

## BOUNDS FOR LOG CANONICAL THRESHOLDS WITH APPLICATIONS TO BIRATIONAL RIGIDITY

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### Introduction

Let  $X$  be a smooth algebraic variety, defined over an algebraically closed field of characteristic zero, and let  $V \subset X$  be a proper closed subscheme. Our main goal in this paper is to study an invariant of the pair  $(X, V)$ , called the log canonical threshold of  $X$  along  $V$ , and denoted by  $\text{lc}(X, V)$ . Interest in bounds for log canonical thresholds is motivated by techniques that have recently been developed in higher dimensional birational geometry. In this paper, we study this invariant using intersection theory, degeneration techniques and jet schemes.

A natural question is how does this invariant behave under basic operations such as restrictions and projections. Restriction properties have been extensively studied in recent years, leading to important results and conjectures. In the first section of this paper, we investigate the behavior under projections, and we prove the following result (see Theorem 1.1 for a more precise statement):

**Theorem 0.1.** *With the above notation, suppose that  $V$  is Cohen-Macaulay, of pure codimension  $k$ , and let  $f : X \rightarrow Y$  be a proper, dominant, smooth morphism of relative dimension  $k - 1$ , with  $Y$  smooth. If  $f|_V$  is finite, then*

$$\text{lc}(Y, f_*[V]) \leq \frac{k! \cdot \text{lc}(X, V)^k}{k^k},$$

*and the inequality is strict if  $k \geq 2$ . Moreover, if  $V$  is locally complete intersection, then*

$$\text{lc}(Y, f_*[V]) \leq \frac{\text{lc}(X, V)^k}{k^k}.$$

Examples show that these bounds are sharp. The proof of the above theorem is based on a general inequality relating the log canonical threshold of a fractional ideal of the form  $h^{-b} \cdot \mathfrak{a}$ , and the colength of  $\mathfrak{a}$ . Here  $\mathfrak{a}$  is a zero dimensional ideal in the local ring of  $X$  at some (not necessarily closed) point,  $b \in \mathbb{Q}_+$ , and  $h$  is the equation of a smooth divisor. We prove this inequality in the second section (see Theorem 2.1), using a degeneration to monomial ideals. It generalizes a result from [dFEM], which was the case  $b = 0$ .

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In the third section, we give lower bounds for the log canonical threshold of affine subschemes defined by homogeneous equations of the same degree. We prove the following

**Theorem 0.2.** *Let  $V \subset X = \mathbb{A}^n$  be a subscheme defined by homogeneous equations of degree  $d$ . Let  $c = \text{lc}(\mathbb{A}^n, V)$ , and let  $Z$  be the non log terminal locus of  $(\mathbb{A}^n, c \cdot V)$ . If  $e = \text{codim}(Z, \mathbb{A}^n)$ , then*

$$\text{lc}(\mathbb{A}^n, V) \geq \frac{e}{d}.$$

*Moreover, we have equality if and only if the following holds:  $Z$  is a linear subspace, and if  $\pi : \mathbb{A}^n \rightarrow \mathbb{A}^n/Z$  is the projection, then there is a subscheme  $V' \subset \mathbb{A}^n/Z$  such that  $V = \pi^{-1}(V')$ ,  $\text{lc}(\mathbb{A}^n/Z, V') = e/d$ , and the non log terminal locus of  $(\mathbb{A}^n/Z, (e/d) \cdot V')$  is the origin.*

The proof of this result is based on the characterization of the log canonical threshold via jet schemes from [Mu2]. In the particular case when  $V$  is the affine cone over a projective hypersurface with isolated singularities, the second assertion in the above result proves a conjecture of Cheltsov and Park from [CP].

In the last section we apply the above bounds in the context of birational geometry. In their influential paper [IM], Iskovskikh and Manin proved that a smooth quartic threefold is what is called nowadays birationally superrigid; in particular, every birational automorphism is regular, and the variety is not rational. There has been a lot of work to extend this result to other Fano varieties of index one, in particular to smooth hypersurfaces of degree  $N$  in  $\mathbb{P}^N$ , for  $N > 4$ . The case  $N = 5$  was done by Pukhlikov in [Pu1], and the cases  $N = 6, 7, 8$  were proven by Cheltsov in [Che]. Moreover, Pukhlikov showed in [Pu2] that a general hypersurface as above is birationally superrigid, for every  $N > 4$ . We use our results to give an easy and uniform proof of birational superrigidity for arbitrary smooth hypersurfaces of degree  $N$  in  $\mathbb{P}^N$  when  $N$  is small.

**Theorem 0.3.** *If  $X \subset \mathbb{P}^N$  is a smooth hypersurface of degree  $N$ , and if  $4 \leq N \leq 12$ , then  $X$  is birationally superrigid.*

Based on previous ideas of Corti, Pukhlikov proposed in [Pu3] a proof of the birational rigidity of every smooth hypersurface of degree  $N$  in  $\mathbb{P}^N$ , for  $N \geq 6$ . Unfortunately, at the moment there is a gap in his arguments (see Remark 4.2 below). Despite this gap, the proof proposed in [Pu3] contains many remarkable ideas, and it seems likely that a complete proof could be obtained in the future along those lines. In fact, the outline of the proof of Theorem 0.3 follows his method, and our contribution is mainly to simplifying and solidifying his argument.

## 1. Singularities of log pairs under projections

Let  $X$  be a smooth algebraic variety, defined over an algebraically closed field of characteristic zero, and let  $V \subset X$  be a proper subscheme. For any rational

number  $c > 0$ , we can consider the pair  $(X, c \cdot V)$ . The usual definitions in the theory of singularities of pairs, for which we refer to [Kol], extend to this context. In particular, we say that an irreducible subvariety  $C \subset X$  is a center of non log canonicity (resp. non log terminality, non canonicity, non terminality) for  $(X, c \cdot V)$  if there is at least one divisorial valuation of  $K(X)$ , with center  $C$  on  $X$ , whose discrepancy along  $(X, c \cdot V)$  is  $< -1$  (resp.  $\leq -1$ ,  $< 0$ ,  $\leq 0$ ). We will denote by  $\text{lc}(X, V)$  the log canonical threshold of the pair  $(X, V)$ , i.e., the largest  $c$  such that  $(X, c \cdot V)$  is log canonical. We will occasionally consider also pairs of the form  $(X, c_1 \cdot V_1 - c_2 \cdot V_2)$ , where  $V_1, V_2 \subset X$  are proper subschemes of  $X$ . The definition of (log) terminal and canonical pairs extends in an obvious way to this setting.

We fix now the set-up for this section. Let  $f : X \rightarrow Y$  be a smooth and proper morphism onto a smooth algebraic variety  $Y$ . We assume that  $V \subset X$  is a pure dimensional, Cohen-Macaulay closed subscheme, such that  $\dim V = \dim Y - 1$ , and such that the restriction of  $f$  to  $V$  is finite. If  $[V]$  denotes the cycle associated to  $V$ , then its push-forward  $f_*[V]$  determines an effective Cartier divisor on  $Y$ . We set  $\text{codim}(V, X) = k$ .

