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A COUNTEREXAMPLE TO A MULTILINEAR ENDPOINT QUESTION OF CHRIST AND KISELEV

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Abstract. Christ and Kiselev [2],[3] have established that the generalized eigenfunctions of one-dimensional Dirac operators with L^p potential F are bounded for almost all energies for $p < 2$. Roughly speaking, the proof involved writing these eigenfunctions as a multilinear series $\sum_n T_n(F, \dots, F)$ and carefully bounding each term $T_n(F, \dots, F)$. It is conjectured that the results in [3] also hold for L^2 potentials F. However in this note we show that the bilinear term $T_2(F, F)$ and the trilinear term $T_3(F, F, F)$ are badly behaved on L^2, which seems to indicate that multilinear expansions are not the right tool for tackling this endpoint case.

1. Introduction

Let $F(x)$ be a real potential on $\mathbb{R}$. For each energy $k^2 > 0$ we can consider the Dirac generalized eigenfunction equation

$$(\frac{d}{dx} + F)(-\frac{d}{dx} + F)\phi(x) = k^2\phi(x)$$

on $\mathbb{R}$. This Dirac equation can be thought of as a Schrödinger equation with potential $V = F' + F^2$. For each k there are two linearly independent eigenfunctions $\phi = \phi_k$. A natural question from spectral theory is to ask whether these eigenfunctions are bounded (i.e. are in L^∞_x) for almost every real k. In [3] Christ and Kiselev[1] showed among other things that this was true when $F \in L^p_x$ for any $1 \le p < 2$. It is well known (see e.g. [12]) that the statement fails when $p > 2$, but the $p = 2$ case remains open. In [5] it is shown that for L^2 potentials one has absolutely continuous spectrum on $[0, \infty)$, but this is a slightly weaker statement.

We briefly outline the arguments in [2],[3]. The method of variation of constants suggests the ansatz

$$\phi(x) = a(x)e^{ikx} + b(x)e^{-ikx}$$
$$(-\frac{d}{dx} + F)\phi(x) = -ika(x)e^{ikx} + ikb(x)e^{-ikx}.$$

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[1]The results cited are phrased for Schrödinger operators but also extend to the slightly simpler case of Dirac operators, see [4].

Substituting this into the previous and simplifying, we reduce to the first-order system

$$a'(x) = F(x)e^{-2ikx}b(x)$$
$$b'(x) = F(x)e^{2ikx}a(x).$$

For simplicity we may assume F is supported on the positive half axis. If we set initial conditions $a(-\infty) = 1$, $b(-\infty) = 0$ for instance, and then solve this system by iteration, we thus obtain the formal multilinear expansions

$$a = 1 + \sum_{n \geq 2, \text{ even}} T_n(F, \dots, F); \quad b = \sum_{n \geq 1, \text{ odd}} T_n(F, \dots, F)$$

where for each $n \geq 1$, T_n is the n-linear operator

$$T_n(F_1, \dots, F_n)(k, x) := \int_{x_1 < \dots < x_n < x} e^{-2ik \sum_{j=1}^n (-1)^j x_j} F_1(x_1) \dots F_n(x_n) \; dx_1 \dots dx_n.$$

For integrable F_j we can define the n-linear operators

$$T_n(F_1, \dots, F_n)(k, +\infty) := \int_{x_1 < \dots < x_n} e^{-2ik \sum_{j=1}^n (-1)^j x_j} F_1(x_1) \dots F_n(x_n) \; dx_1 \dots dx_n.$$

The strategy of Christ and Kiselev was then to control each individual expression T_n on L^p. Specifally, they showed the estimate

$$\| \sup_x |T_n(F, \dots, F)(k, x)| \|_{L_k^{p'/n, \infty}} \leq C_{p,n} \|F\|_{L_x^p}^n \tag{1}$$

for all $n \geq 1$ and $1 \leq p < 2$, where $C_{p,n}$ was a constant which decayed rapidly in n and $1/p + 1/p' := 1$. In particular one has the non-maximal variant

$$\|T_n(F, \dots, F)(k, +\infty)\|_{L_k^{p'/n, \infty}} \leq C_{p,n} \|F\|_{L_x^p}^n. \tag{2}$$

The boundedness of eigenfunctions for almost every k then follows by summing these bounds carefully.

