. The *Dwyer's subspace* $\phi_k(X) \subset H_2(X;\mathbb{Z})$ is defined as the kernel of the composition

$$H_2(X) \longrightarrow H_2(K(\pi_1 X, 1)) = H_2(\pi_1 X)) \longrightarrow H_2(\pi_1(X)/\pi_1(X)^{k-1}).$$

Proposition 4. *Dwyer's subspace $\phi_k(X)$ of $H_2(X)$ coincides with the subset of homology classes represented by maps of closed k-gropes into X.*

Here a *closed k-grope* is a 2-complex obtained by replacing a 2-cell in S^2 with a k-grope. Note again that a homology class is in the ω-term of the Dwyer's filtration if and only if for each $k \geq 2$ it is represented by a map of a closed k-grope into X.

3. Surgery theorem for ϕ_ω.

Before formulating the main theorem (Theorem 1.1 in [FT]), recall the setting for surgery. Let N be a compact topological 4-manifold, possibly with boundary. Suppose $f\colon N \longrightarrow X$ is a degree 1 normal map from N to a Poincaré complex X. Following the higher-dimensional arguments, it is possible to find a map normally bordant to f which is a π_1-isomorphism, and such that the kernel

$$K = \ker(H_2(N;\mathbb{Z}[\pi_1 X]) \longrightarrow H_2(X;\mathbb{Z}[\pi_1 X]))$$

is a free $\mathbb{Z}[\pi_1 X]$-module. Suppose Wall's obstruction vanishes, so there is a preferred basis for the kernel K in which the intersection form is hyperbolic. Then we say that $M \subset N$ *represents the surgery kernel* if M is π_1-null, $H_2(M)$ is free and

$$H_2(M) \otimes_{\mathbb{Z}} \mathbb{Z}[\pi_1 X] \longrightarrow H_2(N; \mathbb{Z}[\pi_1 X])$$

maps isomorphically onto K. Here we assume that $M \subset int(N)$ is a (compact) codimension 0 submanifold. Recall that M is π_1-null means that the inclusion induces the trivial map $\pi_1 M \longrightarrow \pi_1 N$.

Theorem 5. *Suppose a standard surgery kernel is represented by $M \subset N$ which is π_1-null and satisfies $\phi_\omega(M) = H_2(M)$. Then there is a normal bordism from $f\colon N \longrightarrow X$ to a simple homotopy equivalence $f'\colon N' \longrightarrow X$.*

More precisely, the proof shows that there is an integer m depending on the inclusion $M \subset N$ so that the theorem still holds if $H_2(M) \subset \phi_m(M)$.

Proof. Let $\gamma_1, \dots, \gamma_k$ be loops in M representing generators of $\pi_1 M$. Since M is π_1-null, there are null-homotopies $\Delta_1, \dots, \Delta_k$ in N, $\partial\Delta_i = \gamma_i$. Let $f_1, \dots, f_m \in \pi_1 N$ be the group elements represented by $M \cup \Delta_1 \cup \ldots \cup \Delta_k$. These are given by the double point loops of the null-homotopies Δ, and by the intersections $\Delta \cap M$. (Each component of the intersection determines a group element by starting at the basepoint in M, following a path in M to $\Delta \cap M$ and returning via Δ, avoiding its double points. This is well-defined since M is π_1-null.)

The integer m is "universal" for the loops in M: given any loop γ in M, there is a nullhomotopy for γ in N giving rise to at most m group elements, since M is π_1-null in $M \cup \Delta$. (Represent γ as a composition of the generators $\{\gamma_i\}$, so it bounds parallel copies of the singular disks $\{\Delta_i\}$.)

Let $G = \{G_i\}$ be $(m+1)$-gropes representing standard free generators of $H_2(M)$ (corresponding to the preferred basis of $H_2(M) \otimes_{\mathbb{Z}} \mathbb{Z}[\pi_1 X] \cong K$, in which the intersection form is hyperbolic.) Since G is contained in M, all its double point loops are trivial in $\pi_1 N$. Cap G by null-homotopies for its tips in N, to get a collection of capped $(m+1)$-gropes G^c. These are not capped gropes in the conventional sense, because the body has self-intersections (but these are $\pi_1 N$-null), and also the caps may intersect any surface stage – however the total number of group elements represented by the double point loops of G^c is at most m, as observed above.

Split G^c (as in Lemma 2), with respect to the group elements at its caps. In other words, first split the caps, separating different group elements, and then proceed down the grope, splitting surface stages into genus 1 pieces. (When splitting the surface stages, ignore their intersections with any other surfaces and caps.) The result, for each G_i^c, is a capped $(m+1)$-grope with the base surface of high genus, with all surfaces above the first stage of genus 1, and with each cap having double points (intersections with other caps/surface stages) with just one group element.

Consider a genus 1 piece of the base surface. It is a base of a capped "dyadic" $(m+1)$-grope (all surface stages have genus 1), so has $m+1$ caps. There are at most m group elements present at the caps, so two of the caps must have the same group element. Contract the grope along these two caps, and push off all other caps/surfaces intersecting them, thus creating only π_1-null intersections. This produces a collection of π_1-null transverse pairs of spheres, and reduces the problem to Chapter 6 of [FQ]. $\square$

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