**Theorem 1.1.** *With the above notation, let  $C \subset X$  be an irreducible center of non log terminality for  $(X, c \cdot V)$ , for some  $c > 0$ . Then  $f(C)$  is a center of non log terminality (even non log canonicity, if  $k \geq 2$ ) for the pair*

$$(1.1) \quad \left( Y, \frac{k! \cdot c^k}{k^k} \cdot f_*[V] \right).$$

Moreover, if  $V$  is locally complete intersection (l.c.i. for short) then  $f(C)$  is a center of non log terminality for the pair

$$(1.2) \quad \left( Y, \frac{c^k}{k^k} \cdot f_*[V] \right).$$

**Example 1.2.** Let  $k$  and  $n$  be two positive integers with  $n > k$ , and let  $R = K[x_k, \dots, x_n]$ . We take  $X = \mathbb{P}_R^{k-1} = \text{Proj } R[x_0, \dots, x_{k-1}]$ ,  $Y = \text{Spec } R$ , and let  $f$  be the natural projection from  $X$  to  $Y$ . For any  $t > 0$ , let  $V_t$  be the subscheme of  $X$  defined by the homogeneous ideal  $(x_1, \dots, x_k)^t$ . Note that  $\text{lc}(X, V_t) = k/t$ , and that if  $c = k/t$ , then  $V_1$  is a center of non log terminality for  $(X, c \cdot V_t)$ . Since  $l(\mathcal{O}_{V_t, V_1}) = \binom{k+t-1}{k}$ , we see that

$$\lim_{t \rightarrow \infty} \frac{k! \cdot c^k / k^k}{\text{lc}(Y, f_*[V_t])} = \lim_{t \rightarrow \infty} \frac{t(t+1) \dots (t+k-1)}{t^k} = 1,$$

so the bound in (1.1) is sharp (at least asymptotically).

To prove sharpness in the l.c.i. case, let  $W_t \subset X$  be the complete intersection subscheme defined by  $(x_1^t, \dots, x_k^t)$ . This time  $l(\mathcal{O}_{W_t, W_1}) = t^k$ , and  $\text{lc}(Y, f_*[W_t]) = 1/t^k = \text{lc}(X, W_t)^k / k^k$ .

*Proof of Theorem 1.1.* By hypothesis, there is a proper birational morphism  $\nu : W \rightarrow X$ , where  $W$  can be chosen to be smooth, and a smooth irreducible divisor

$E$  on  $W$ , such that  $\nu(E) = C$ , and such that the discrepancy of  $(X, c \cdot V)$  at  $E$  is

$$(1.3) \quad a_E(X, c \cdot V) \leq -1.$$

The surjection  $f$  induces an inclusion of function fields  $f^* : K(Y) \hookrightarrow K(X)$ . Let  $R_E := \mathcal{O}_{W,E} \subset K(X)$  be the discrete valuation ring associated to the valuation along  $E$ , and let  $R = (f^*)^{-1}R_E$ . Note that  $R$  is a non-trivial discrete valuation ring.

**Lemma 1.3.**  *$R$  corresponds to a divisorial valuation.*

*Proof.* It is enough to show that the transcendence degree of the residue field of  $R$  over the ground field is  $\dim Y - 1$  (see [KM], Lemma 2.45). This follows from [ZS], VI.6, Corollary 1.  $\square$

The lemma implies that there is a proper birational morphism  $\gamma : Y' \rightarrow Y$  and an irreducible divisor  $G$  on  $Y'$  such that  $R = \mathcal{O}_{Y',G}$ . By Hironaka’s theorem, we may assume that both  $Y'$  and  $G$  are smooth, and moreover, that the union between  $G$  and the exceptional locus of  $\gamma$  has simple normal crossings. Since the center of  $R_E$  on  $X$  is  $C$ , we deduce that  $R$  has center  $f(C)$  on  $Y$ , so  $\gamma(G) = f(C)$ .

Consider the fibered product  $X' = Y' \times_Y X$ . We may clearly assume that  $\nu$  factors through the natural map  $\phi : X' \rightarrow X$ . Therefore we have the following commutative diagram:

$$\begin{array}{ccccc} W & \xrightarrow{\eta} & X' & \xrightarrow{\phi} & X \\ & & \downarrow g & & \downarrow f \\ & & Y' & \xrightarrow{\gamma} & Y, \end{array}$$

where  $\phi \circ \eta = \nu$ . Note that  $X'$  is a smooth variety,  $g$  is a smooth, proper morphism, and  $\eta$  and  $\phi$  are proper, birational morphisms. Let  $V' = \phi^{-1}(V)$  be the scheme theoretic inverse image of  $V$  in  $X'$ , i.e., the subscheme of  $X'$  defined by the ideal sheaf  $I_V \cdot \mathcal{O}_{X'}$ .

**Lemma 1.4.**  *$V'$  is pure dimensional,  $\text{codim}(V', X') = k$ , and  $\phi^*[V]$  is the class of  $[V']$ . Moreover, if  $V$  is l.c.i., then so is  $V'$ .*

*Proof.* Note that both  $\gamma$  and  $\phi$  are l.c.i. morphisms, because they are morphisms between smooth varieties. The pull-back in the statement is the pull-back by such a morphism (see [Ful], Section 6.6). Recall how this is defined. We factor  $\gamma$  as  $\gamma_1 \circ \gamma_2$ , where  $\gamma_1 : Y' \times Y \rightarrow Y$  is the projection, and  $\gamma_2 : Y' \hookrightarrow Y' \times Y$  is the graph of  $\gamma$ . By pulling-back, we get a corresponding decomposition  $\phi = \phi_1 \circ \phi_2$ , with  $\phi_1$  smooth, and  $\phi_2 : X' \hookrightarrow Y' \times X$  a regular embedding of codimension  $\dim Y'$ . Then  $\phi^*[V] = \phi_2^!([Y' \times V])$ .

Since  $f|_V$  is finite and  $V' = Y' \times_Y V$ ,  $g|_{V'}$  is also finite. Moreover, since  $g(V')$  is a proper subset of  $Y'$ , we see that  $\dim V' \leq \dim Y' - 1$ . On the other hand,  $V'$  is locally cut in  $Y' \times V$  by  $\dim Y'$  equations, so that every irreducible

component of  $V'$  has dimension at least  $\dim V$ . Therefore  $V'$  is pure dimensional, and  $\dim V' = \dim V$ .

Since  $Y' \times V$  is Cohen-Macaulay, this also implies that  $\phi_2^1([Y' \times V])$  is equal to the class of  $[V']$ , by Proposition 7.1 in [Ful]. This proves the first assertion. Moreover, if  $V$  is l.c.i., then it is locally defined in  $X$  by  $k$  equations. The same is true for  $V'$ , hence  $V'$  is l.c.i., too.  $\square$

We will use the following notation for multiplicities. Suppose that  $W$  is an irreducible subvariety of a variety  $Z$ . Then the multiplicity of  $Z$  along  $W$  is denoted by  $e_W Z$  (we refer to [Ful], Section 4.3, for definition and basic properties). If  $\alpha = \sum_i n_i [T_i]$  is a pure dimensional cycle on  $Z$ , then  $e_W \alpha := \sum_i n_i e_W T_i$  (if  $W \not\subseteq T_i$ , then we put  $e_W T_i = 0$ ). Note that if  $W$  is a prime divisor, and if  $D$  is an effective Cartier divisor on  $Z$ , then we have  $e_W [D] = \text{ord}_W(D)$ , where  $[D]$  is the cycle associated to  $D$ , and  $\text{ord}_W(D)$  is the coefficient of  $W$  in  $[D]$ . As we work on smooth varieties, from now on we will identify  $D$  with  $[D]$ .

Let  $F = \eta(E)$ . Note that by construction, we have  $g(F) = G$ . Since  $F \subseteq V'$ , and  $g|_{V'}$  is finite, and  $\dim G = \dim V'$ , it follows that  $F$  is an irreducible component of  $V'$ , hence  $\text{codim}(F, X') = k$ . We set  $a = e_F(K_{X'/X})$ .