It is tempting to try this approach for the endpoint $p = 2$. For $n = 1$ we see that $T_1(F)(k, +\infty)$ is essentially the Fourier co-efficient $\hat{F}(k)$, while $\sup_x |T_1(F)(k, x)|$ is essentially the Carleson maximal operator $CF(k)$. The estimates (2), (1) for $p = 2$ then follow from Plancherel's theorem and the Carleson-Hunt theorem [1], [6] respectively.

For $n = 2$ the expression $T_2(F, F)(k, +\infty)$ is essentially $H_-(|\hat{F}|^2)(k)$, where H_- is the Riesz projection

$$\widehat{H_- F} := \chi_{(-\infty, 0]} \widehat{F},$$

and so (2) follows for $p = 2$ by Hölder's inequality and the weak-type $(1, 1)$ of the Riesz projections. We also remark that if the phase function $x_1 - x_2$ in the definition of T_2 were replaced by $\alpha_1 x_1 + \alpha_2 x_2$ for generic numbers α_1, α_2

then the operator is essentially a bilinear Hilbert transform and one still has boundedness from the results in [7], [8], [13].

It may thus appear encouraging to try to estimate the higher order multilinear operators for L^2 potentials F. However, in this note we show

Theorem 1.1. *When $p = 2$ and $n = 2$, the estimate* (1) *fails. When $p = 2$ and $n = 3$, the estimate* (2) *fails.*

Because of this, we believe that it is not possible to prove the almost everywhere boundedness of eigenfunctions for Dirac or Schrödinger operators with L^2 potential purely by multilinear expansions; we discuss this further in the remarks section.

The counterexample has a logarithmic divergence, and essentially relies on the fact that while convolution with the Hilbert kernel $p.v.\frac{1}{x}$ is bounded, convolution with $\frac{\mathrm{sgn}(x)}{x}$ or $\frac{\chi_{(-\infty,0]}(x)}{x}$ is not. It may be viewed as an assertion that L^2 potentials create significant long-range interaction effects which are not present for more rapidly decaying potentials.

Interestingly, our counterexamples rely strongly on a certain degeneracy in the phase function $\sum_j (-1)^j x_j$ on the boundary of the simplex $x_1 < \ldots < x_n$. If one replaced this phase by $\sum_j x_j$, then we have shown in [9], [10] that the bound (2) in fact holds when $p = 2$ and $n = 3$. Indeed this statement is true for generic phases of the form $\sum_j \alpha_j x_j$. A similar statement holds for (1) when $p = 2$ and $n = 2$ and will appear elsewhere.

2. Proof of Theorem 1.1

The letter C may denote different large constants in the sequel. To be consistent with the previous notation we shall define the Fourier transform as

$$\hat{F}(k) := \int e^{-2ikx} F(x)\, dx.$$

We let $N \gg 1$ be a large integer parameter, which we shall take to be a square number, and test (1), (2) with the real-valued potential

$$F(x) := \sum_{j=N}^{2N} F_j(x)$$

where the F_j are given by

$$F_j(x) := N^{-1} \cos(2\frac{Aj}{N}x)\phi(\frac{x}{N} - j),$$

ϕ is a smooth real valued function supported in $[-\frac{1}{4}, \frac{1}{4}]$ with total mass $\int \phi = 1$ such that $\widehat{\phi}$ stays away from 0 in $[-1, 1]$, and A is a sufficiently large absolute constant whose purpose is to ensure that

$$4 \sum_{j \in \mathbf{Z} \setminus \{0\}} \left| \widehat{\phi}(\xi - Aj) \right| \leq |\widehat{\phi(\xi)}|$$

for $\xi \in [-1,1]$. Informally, F is a "chirp" which is localized in phase space to the region

$$\{(k,x) : k = \pm\frac{Aj}{N} + O(\frac{1}{N}); x = Nj + O(N), N \leq j \leq 2N\}.$$

We may compute the Fourier transform of the F_j using the rapid decay of $\hat{\phi}$ as

$$\hat{F}_j(k) = \frac{1}{2}e^{-2i(Nk-Aj)j}\hat{\phi}(Nk - Aj) + O(N^{-200}) \tag{3}$$

in the region $\frac{A}{2} < k < 3A$. We remark that the error term $O(N^{-200})$ has a gradient which is also $O(N^{-200})$.

Clearly we have $\|F_j\|_2 = O(N^{-1/2})$, and hence that

$$\|F\|_2 = O(1).$$

We now compute

$$T_2(F,F)(k,x) = \int_{x_1<x_2<x} e^{2ik(x_1-x_2)}F(x_1)F(x_2)\ dx_1dx_2 \tag{4}$$

in the region

$$|Nk - Aj_0| \leq 1; \quad x = N(j_0 - \sqrt{N} + \frac{1}{2}) \tag{5}$$

for some integer $\frac{3N}{2} < j_0 < 2N$. In this region we show that

$$|T_2(F,F)(k,x)| \geq C^{-1}\log N, \tag{6}$$

which will imply that

$$\|\sup_x |T_2(F,F)|(k,x)\|_{L_k^{2,\infty}} \geq C^{-1}\log N$$

and thus contradict (1) for $n = 2$ and $p = 2$ by letting N go to infinity.

We now prove (6). Fix k, j_0, x. Observe from (4) that $T_2(F_j, F_{j'})(k,x)$ vanishes unless $j \leq j' \leq j_0 - \sqrt{N}$. Thus we may expand

$$T_2(F,F)(k,x) = \sum_{N\leq j\leq j_0-\sqrt{N}} T_2(F_j,F_j)(k,x) \tag{7}$$

$$+ \sum_{N\leq j<j'\leq j_0-\sqrt{N}} T(F_j,F_{j'})(k,x). \tag{8}$$

We first dispose of the error term (8). In the region $j < j' \leq j_0 - \sqrt{N}$, the conditions $x_1 < x_2 < x$ in (4) become superfluous, so we may factor

$$T_2(F_j,F_{j'})(k,x) = \overline{\hat{F}_j(k)}\hat{F}_{j'}(k).$$

However, since $\hat{\phi}$ is rapidly decreasing and $|j - j_0|, |j' - j_0| \geq \sqrt{N}$, we see from (3) that

$$|\hat{F}_j(k)|, |\hat{F}_{j'}(k)| \leq CN^{-100}.$$

Summing this, we see that the total contribution of (8) is $O(N^{-198})$.

Now we consider the contribution of (7). We use the identity

$$T_2(F_j, F_j)(k, x) = T_2(F_j, F_j)(k, +\infty) = H_-(|\hat{F}_j|^2)(k) \tag{9}$$

combined with (3). The operator H_- is a non-trivial linear combination of the identity and the Hilbert transform, while $|\hat{F}_j|^2$ is essentially a non-negative bump function rapidly decreasing away from the interval $[jA/N - O(1/N), jA/N + O(1/N)]$. Because of this we see that for $j \neq j_0$ we have

$$H_-(|\hat{F}_j|^2)(k) = \frac{c}{j - j_0} + O(|j - j_0|^{-2}) \tag{10}$$

where c is a non-zero absolute constant. Summing this over all $j \leq j_0 - \sqrt{N}$ and observing that $j - j_0$ has a consistent sign we see that the contribution of (7) has magnitude at least $C^{-1} \log N$, and (6) follows.

We now compute $T_3(F, F, F)(k, +\infty)$ in the region

$$|Nk - Aj_0| \leq 1; \quad 1.4N < j_0 < 1.6N. \tag{11}$$

We will show that

$$|T_3(F, F, F)(k, +\infty)| \geq C^{-1} \log N \tag{12}$$

in this region, which will disprove (2) for $n = 3$ and $p = 2$ similarly to before.