To simplify the statements, we put

$$\delta = \begin{cases} 1 & \text{if } V \text{ is a l.c.i.,} \\ k! & \text{otherwise.} \end{cases}$$

**Lemma 1.5.** *We have*

$$\text{ord}_G(\gamma^* f_* [V]) \geq \frac{(a+1)k^k}{\delta c^k},$$

and the inequality is strict in the case  $\delta = k!$ , if  $k \geq 2$ .

*Proof.* Since  $\phi$  and  $\gamma$  are l.c.i. morphisms of the same relative dimension, it follows from [Ful], Example 17.4.1, and Lemma 1.4 that  $g_* [V']$  and  $\gamma^* f_* [V]$  are linearly equivalent, as divisors on  $Y'$ . As the two divisors are equal outside the exceptional locus of  $\gamma$ , we deduce from the Negativity Lemma (see [KM], Lemma 3.39) that also their  $\gamma$ -exceptional components must coincide. This gives  $g_* [V'] = \gamma^* f_* [V]$ .

In particular,  $\text{ord}_G(\gamma^* f_* [V])$  is greater or equal to the coefficient of  $F$  in  $[V']$ . Lemma 1.4 implies

$$\text{ord}_G(\gamma^* f_* [V]) \geq l(\mathcal{O}_{V',F}),$$

so that it is enough to show that

$$(1.4) \quad l(\mathcal{O}_{V',F}) \geq \frac{(a+1)k^k}{\delta c^k},$$

and that the inequality is strict in the case  $\delta = k!$ , if  $k \geq 2$ .

By replacing  $W$  with a higher model, we may clearly assume that  $\nu^{-1}(V)$  is an effective divisor on  $W$ . If  $I_V \subseteq \mathcal{O}_X$  is the ideal defining  $V$ , then we put  $\text{ord}_E(I_V) := \text{ord}_E \nu^{-1}(V)$ . It follows from (1.3) that we have

$$-1 \geq \text{ord}_E(K_{W/X}) - c \cdot \text{ord}_E(I_V) = \text{ord}_E(K_{W/X'}) - (c \cdot \text{ord}_E(I_{V'}) - \text{ord}_E(K_{X'/X})).$$

Therefore  $F$  is a center of non log terminality for the pair  $(X', c \cdot V' - K_{X'/X})$ . Since  $g(F) = G$  is a divisor on  $Y'$ , it follows that  $F$  can not be contained in the intersection of two distinct  $\phi$ -exceptional divisors. Hence the support of  $K_{X'/X}$  is smooth at the generic point of  $F$ . Then (1.4) follows from Theorem 2.1 below (note that the length of a complete intersection ideal coincides with its Samuel multiplicity).  $\square$

We continue the proof of Theorem 1.1. Note that  $\text{ord}_G K_{Y'/Y} \leq e_F(g^* K_{Y'/Y})$ . Since  $K_{X'/X} = g^* K_{Y'/Y}$  (see [Har], Proposition II 8.10), we deduce

$$\text{ord}_G K_{Y'/Y} \leq a.$$

In conjunction with Lemma 1.5, this gives

$$a_G \left( Y, \frac{\delta c^k}{k^k} \cdot f_*[V] \right) = \text{ord}_G \left( K_{Y'/Y} - \frac{\delta c^k}{k^k} \cdot \gamma^* f_*[V] \right) \leq -1.$$

Moreover, this inequality is strict in the case when  $\delta = k!$ , if  $k \geq 2$ . This completes the proof of Theorem 1.1.  $\square$

**Remark 1.6.** We refer to [Pu3] for a result on the canonical threshold of complete intersection subschemes of codimension 2, via generic projection.

## 2. Multiplicities of fractional ideals

In this section we extend some of the results of [dFEM], as needed in the proof of Theorem 1.1. More precisely, we consider the following set-up. Let  $X$  be a smooth variety,  $V \subset X$  a closed subscheme, and let  $Z$  be an irreducible component of  $V$ . We denote by  $n$  the codimension of  $Z$  in  $X$ , and by  $\mathfrak{a} \subset \mathcal{O}_{X,Z}$  the image of the ideal defining  $V$ . Let  $H \subset X$  be a prime divisor containing  $Z$ , such that  $H$  is smooth at the generic point of  $Z$ . We consider the pair

$$(X, V - b \cdot H),$$

for a given  $b \in \mathbb{Q}_+$ .

**Theorem 2.1.** *With the above notation, suppose that for some  $\mu \in \mathbb{Q}_+^*$ ,  $(X, \frac{1}{\mu}(V - b \cdot H))$  is not log terminal at the generic point of  $Z$ . Then*

$$(2.1) \quad l(\mathcal{O}_{X,Z}/\mathfrak{a}) \geq \frac{n^n \mu^{n-1}(\mu + b)}{n!},$$

*and the inequality is strict if  $n \geq 2$ . Moreover, if  $e(\mathfrak{a})$  denotes the Samuel multiplicity of  $\mathcal{O}_{X,Z}$  along  $\mathfrak{a}$ , then*

$$(2.2) \quad e(\mathfrak{a}) \geq n^n \mu^{n-1}(\mu + b).$$

**Remark 2.2.** For  $n = 2$ , inequality (2.2) gives a result of Corti from [Co2]. On the other hand, if  $b = 0$ , then the statement reduces to Theorems 1.1 and 1.2 in [dFEM].

*Proof of Theorem 2.1.* We see that (2.1) implies (2.2) as follows. If we apply the first formula to the subscheme  $V_t \subseteq X$  defined by  $\mathfrak{a}^t$ , to  $\mu_t = \mu t$ , and to  $b_t = bt$ , we get

$$l(\mathcal{O}_{X,Z}/\mathfrak{a}^t) \geq \frac{n^n \mu^{n-1} (\mu + b)}{n!} t^n.$$

Dividing by  $t^n$  and passing to the limit as  $t \rightarrow \infty$  gives (2.2).

In order to prove (2.1), we proceed as in [dFEM]. Passing to the completion, we obtain an ideal  $\widehat{\mathfrak{a}}$  in  $\widehat{\mathcal{O}}_{X,Z}$ . We identify  $\widehat{\mathcal{O}}_{X,Z}$  with  $K[[x_1, \dots, x_n]]$  via a fixed isomorphism, where  $K$  is the residue field of  $\mathcal{O}_{X,Z}$ . Moreover, we may choose the local coordinates so that the image of an equation  $h$  defining  $H$  in  $\mathcal{O}_{X,Z}$  is  $x_n$ . Since  $\widehat{\mathfrak{a}}$  is zero dimensional, we can find an ideal  $\mathfrak{b} \subset R = K[x_1, \dots, x_n]$ , which defines a scheme supported at the origin, and such that  $\widehat{\mathfrak{b}} = \widehat{\mathfrak{a}}$ .

If  $V', H' \subset \mathbb{A}^n$  are defined by  $\mathfrak{b}$  and  $x_n$ , respectively, then  $(\mathbb{A}^n, \frac{1}{\mu}(V' - b \cdot H'))$  is not log terminal at the origin. We write  $\mu = r/s$ , for some  $r, s \in \mathbb{N}$ , and we may clearly assume that  $sb \in \mathbb{N}$ . Consider the ring  $S = K[x_1, \dots, x_{n-1}, y]$ , and the inclusion  $R \subseteq S$  which takes  $x_n$  to  $y^r$ . This determines a cyclic covering of degree  $r$

$$M := \text{Spec } S \rightarrow N := \mathbb{A}^n = \text{Spec } R,$$

with ramification divisor defined by  $(y^{r-1})$ .

For any ideal  $\mathfrak{c} \subset R$ , we put  $\widetilde{\mathfrak{c}} := \mathfrak{c}S$ . If  $W$  is the scheme defined by  $\mathfrak{c}$ , then we denote by  $\widetilde{W}$  the scheme defined by  $\widetilde{\mathfrak{c}}$ . In particular, if  $H'' \subset M$  is defined by  $(y)$ , then  $\widetilde{H}' = rH''$ . It follows from [Ein], Proposition 2.8 (see also [Laz], Section 9.5.E) that  $(N, \frac{1}{\mu}(V' - b \cdot H'))$  is not log terminal at the origin in  $N$  if and only if  $(M, \frac{1}{\mu} \cdot \widetilde{V}' - (sb + r - 1)H'')$  is not log terminal at the origin in  $M$ .

We write the rest of the proof in the language of multiplier ideals, for which we refer to [Laz]. We use the formal exponential notation for these ideals. If  $\widetilde{\mathfrak{b}}$  is the ideal defining  $\widetilde{V}'$ , then the above non log terminality condition on  $M$  can be interpreted as saying that

$$(2.3) \quad y^{bs+r-1} \notin \mathcal{J}(\widetilde{\mathfrak{b}}^{1/\mu}).$$

We choose a monomial order in  $S$ , with the property that

$$x_1 > \dots > x_{n-1} > y^{bs+r-1}.$$

This induces flat deformations to monomial ideals (see [Eis], Chapter 15). For an ideal  $\mathfrak{d} \subseteq S$ , we write the degeneration as  $\mathfrak{d}_t \rightarrow \mathfrak{d}_0$ , where  $\mathfrak{d}_t \cong \mathfrak{d}$  for  $t \neq 0$  and  $\mathfrak{d}_0 =: \text{in}(\mathfrak{d})$  is a monomial ideal.

We claim that

$$(2.4) \quad y^{bs+r-1} \notin \text{in}(\mathcal{J}(\widetilde{\mathfrak{b}}^{1/\mu})).$$

Indeed, suppose that  $y^{bs+r-1} \in \text{in}(\mathcal{J}(\widetilde{\mathfrak{b}}^{1/\mu}))$ . Then we can find an element  $f \in \mathcal{J}(\widetilde{\mathfrak{b}}^{1/\mu})$  such that  $\text{in}(f) = y^{bs+r-1}$ . Because of the particular monomial order we have chosen,  $f$  must be a polynomial in  $y$  of degree  $bs + r - 1$ . On the other hand,  $\mathcal{J}(\widetilde{\mathfrak{b}}^{1/\mu})$  defines a scheme which is supported at the origin (or

empty), since so does  $\tilde{\mathfrak{b}}$ . We deduce that  $y^i \in \mathcal{J}(\tilde{\mathfrak{b}}^{1/\mu})$ , for some  $i \leq bs + r - 1$ , which contradicts (2.3).

**Lemma 2.3.** *For every ideal  $\mathfrak{d} \subseteq S$ , and every  $c \in \mathbb{Q}_+^*$ , we have*

$$\text{in}(\mathcal{J}(\mathfrak{d}^c)) \supseteq \mathcal{J}(\text{in}(\mathfrak{d})^c).$$

*Proof.* Consider the family  $\pi : \mathfrak{M} = \mathbb{A}^n \times T \rightarrow T$ , with  $T = \mathbb{A}^1$ , and the ideal  $\mathfrak{D} \subset \mathcal{O}_{\mathfrak{M}}$  corresponding to the degeneration of  $\mathfrak{d}$  described above. If  $U$  is the complement of the origin in  $T$ , then there is an isomorphism

$$(\pi^{-1}(U), \mathfrak{D}|_{\pi^{-1}(U)}) \simeq (\mathbb{A}^n \times U, \text{pr}_1^{-1}\mathfrak{d}).$$

Via this isomorphism we have  $\mathcal{J}(\pi^{-1}(U), \mathfrak{D}^c) \simeq \text{pr}_1^{-1}(\mathcal{J}(\mathfrak{d}^c))$ . Since the family degenerating to the initial ideal is flat, we deduce easily that

$$\mathcal{J}(\mathfrak{M}, \mathfrak{D}^c) \cdot \mathcal{O}_{\pi^{-1}(0)} \subseteq \text{in}(\mathcal{J}(\mathfrak{d}^c)).$$

On the other hand, the Restriction Theorem (see [Laz]) gives

$$\mathcal{J}(\text{in}(\mathfrak{d})^c) = \mathcal{J}((\mathfrak{D}|_{\pi^{-1}(0)})^c) \subseteq \mathcal{J}(\mathfrak{M}, \mathfrak{D}^c) \cdot \mathcal{O}_{\pi^{-1}(0)}.$$

If we put together the above inclusions, we get the assertion of the lemma.  $\square$

Note that the monomial order on  $S$  induces a monomial order on  $R$ , and that  $\widetilde{\text{in}(\mathfrak{b})} = \text{in}(\tilde{\mathfrak{b}})$ . Indeed, the inclusion  $\widetilde{\text{in}(\mathfrak{b})} \subseteq \text{in}(\tilde{\mathfrak{b}})$  is obvious, and the corresponding subschemes have the same length  $r \cdot l(R/\mathfrak{b})$ .

On the other hand, Lemma 2.3 and (2.4) give

$$y^{bs+r-1} \notin \mathcal{J}(\text{in}(\tilde{\mathfrak{b}})^{1/\mu}).$$

Applying again Proposition 2.8 in [Ein], in the other direction, takes us back in  $R$ : we deduce that  $(N, \frac{1}{\mu}(W - b \cdot H'))$  is not log terminal at the origin, where  $W \subset N$  is defined by  $\text{in}(\mathfrak{b})$ . Since  $l(\mathcal{O}_{X,Z}/\mathfrak{a}) = l(R/\mathfrak{b}) = l(R/\text{in}(\mathfrak{b}))$ , we have reduced the proof of (2.1) to the case when  $\mathfrak{a}$  is a monomial ideal. In this case, we have in fact a stronger statement, which we prove in the lemma below; therefore the proof of Theorem 2.1 is complete.  $\square$

The following is the natural generalization of Lemma 2.1 in [dFEM].

**Lemma 2.4.** *Let  $\mathfrak{a}$  be a zero dimensional monomial ideal in the ring  $R = K[x_1, \dots, x_n]$ , defining a scheme  $V$ . Let  $H_i$  be the hyperplane defined by  $x_i = 0$ . We consider  $\mu \in \mathbb{Q}_+^*$  and  $b_i \in \mathbb{Q}$ , such that  $\mu \geq \max_i \{b_i\}$ . If the pair  $(\mathbb{A}^n, \frac{1}{\mu}(V + \sum_i b_i H_i))$  is not log terminal, then*

$$l(R/\mathfrak{a}) \geq \frac{n^n}{n!} \cdot \prod_{i=1}^n (\mu - b_i),$$

and the inequality is strict if  $n \geq 2$ .

*Proof.* We use the result in [ELM] which gives the condition for a monomial pair, with possibly negative coefficients, to be log terminal. This generalizes the formula for the log canonical threshold of a monomial ideal from [How]. It follows from [ELM] that  $(X, \frac{1}{\mu}(V + \sum_i b_i H_i))$  is not log terminal if and only if there is a facet of the Newton polytope associated to  $\mathfrak{a}$  such that, if  $\sum_i u_i/a_i = 1$  is the equation of the hyperplane supporting it, then

$$\sum_{i=1}^n \frac{\mu - b_i}{a_i} \leq 1.$$

Applying the inequality between the arithmetic mean and the geometric mean of the set of nonnegative numbers  $\{(\mu - b_i)/a_i\}_i$ , we deduce

$$\prod_i a_i \geq n^n \cdot \prod_i (\mu - b_i).$$

We conclude using the fact that  $n! \cdot l(R/\mathfrak{a}) \geq \prod_i a_i$ , and the inequality is strict if  $n \geq 2$  (see, for instance, Lemma 1.3 in [dFEM]).  $\square$

### 3. Log canonical thresholds of affine cones

In this section we give a lower bound for the log canonical threshold of a subscheme  $V \subset \mathbb{A}^n$ , cut out by homogeneous equations of the same degree. The bound involves the dimension of the non log terminal locus of  $(\mathbb{A}^n, c \cdot V)$ , where  $c = \text{lc}(\mathbb{A}^n, V)$ . Moreover, we characterize the case when we have equality. In the particular case when  $V$  is the affine cone over a projective hypersurface with isolated singularities, this proves a conjecture of Cheltsov and Park from [CP].