It remains to prove (12). Fix j_0. Observe that $T_3(F_j, F_{j'}, F_{j''})(k, +\infty)$ vanishes unless $j \leq j' \leq j''$. Thus we can split

$$T_3(F, F, F)(k, +\infty) = \sum_{N \leq j \leq 2N} T_3(F_j, F_j, F_j)(k, +\infty) \tag{13}$$

$$+ \sum_{N \leq j < j' \leq 2N} T_3(F_j, F_j, F_{j'})(k, +\infty) \tag{14}$$

$$+ \sum_{N \leq j' < j \leq 2N} T_3(F_{j'}, F_j, F_j)(k, +\infty) \tag{15}$$

$$+ \sum_{N \leq j < j' < j'' \leq 2N} T_3(F_j, F_{j'}, F_{j''})(k, +\infty). \tag{16}$$

We first consider (13). We expand

$$T_3(F_j, F_j, F_j)(k, +\infty) = \int_{x_1 < x_2 < x_3} e^{2ik(x_1 - x_2 + x_3)} F_j(x_1) F_j(x_2) F_j(x_3)\, dx_1 dx_2 dx_3.$$

This is a linear combination of eight terms of the form

$$N^{-3} \int_{x_1 < x_2 < x_3} e^{2ik(x_1 - x_2 + x_3)} e^{2i\frac{Aj}{N}(\pm x_1 \pm x_2 \pm x_3)} \cdot \phi(\frac{x_1}{N} - j)\phi(\frac{x_2}{N} - j)\phi(\frac{x_3}{N} - j)\; dx_1 dx_2 dx_3;$$

making the substitutions $y_s := \frac{x_s}{N} - j$ for $s = 1, 2, 3$, this becomes

$$e^{i\theta} \int_{y_1 < y_2 < y_3} e^{2ikN(y_1 - y_2 + y_3)} e^{2iAj(\pm y_1 \pm y_2 \pm y_3)} \phi(y_1)\phi(y_2)\phi(y_3)\; dy_1 dy_2 dy_3$$

for some phase $e^{i\theta}$ depending on all the above variables.

We shall only consider the choice of signs $(-y_1 + y_2 - y_3)$; the reader may easily verify that the other choices of signs are much smaller thanks to stationary phase. In this case we can write the above as

$$e^{i\theta} \int_{y_1 < y_2 < y_3} e^{2i(kN - Aj)(y_1 - y_2 + y_3)} \phi(y_1)\phi(y_2)\phi(y_3)\ dy_1 dy_2 dy_3.$$

If $kN - Aj = O(1)$ we estimate this crudely by $O(1)$. Otherwise we can perform the y_1 integral using stationary phase to obtain

$$e^{i\theta} \frac{1}{2i(kN - Aj)} \int_{y_2 < y_3} e^{2i(kN - Aj)y_3} \phi(y_2)\phi(y_2)\phi(y_3)\ dy_2 dy_3 + O(|kN - Aj|^{-2}).$$

Performing another stationary phase we see that this quantity is $O(|kN - Aj|^{-2})$. Summing over all j we see that (13) is $O(1)$.

Let us now consider (16). When $j < j' < j''$, the constraints $x_1 < x_2 < x_3$ in the definition of T_3 are redundant, and we can factorize

$$T_3(F_j, F_{j'}, F_{j''})(k, +\infty) = \overline{\hat{F}_j(k)} \hat{F}_{j'}(k) \overline{\hat{F}_{j''}(k)}.$$

Applying (3) and using the rapid decay of $\hat{\phi}$ we see that

$$|T_3(F_j, F_{j'}, F_{j''})(k, +\infty)| \leq C(1 + |j - j_0| + |j' - j_0| + |j'' - j_0|)^{-10} + CN^{-100}.$$

Summing over all j, j', j'' we see that (16) is $O(1)$.

It remains to control (15) + (14). First we consider (14). For this term the condition $x_2 < x_3$ is redundant, so we can factorize

$$T_3(F_j, F_j, F_{j'})(k, +\infty) = T_2(F_j, F_j)(k, +\infty) \overline{\hat{F}_{j'}(k)}.$$

Now consider (15). For this term the condition $x_1 < x_2$ is redundant, so we can factorize

$$T_3(F_{j'}, F_j, F_j)(k, +\infty) = \overline{\hat{F}_{j'}(k)} \int_{x_2 < x_3} e^{2ik(x_3 - x_2)} F_j(x_2) F_j(x_3)\ dx_3 dx_2.$$

Writing x_1 instead of x_3 we thus have

$$T_3(F_{j'}, F_j, F_j)(k, +\infty) = \overline{\hat{F}_{j'}(k)} (|\hat{F}_j(k)|^2 - T_2(F_j, F_j)(k, +\infty)).$$