The main ingredient we use for this bound is a formula for the log canonical threshold in terms of jet schemes, from [Mu2]. Recall that for an arbitrary scheme  $W$ , of finite type over the ground field  $k$ , the  $m$ th jet scheme  $W_m$  is again a scheme of finite type over  $k$  characterized by

$$\text{Hom}(\text{Spec } A, W_m) \simeq \text{Hom}(\text{Spec } A[t]/(t^{m+1}), W),$$

for every  $k$ -algebra  $A$ . Note that  $W_m(k) = \text{Hom}(\text{Spec } k[t]/(t^{m+1}), W)$ , and in fact, we will be interested only in the dimensions of these spaces. For the basic properties of the jet schemes, we refer to [Mu1] and [Mu2].

**Theorem 3.1.** ([Mu2], 3.4) *If  $X$  is a smooth, connected variety of dimension  $n$ , and if  $V \subset X$  is a subscheme, then the log canonical threshold of  $(X, V)$  is given by*

$$\text{lc}(X, V) = n - \sup_{m \in \mathbb{N}} \frac{\dim V_m}{m + 1}.$$

*Moreover, there is  $p \in \mathbb{N}$ , depending on the numerical data given by a log resolution of  $(X, V)$ , such that  $\text{lc}(X, V) = n - (\dim V_m)/(m + 1)$  whenever  $p \mid (m + 1)$ .*

For every  $W$  and every  $m \geq 1$ , there are canonical projections  $\phi_m^W : W_m \rightarrow W_{m-1}$  induced by the truncation homomorphisms  $k[t]/(t^{m+1}) \rightarrow k[t]/(t^m)$ . By

composing these projections we get morphisms  $\pi_m^W : W_m \rightarrow W$ . When there is no danger of confusion, we simply write  $\phi_m$  and  $\pi_m$ .

If  $W$  is a smooth, connected variety, then  $W_m$  is smooth, connected, and  $\dim W_m = (m+1)\dim W$ , for all  $m$ . It follows from definition that taking jet schemes commutes with open immersions. In particular, if  $W$  has pure dimension  $n$ , then  $\pi_m^{-1}(W_{\text{reg}})$  is smooth, of pure dimension  $(m+1)n$ .

Recall that the non log terminal locus of a pair is the union of all centers of non log terminality. In other words, its complement is the largest open subset over which the pair is log terminal. Theorem 3.1 easily gives a description via jet schemes of the non log terminal locus of a pair which is log canonical, but is not log terminal. Suppose that  $(X, V)$  is as in the theorem, and let  $c = \text{lc}(X, V)$ . We say that an irreducible component  $T$  of  $V_m$  (for some  $m$ ) computes  $\text{lc}(X, V)$  if  $\dim(T) = (m+1)(n-c)$ . Note that basic results on jet schemes show that for every irreducible component  $T$  of  $V_m$ , the projection  $\pi_m(T)$  is closed in  $V$  (see [Mu1]). It follows from Theorem 3.1 that if  $W$  is an irreducible component of  $V_m$  that computes the log canonical threshold of  $(X, V)$  then  $\pi_m(W)$  is contained in the non log terminal locus of  $(X, c \cdot V)$  (see also [ELM]).

For future reference, we record here two lemmas. For  $x \in \mathbb{R}$ , we denote by  $[x]$  the largest integer  $p$  such that  $p \leq x$ .

**Lemma 3.2.** ([Mu1], 3.7) *If  $X$  is a smooth, connected variety of dimension  $n$ ,  $D \subset X$  is an effective divisor, and  $x \in D$  is a point with  $e_x D = q$ , then*

$$\dim(\pi_m^D)^{-1}(x) \leq mn - [m/q],$$

for every  $m \in \mathbb{N}$ .

In fact, the only assertion we will need from Lemma 3.2 is that  $\dim(\pi_m^D)^{-1}(x) \leq mn - 1$ , if  $m \geq q$ , which follows easily from the equations describing the jet schemes (see [Mu1]).

**Lemma 3.3.** ([Mu2] 2.3) *Let  $\Phi : \mathcal{W} \rightarrow S$  be a family of schemes, and let us denote the fiber  $\Phi^{-1}(s)$  by  $\mathcal{W}_s$ . If  $\tau : S \rightarrow \mathcal{W}$  is a section of  $\Phi$ , then the function*

$$f(s) = \dim(\pi_m^{\mathcal{W}_s})^{-1}(\tau(s))$$

is upper semi-continuous on the set of closed points of  $S$ , for every  $m \in \mathbb{N}$ .

The following are the main results in this section.

**Theorem 3.4.** *Let  $V \subset \mathbb{A}^n$  be a subscheme whose ideal is generated by homogeneous polynomials of degree  $d$ . Let  $c = \text{lc}(\mathbb{A}^n, V)$ , and let  $Z$  be the non log terminal locus of  $(\mathbb{A}^n, c \cdot V)$ . If  $e = \text{codim}(Z, \mathbb{A}^n)$ , then  $c \geq e/d$ .*

**Theorem 3.5.** *With the notation in the previous theorem,  $c = e/d$  if and only if  $V$  satisfies the following three properties:*

- (a)  $Z = L$  is a linear subspace of codimension  $e$ .

- (b)  $V$  is the pull back of a closed subscheme  $V' \subset \mathbb{A}^n/L$ , which is defined by homogeneous polynomials of degree  $d$  and such that  $\text{lc}(\mathbb{A}^n/L, V') = e/d$ .
- (c) The non log terminal locus of  $(\mathbb{A}^n/L, e/d \cdot V')$  is just the origin.

*Proof of Theorem 3.4.* If  $\pi_m : V_m \rightarrow V$  is the canonical projection, then we have an isomorphism

$$(3.1) \quad \pi_m^{-1}(0) \simeq V_{m-d} \times \mathbb{A}^{n(d-1)},$$

for every  $m \geq d-1$  (we put  $V_{-1} = \{0\}$ ). Indeed, for a  $k$ -algebra  $A$ , an  $A$ -valued point of  $\pi_m^{-1}(0)$  is a ring homomorphism

$$\phi : k[X_1, \dots, X_n]/(F_1, \dots, F_s) \rightarrow A[t]/(t^{m+1}),$$

such that  $\phi(X_i) \in (t)$  for all  $i$ . Here  $F_1, \dots, F_s$  are homogeneous equations of degree  $d$ , defining  $V$ . Therefore we can write  $\phi(X_i) = tf_i$ , and  $\phi$  is a homomorphism if and only if the classes of  $f_i$  in  $A[t]/(t^{m+1-d})$  define an  $A$ -valued point of  $V_{m-d}$ . But  $\phi$  is uniquely determined by the classes of  $f_i$  in  $A[t]/(t^m)$ , so this proves the isomorphism in equation (3.1).

By Theorem 3.1, we can find  $p$  such that

$$\dim V_{pd-1} = pd(n-c).$$

Let  $W$  be an irreducible component of  $V_{pd-1}$  computing  $\text{lc}(X, V)$ , so  $\dim W = pd(n-c)$  and  $\pi_{pd-1}(W) \subset Z$ . By our hypothesis,  $\dim \pi_{pd-1}(W) \leq n-e$ . Therefore Lemma 3.3 gives

$$(3.2) \quad pd(n-c) = \dim W \leq \dim \pi_{pd-1}^{-1}(0) + n - e = \dim V_{(p-1)d-1} + (d-1)n + n - e,$$

where the last equality follows from (3.1). Another application of Theorem 3.1 gives

$$(3.3) \quad \dim V_{(p-1)d-1} \leq (p-1)d(n-c).$$

Using this and (3.2), we get  $c \geq e/d$ . □

*Proof of Theorem 3.5.* We use the notation in the above proof. Since  $c = e/d$ , we see that in both equations (3.2) and (3.3) we have, in fact, equalities. The equality in (3.3) shows that  $\dim V_{(p-1)d-1} = (p-1)d(n-c)$ , so we may run the same argument with  $p$  replaced by  $p-1$ . Continuing in this way, we see that we may suppose that  $p = 1$ . In this case, the equality in (3.2) shows that for some irreducible component  $W$  of  $V_{d-1}$ , with  $\dim W = dn - e$ , we have  $\dim \pi_{d-1}(W) = n - e$ . It follows that if  $Z_1 := \pi_{d-1}(W)$ , then  $Z_1$  is an irreducible component of  $Z$ .

Fix  $x \in Z_1$ . If  $\text{mult}_x F \leq d-1$ , for some degree  $d$  polynomial  $F$  in the ideal of  $V$ , then Lemma 3.2 would give  $\dim \pi_{d-1}^{-1}(x) \leq (d-1)n - 1$ . This would imply  $\dim W \leq n - e + (d-1)n - 1$ , a contradiction. Therefore we must have  $\text{mult}_x F \geq d$ , for every such  $F$ .

Recall that we have degree  $d$  generators of the ideal of  $V$ , denoted by  $F_1, \dots, F_s$ . Let  $L_i = \{x \in \mathbb{A}^n \mid \text{mult}_x F_i = d\}$ , for  $i \leq s$ . By the Bézout theorem,  $L_i$  is a linear space. If  $L = \bigcap_{i=1}^s L_i$ , then  $Z_1 \subset L$ . On the other hand, by blowing-up along  $L$ , we see that  $L$  is contained in the non log terminal locus of  $(\mathbb{A}^n, c \cdot V)$ . Therefore  $Z_1 = L$ . Let  $z_1, \dots, z_e$  be the linear forms defining  $L$ . Then each  $F_i$  is a homogeneous polynomial of degree  $d$  in  $z_1, \dots, z_e$ . This shows that  $V$  is the pull back of a closed subscheme  $V' \subset \mathbb{A}^n/L$ , defined by  $F_1, \dots, F_s$ . Since the projection map  $\pi : \mathbb{A}^n \rightarrow \mathbb{A}^n/L$  is smooth and surjective, we see that  $\text{lc}(\mathbb{A}^n/L, V') = \text{lc}(\mathbb{A}^n, V)$  and that the non log terminal locus of  $(\mathbb{A}^n, \frac{e}{d} \cdot V)$  is just the pull-back of the corresponding locus for the pair  $(\mathbb{A}^n/L, \frac{e}{d} \cdot V')$ . Note that the non log terminal locus of  $(\mathbb{A}^n/L, \frac{e}{d} \cdot V')$  is defined by an homogeneous ideal. By dimension considerations, we conclude that this locus consists just of the origin, so  $Z = L$ .

Conversely, if  $V$  is the pull back of a closed subscheme from  $\mathbb{A}^n/L$  as described in the theorem, one checks that  $\text{lc}(\mathbb{A}^n, V) = e/d$  and that the corresponding non log terminal locus is just  $L$ .  $\square$

Let  $V'$  be a closed subscheme of  $\mathbb{P}^{n-1}$  defined by degree  $d$  homogeneous polynomials  $F_1, \dots, F_s$ , and let  $V$  be the closed subscheme in  $\mathbb{A}^n$  defined by the same set of polynomials. Let  $c = \text{lc}(\mathbb{P}^{n-1}, V')$ , and let  $Z'$  be the non log terminal locus of  $(\mathbb{P}^{n-1}, c \cdot V')$ . Suppose that the codimension of  $Z'$  in  $\mathbb{P}^{n-1}$  is  $e$ .

**Corollary 3.6.** *With the above notation,  $\text{lc}(\mathbb{P}^{n-1}, V') \geq e/d$ . Moreover, if we have equality, then  $V'$  is the cone over a scheme in some  $\mathbb{P}^{e-1}$ .*

*Proof.* Note that

$$\text{lc}(\mathbb{P}^{n-1}, V') = \text{lc}(\mathbb{A}^n - \{0\}, V - \{0\}) \geq \text{lc}(\mathbb{A}^n, V).$$

Now the first assertion follows from Theorem 3.4.

If  $\text{lc}(\mathbb{P}^{n-1}, V') = e/d$ , then  $\text{lc}(\mathbb{A}^n, V) = e/d$  and the non log terminal locus of  $(\mathbb{A}^n, \frac{e}{d} \cdot V)$  is a linear space  $L$  of codimension  $e$ . If  $z_1, \dots, z_e$  are the linear forms defining  $L$ , then each  $F_i$  is a homogeneous polynomial of degree  $d$  in  $z_1, \dots, z_e$ . Therefore  $V'$  is the cone with center  $L$  over the closed subscheme of  $\mathbb{P}^{e-1}$  defined by  $F_1, \dots, F_s$ .  $\square$

**Remark 3.7.** In [CP], Cheltsov and Park studied the log canonical threshold of singular hyperplane sections of smooth, projective hypersurfaces. If  $X \subset \mathbb{P}^n$  is a smooth hypersurface of degree  $d$ , and if  $V = X \cap H$ , for a hyperplane  $H$ , then they have shown that

$$(3.4) \quad \text{lc}(X, V) \geq \min\{(n-1)/d, 1\}.$$

It follows from Theorem 3.1 that  $\text{lc}(X, V) = \text{lc}(\mathbb{P}^{n-1}, V)$ . As it is well known that  $V$  has isolated singularities, if we apply the first assertion in Corollary 3.6, then we recover the result in [CP].

Cheltsov and Park have conjectured in their setting that if  $d \geq n$ , then equality holds in (3.4) if and only if  $V$  is a cone. They have shown that their conjecture

would follow from the Log Minimal Model Program. The second assertion in Corollary 3.6 proves, in particular, their conjecture.

#### 4. Application to birational rigidity

Using the bounds on log canonical thresholds from the previous sections, we prove now the birational rigidity of certain Fano hypersurfaces. We recall that a Mori fiber space  $X$  is called *birationally superrigid* if any birational map  $\phi : X \dashrightarrow X'$  to another Mori fiber space  $X'$  is an isomorphism. For the definition of Mori fiber space and for another notion of rigidity, we refer to [Co2]. Note that Fano manifolds having Néron-Severi group of rank 1 are trivially Mori fiber spaces. Birational superrigidity is a very strong condition: it implies that  $X$  is not rational, and that  $\text{Bir}(X) = \text{Aut}(X)$ . Note that if  $X$  is a smooth hypersurface of degree  $N$  in  $\mathbb{P}^N$  ( $N \geq 4$ ), then  $X$  has no nonzero vector fields. Therefore if  $X$  is birationally superrigid, then the birational invariant  $\text{Bir}(X)$  is a finite group.

The following theorem is the main result of this section.