Combining this with the previous, we thus see that

$$(15) + (14) = \sum_{N \leq j, j' \leq 2N} \operatorname{sgn}(j' - j) T_2(F_j, F_j)(k, +\infty) \overline{\hat{F}_{j'}(k)} + \sum_{N \leq j' < j \leq 2N} \overline{\hat{F}_{j'}(k)} |\hat{F}_j(k)|^2.$$

Using (3) as in (16) we see the second term is $O(1)$, so to prove (12) it will suffice to show that

$$\Big| \sum_{N \leq j, j' \leq 2N, j \neq j'} \operatorname{sgn}(j' - j) T_2(F_j, F_j)(k, +\infty) \overline{\hat{F}_{j'}(k)} \Big| \geq C^{-1} \log N. \tag{17}$$

We first consider the terms with $j' = j_0$. We claim these terms are the dominant contribution. From (9), (10) we conclude

$$\sum_{N \le j \le 2N, j \ne j_0} \operatorname{sgn}(j_0 - j) T_2(F_j, F_j)(k, +\infty) \overline{\hat{F}_{j_0}(k)} = \sum_{N \le j \le 2N, j \ne j_0} c \frac{\operatorname{sgn}(j_0 - j)}{j_0 - j} \overline{\hat{F}_{j_0}(k)} \; + \; O(1) \quad . \tag{18}$$

Here c is the same non-zero constant as in (10), and $\hat{F}_{j_0}(k)$ is bounded away from 0 by choice of ϕ. Thus the first term is greater than $C^{-1} \log N$, so it suffices indeed to show that this term is the dominant contribution to (17).

We consider the terms with $j = j_0$. Using that $|T_2(F_j, F_j)(k, \infty)| \le C$ we obtain

$$\sum_{N \le j' \le 2N, j_0 \ne j'} |T_2(F_{j_0}, F_{j_0})(k, +\infty) \overline{\hat{F}_{j'}(k)}| \le C$$

This term is therefore negligible.

Finally, we have to consider the terms with $j, j' \ne j_0$. We have by the choice of A,

$$\sum_{N \le j, j' \le 2N, j, j' \ne j_0} |T_2(F_j, F_j)(k, +\infty)| |\overline{\hat{F}_{j'}(k)}| \le \frac{1}{2} \sum_{N \le j \le 2N, j \ne j_0} \frac{c}{|j - j_0|} |\overline{\hat{F}_{j_0}(k)}| + C$$

This term is dominated by (18). This completes the proof of (12).

3. Remarks

- The counterexample can easily be extended to larger n (e.g. by appending some bump functions to the left or right of F).
- The counterexample above involved a potential F which was bounded in L^2, but for which $\sup_x |T_2(F, F)(k, x)|$ and $|T_3(F, F, F)(k, +\infty)|$ were large (about $\log N$) on a large subset of $[A, 2A]$. By letting N vary and taking suitable linear combinations of such variants of the above counterexample, one can in fact generate a potential F bounded in L^2 for which $\sup_x |T_2(F, F)(k, x)|$ is infinite and $|T_3(F, F, F)(k, x)|$ accumulates at ∞ for $x \to \infty$ for all k in a set of positive measure (one can even achieve blow-up almost everywhere). Thus it is not possible to estimate these multilinear expansions in any reasonable norm if one only assumes the potential to be in L^2. Similarly if F had a derivative in L^2; it is the decay of F which is relevant here, not the regularity.
- The unboundedness of T_3 on L^2 can be interpreted as stating that the (non-linear) scattering map $F \mapsto b_k(+\infty)$ from potentials to reflection coefficients is not C^3 on the domain of L^2 potentials. Similarly the map $F \mapsto a_k(+\infty)$ from potentials to transmission coefficients is not C^4 on

the domain of L^2 potentials. In particular these scattering maps are not analytic.

- Despite the bad behavior of the individual terms $T_k(F, \dots, F)$, the transmission and reflection coefficients $a_k(x)$, $b_k(x)$ are still bounded for the counterexample given above. This phenomenon is similar to the observation that the function $e^{ix} = 1 + ix - x^2/2 - \dots$ is bounded for arbitrarily large *real* x, even if the individual terms $(ix)^n/n!$ are not.