**Theorem 4.1.** *For any integer  $4 \leq N \leq 12$ , every smooth hypersurface  $X = X_N \subset \mathbb{P}^N$  of degree  $N$  is birationally superrigid.*

The case  $N = 4$  of the above theorem is due to Iskovskikh and Manin (see [IM]). The case  $N = 5$  was proven by Pukhlikov in [Pu1], while the cases  $N = 6, 7, 8$  were established by Cheltsov in [Che]. Birational superrigidity of smooth hypersurfaces of degree  $N$  in  $\mathbb{P}^N$  (for  $N \geq 5$ ) was conjectured by Pukhlikov in [Pu2], where the result is established under a suitable condition of regularity on the equation defining the hypersurface. We remark that there is an attempt due to Pukhlikov in [Pu3] to prove the general case (for  $N \geq 6$ ). Despite a gap in the proof (see the remark below), we believe that the method therein could lead in the future to the result. In fact, the proof given below for Theorem 4.1 follows his method, and our contribution is mainly in simplifying and solidifying his argument.

**Remark 4.2.** The following gives a counterexample to Corollary 2 in [Pu3]. Let  $Q \subset \mathbb{P}^4$  be a cone over a twisted cubic, and let  $\pi_a : Q \rightarrow R = \pi_a(Q)$  be the projection from an arbitrary point  $a \in \mathbb{P}^4 \setminus Q$ ; note that  $R$  is the cone over a singular plane cubic. If  $p$  is the vertex of  $Q$ , then the restriction of  $\pi_a$  to any punctured neighbourhood of  $p$  in  $Q$  can not preserve multiplicities, as  $q = \pi_a(p)$  lies on a one dimensional component of the singular locus of  $R$ .

Before proving the above theorem, we recall the following result, due to Pukhlikov:

**Proposition 4.3.** ([Pu3], Proposition 5) *Let  $X \subset \mathbb{P}^N$  be a smooth hypersurface, and let  $Z$  be an effective cycle on  $X$ , of pure codimension  $k < \frac{1}{2} \dim X$ . If  $m \in \mathbb{N}$  is such that  $Z \equiv m \cdot c_1(\mathcal{O}_X(1))^k$ , then  $\dim\{x \in Z \mid e_x Z > m\} < k$ .*

**Remark 4.4.** Because we have assumed  $k < \frac{1}{2} \dim X$ , the existence of  $m$  as in the proposition follows from Lefschetz Theorem. One can check that the proof of Proposition 4.3 extends to the case  $k = \frac{1}{2} \dim X$ , if we assume that such  $m$  exists. Note also that the statement is trivially true if  $k > \frac{1}{2} \dim X$ .

We need first a few basic properties which allow us to control multiplicities when restricting to general hyperplane sections, and when projecting to lower dimensional linear subspaces. The following proposition must be well known, but we include a proof for the convenience of the readers. We learned this proof, which simplifies our original arguments, from Steve Kleiman.

**Proposition 4.5.** *Let  $Z \subset \mathbb{P}^n$  be an irreducible projective variety. If  $H \subset Z$  is a general hyperplane section, then  $e_p H = e_p Z$  for every  $p \in H$ .*

*Proof.* As observed by Whitney (e.g., see [Kle], page 219), at any point  $p \in Z$ , the fiber over  $p$  of the conormal variety of  $Z$ , viewed as a linear subspace of  $(\mathbb{P}^n)^*$ , contains the dual variety of every component of the embedded projective tangent cone  $C_p Z$  of  $Z$  at  $p$ . A hyperplane section  $H$  of  $Z$  satisfies  $e_p H = e_p Z$  if the hyperplane meets  $C_p Z$  properly. Therefore, this equality holds for every point  $p$  in  $H$  whenever  $H$  is cut out by a hyperplane not in the dual variety of  $Z$ .  $\square$

In the next two propositions, we consider a (possibly reducible) subvariety  $Z \subset \mathbb{P}^{n+s}$ , of pure dimension  $n - 1$ , for some  $n \geq 2$  and  $s \geq 1$ , and take a general linear projection  $\pi : \mathbb{P}^{n+s} \setminus \Lambda \rightarrow \mathbb{P}^n$ . Here  $\Lambda$  denotes the center of projection, that is an  $(s - 1)$  dimensional linear space. We put  $T = \pi(Z)$  and  $g = \pi|_Z : Z \rightarrow T$ . It is easy to see that since  $\Lambda$  is general,  $g$  is a finite birational map. For convenience, we put  $\dim(\emptyset) = -1$ .

**Proposition 4.6.** *With the above notation, consider the set*

$$\Delta = \left\{ q \in T \mid e_q T > \sum_{p \in g^{-1}(q)} e_p Z \right\}.$$

*If the projection is chosen with suitable generality, then  $\text{codim}(\Delta, \mathbb{P}^n) \geq 3$ .*

*Proof.* Note that  $e_q T \geq \sum e_p Z$  for every  $q \in T$ , the sum being taken over all points  $p$  over  $q$ . Moreover, for a generic projection, every irreducible component of  $Z$  is mapped to a distinct component of  $T$ . Therefore, by the linearity of the multiplicity, we may assume that  $Z$  is irreducible.

Let  $\Delta' \subset T$  be the set of points  $q$ , such that for some  $p$  over  $q$ , the intersection of the  $s$  dimensional linear space  $\overline{\Lambda}q$  with the embedded projective tangent cone  $C_p Z$  of  $Z$  at  $p$ , is at least one dimensional. We claim that  $\text{codim}(\Delta', \mathbb{P}^n) \geq 3$ . Indeed, it follows from the theorem on generic flatness that there is a stratification  $Z = Z_1 \sqcup \cdots \sqcup Z_t$  by locally closed subsets such that, for every  $1 \leq j \leq t$ , the incidence set

$$I_j = \{(p, x) \in Z_j \times \mathbb{P}^{n+s} \mid x \in C_p Z\}$$

is a (possibly reducible) quasi-projective variety of dimension no more than  $2 \dim Z = 2n - 2$ . Let  $\text{pr}_1$  and  $\text{pr}_2$  denote the projections of  $I_j$  to the first and to the second factor, respectively. It is clear that the set of those  $y \in \mathbb{P}^{n+s}$ , with  $\dim \text{pr}_2^{-1}(y) = \tau$  has dimension at most  $\max\{2n - 2 - \tau, -1\}$ , for every  $\tau \in \mathbb{N}$ . Since  $\Lambda$  is a general linear subspace of dimension  $s - 1$ , it intersects a given  $d$  dimensional closed subset in a set of dimension  $\max\{d - n - 1, -1\}$ . Hence  $\dim \text{pr}_2^{-1}(\Lambda) \leq n - 3$ , and therefore  $\dim(\text{pr}_1(\text{pr}_2^{-1}(\Lambda))) \leq n - 3$ . As this is true for every  $j$ , we deduce  $\text{codim}(\Delta', \mathbb{P}^n) \geq 3$ . Thus, in order to prove the proposition, it is enough to show that  $\Delta \subseteq \Delta'$ .

For a given point  $p \in Z$ , let  $L_p \subset \mathbb{P}^{n+s}$  be an  $(s + 1)$  dimensional linear subspace passing through  $p$ . Let  $\mathfrak{m}_p$  be the maximal ideal of  $\mathcal{O}_{Z,p}$ , and let  $\mathcal{P} \subset \mathcal{O}_{Z,p}$  be the ideal locally defining  $L_p \cap Z$ . If  $L_p$  meets the tangent cone  $C_p Z$  of  $Z$  at  $p$  properly, then the linear forms defining  $L_p$  generate the ideal of the exceptional divisor of the blow up of  $Z$  at  $p$ . Therefore  $e(\mathfrak{m}_p) = e(\mathcal{P})$ .

Consider now some  $q \in T \setminus \Delta'$ . Let  $L_q \subset \mathbb{P}^n$  be a general line passing through  $q$ , and let  $\mathcal{Q} \subset \mathcal{O}_{T,q}$  be the ideal generated by the linear forms vanishing along  $L_q$ . We denote by  $L$  the closure of  $\pi^{-1}(L_q)$  in  $\mathbb{P}^{n+q}$ . For every  $p \in g^{-1}(q)$ , let  $\mathcal{P} \subset \mathcal{O}_{Z,p}$  be the ideal generated by the linear forms vanishing along  $L$ . Since  $L_q$  is general and  $q \notin \Delta'$ , we may assume that  $L$  intersects  $C_p Z$  properly, hence  $e(\mathfrak{m}_p) = e(\mathcal{P})$ . On the other hand, if  $\mathfrak{m}_q$  is the maximal ideal of  $\mathcal{O}_{T,q}$ , then  $\mathcal{Q} \subseteq \mathfrak{m}_q$ , which gives

$$\mathcal{P} = \mathcal{Q} \cdot \mathcal{O}_{Z,p} \subseteq \mathfrak{m}_q \cdot \mathcal{O}_{Z,p} \subseteq \mathfrak{m}_p.$$

Therefore  $e(\mathfrak{m}_p) = e(\mathfrak{m}_q \cdot \mathcal{O}_{Z,p})$  for every  $p$  as above, hence  $q \notin \Delta$ , by [Ful], Example 4.3.6. □

**Proposition 4.7.** *With the notation in Proposition 4.6, consider the set*

$$\Sigma = \Sigma(Z, \pi) := \{q \in T \mid g^{-1}(q) \text{ has at least 3 distinct points}\}.$$

*If the projection is sufficiently general, then  $\text{codim}(\Sigma, \mathbb{P}^n) \geq 3$ .*

*Proof.* We have  $\text{codim}(\Sigma, \mathbb{P}^n) \geq 3$  if and only if  $\Sigma \cap P = \emptyset$  for every general plane  $P \subset \mathbb{P}^n$ . Pick one general plane  $P$ , let  $P' (\cong \mathbb{P}^{s+2})$  be the closure of  $\pi^{-1}(P)$  in  $\mathbb{P}^{n+s}$ , and let  $\pi'$  be the restriction of  $\pi$  to  $P' \setminus \Lambda$ . If  $Z' = Z \cap P'$ , then  $Z'$  is a (possibly reducible) curve, and its multisequant variety is at most two dimensional (see, for example, [FOV], Corollary 4.6.17). Note that  $\Lambda$  is general in  $P'$ . Indeed, choosing the center of projection  $\Lambda$  general in  $\mathbb{P}^{n+s}$ , and then picking  $P$  general in  $\mathbb{P}^n$  is equivalent to first fixing a general  $(s + 2)$ -plane  $P'$  in  $\mathbb{P}^{n+s}$  and then choosing  $\Lambda$  general in  $P'$ . Therefore we conclude that  $\Sigma \cap P$ , which is the same as  $\Sigma(Z', \pi')$ , is empty. □

*Proof of Theorem 4.1.* By adjunction,  $\mathcal{O}_X(-K_X) \simeq \mathcal{O}_X(1)$ . Let  $\phi : X \dashrightarrow X'$  be a birational map from  $X$  to a Mori fiber space  $X'$ , and assume that  $\phi$  is not an isomorphism. By the Noether-Fano inequality (see [Co1] and [Isk], or [Mat]), we find a linear subsystem  $\mathcal{H} \subset |\mathcal{O}_X(r)|$ , with  $r \geq 1$ , whose base scheme  $B$  has codimension  $\geq 2$ , and such that the pair  $(X, \frac{1}{r} \cdot B)$  is not canonical. We choose

$c < \frac{1}{r}$ , such that  $(X, c \cdot B)$  is still not canonical, and let  $C \subset X$  be a center of non canonicity for  $(X, c \cdot B)$ . Note that  $C$  is a center of non canonicity also for the pairs  $(X, c \cdot D)$  and  $(X, c \cdot V)$ , where  $V = D \cap D'$  and  $D, D' \in \mathcal{H}$  are two general members. Applying Proposition 4.3 for  $Z = D$  and  $k = 1$ , we see that the multiplicity of  $D$  is  $\leq r$  on an open subset whose complement has dimension zero. On this open subset  $(X, c \cdot D)$  is canonical (see, for example, [Kol] 3.14.1). Therefore  $C = p$ , a point of  $X$ .

Let  $Y$  be a general hyperplane section of  $X$  containing  $p$ . Then  $p$  is a center of non log canonicity for  $(Y, c \cdot B|_Y)$ . Note that  $Y$  is a smooth hypersurface of degree  $N$  in  $\mathbb{P}^{N-1}$ . Let  $\pi : \mathbb{P}^{N-1} \setminus \Lambda \rightarrow \mathbb{P}^{N-3}$  be a general linear projection, where the center of projection  $\Lambda$  is a line. We can assume that the restriction of  $\pi$  to each irreducible component of  $V|_Y$  is finite and birational. Note that  $\pi_*[V|_Y]$  is a divisor in  $\mathbb{P}^{N-3}$  of degree  $Nr^2$ . If  $\tilde{Y} = \text{Bl}_{\Lambda \cap Y} Y$ , then we get a morphism  $f : \tilde{Y} \rightarrow \mathbb{P}^{N-3}$ . If we choose  $\Lambda$  general enough, then we can find an open set  $U \subset \mathbb{P}^{N-3}$ , containing the image  $q$  of  $p$ , such that  $f$  restricts to a smooth (proper) morphism  $f^{-1}(U) \rightarrow U$ . Applying Theorem 1.1, we deduce that the pair

$$(4.1) \quad \left( \mathbb{P}^{N-3}, \frac{c^2}{4} \cdot \pi_*[V|_Y] \right)$$

is not log terminal at  $q$ .

We claim that

$$(4.2) \quad \dim\{y \in \pi(V|_Y) \mid e_y(\pi_*[V|_Y]) > 2r^2\} \leq \max\{N - 6, 0\}.$$

Indeed, by Propositions 4.7 and 4.6, the map  $\text{Supp}([V|_Y]) \rightarrow \text{Supp}(\pi_*[V|_Y])$  is at most 2 to 1 and preserves multiplicities outside a set, say  $\Delta \cup \Sigma$ , of dimension  $\leq \max\{N - 6, -1\}$ . This implies that, for each  $y$  outside the set  $\Delta \cup \Sigma$ ,  $e_y(\pi_*[V|_Y]) = \sum e_x([V|_Y])$ , where the sum is taken over the points  $x$  over  $y$ , and this sum involves at most two non-zero terms. Then (4.2) follows from the fact that, by Propositions 4.3 and 4.5 (see also Remark 4.4), the set of points  $x$  for which  $e_x[V|_Y] > r^2$  is at most zero dimensional.

Note that the pair (4.1) is log terminal at every point  $y$  where  $e_y(\pi_*[V|_Y]) \leq 4r^2$ . If  $4 \leq N \leq 6$ , we deduce that the pair is log terminal outside a zero dimensional closed subset. In this case, Corollary 3.6 gives  $c^2/4 \geq (N-3)/(Nr^2)$ . Since  $c < 1/r$ , this implies  $N < 4$ , a contradiction. If  $7 \leq N \leq 12$ , then we can only conclude that the pair (4.1) is log terminal outside a closed subset of codimension at least 3. This time the same corollary gives  $c^2/4 \geq 3/(Nr^2)$ , which implies  $N > 12$ . This again contradicts our assumptions, so the proof is complete.  $\square$

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