We now sketch the proof of boundedness of a_k, b_k. Suppose that $k = Aj_0/N + O(1/N)$ for some $N \leq j_0 \leq 2N$; we now fix j_0 and k. We can write

$$\begin{pmatrix} a_k(x) \\ b_k(x) \end{pmatrix} = G(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

where G is the 2×2 matrix solving the ODE

$$G'(x) = \begin{pmatrix} 0 & F(x)e^{-2ikx} \\ F(x)e^{2ikx} & 0 \end{pmatrix} G(x); \quad G(-\infty) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We define the matrices G_j similarly by

$$G_j'(x) = \begin{pmatrix} 0 & F_j(x)e^{-2ikx} \\ F_j(x)e^{2ikx} & 0 \end{pmatrix} G_j(x); \quad G_j(-\infty) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We observe the identity

$$G(x) = G_{j_1}(+\infty)G_{j_1-1}(+\infty)\dots G_N(+\infty)$$

whenever $N \leq j_1 \leq 2N$ and $x = N(j_1 + \frac{1}{2})$; this can be proven by an easy induction on j_1 and the observation that the above ODE are invariant under right-multiplication.

One can compute the $G_j(+\infty)$ using multilinear expansions (or using Gronwall's inequality), eventually obtaining

$$G_j(+\infty) = \begin{pmatrix} 1 + \frac{iC}{j-j_0} & 0 \\ 0 & 1 - \frac{iC}{j-j_0} \end{pmatrix} + O(|j - j_0|^{-2})$$

for all $j \neq j_0$, where C is a non-zero real constant. Because of the crucial factor of i in the diagonal entries we see that the operator norm $\|G_j(+\infty)\|$ of G_j is

$$\|G_j(+\infty)\| = 1 + O(|j - j_0|^{-2}).$$

This allows one to multiply the $G_j(+\infty)$ together and obtain boundedness of $G(x)$ and hence $a_k(x)$, $b_k(x)$.

In analogy with the observation concerning e^{ix}, one may need to use the fact that F is real in order to obtain boundedness of eigenfunctions in the L^2 case. When F is real there are additional estimates available, such as the scattering identity

$$\int \log |a_k(+\infty)| \, dk = C \int |F(x)|^2 \, dx$$

for some absolute constant C; see for instance [5].

We do not yet know how to obtain boundedness of eigenfunctions for L^2 potentials F. However we have been able to achieve this for a model problem in which the Fourier phases e^{2ikx} are replaced by a dyadic Walsh variant $e(k,x)$. See [11].

- One can modify the counterexample to provide similar counterexamples for Schrödinger operators $-\frac{d^2}{dx^2}+V$ with $V\in L^2$, either by using the Miura transform $V=F'+F^2$ mentioned in the introduction, or by inserting the standard WKB phase modification to the operators T_k as in [2]. We omit the details.
- The multilinear expansion of a leads to an expansion of $|a|^2$, whose quadratic term is equal to

$$2Re(T_2(F,F)) = 2Re(H_-(|\widehat{F}|^2)) = |\widehat{F}|^2$$

This term is in L^1, which is better than the term $T_2(F,F)$, which is in general only in the Lorentz space $L^{1,\infty}$. The higher order terms of the expansion of $|a|^2$ are however unbounded again. Using the identity $|a|^2 = 1+|b|^2$ we see that the fourth order term of $|a|^2$ is equal to

$$2Re(\overline{T_1(F)}T_3(F,F,F))$$

We now define the modified potential

$$G(x) = F(x) + G_0(x)$$

where F is as in the proof of Theorem 1.1 and $G_0(x)=\phi(x-N^3)$. Expanding the fourth order term by multilinearity, one observes that all terms can be estimated from above nicely with the exception of

$$2Re(\overline{T_1(G)}T_3(F,F,F))$$

Since $T_1(G)=\widehat{G}$ has more rapidly changing phase than $T_3(F,F,F)$, the real part and the modulus $\overline{T_1(G)}T_3(F,F,F)$ are of comparable size on a large set, and so this term is of the order $\log(N)$ on a large set just like $T_3(F,F,F)$ itself.

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