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SEMISTABLE REDUCTION FOR OVERCONVERGENT F-ISOCRYSTALS ON A CURVE

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Abstract. Let X be a smooth affine curve over a field k of characteristic $p > 0$ and $\mathcal{E}$ an overconvergent F^a-isocrystal on X for some positive integer a. We prove that after replacing k by some finite purely inseparable extension, there exists a finite separable morphism $X' \to X$, the pullback of $\mathcal{E}$ along which extends to a log-F^a-isocrystal on a smooth compactification of X'. This resolves a weak form of the global version of a conjecture of Crew; the proof uses the local version of the conjecture, established (separately) by André, Mebkhout and the author.

1. Introduction

The purpose of this paper is to prove the following semistable reduction theorem for overconvergent F-isocrystals on a curve, answering a conjecture of de Jong [dJ, Section 5], a reformulation of a conjecture of Crew (of which more about below).

Theorem 1.1 (Semistable reduction)**.** *Let X be a smooth, geometrically connected curve over a field k of characteristic $p > 0$, let K be a complete discrete valuation field with residue field k, admitting a lift of the p-power Frobenius on k, and let $\mathcal{E}$ be an overconvergent F^a-isocrystal on X with respect to K for some $a \in \mathbb{N}$. Then after replacing k by a suitable finite purely inseparable extension (depending on $\mathcal{E}$), there exist a finite generically étale morphism $f : X_1 \to X$, a smooth compactification $j : X_1 \hookrightarrow \overline{X_1}$ of X_1, and a log-F^a-isocrystal $\mathcal{F}$ on $(\overline{X_1}, \overline{X_1} \setminus X_1)/K$ such that $j^*\mathcal{F} \cong f^*\mathcal{E}$.*

Our basic approach is to use the quasi-unipotence theorem (p-adic local monodromy theorem) for F^a-isocrystals, plus a matrix factorization argument from [Ke1], to "fill in" $\mathcal{E}$ at each of the points of $\overline{X_1} \setminus X_1$. The quasi-unipotence theorem, conjectured by Crew [Cr, Section 10], follows from the work of any of André [A], Mebkhout [M], or the author [Ke2]; see Proposition 3.1 for the formulation we need here.

Theorem 1.1 "almost" yields a more precise statement proposed by Crew [Cr], by implying that there exists a finite morphism $f : X_1 \to X$ such that $f^*\mathcal{E}$ is unipotent at each point of $\overline{X_1} \setminus X_1$. (In fact, we will prove Theorem 1.1 by proving this first.) The caveat is that Crew actually wanted f to be étale.

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We have only been able to achieve this in the unit-root case, and do not know whether it should be possible in general. We discuss these matters in Section 6.

We hope to extend Theorem 1.1 to higher dimensional varieties in subsequent work. In this case, we must allow the morphism $f : X_1 \to X$ to be generically étale, but not necessarily finite, because the normalization of X in a finite extension of its function field need not be smooth. For this and other reasons, the argument in higher dimensions will be technically more involved.

The precise higher-dimensional analogue of Theorem 1.1 has been formulated conjecturally by Shiho [S, Conjecture 3.1.8]; by Shiho's work, proving this statement would provide good comparison results between rigid and crystalline cohomology. For example, it would yield an alternate proof of finite dimensionality of rigid cohomology of a curve with coefficients in an overconvergent F-isocrystal, via comparison with crystalline cohomology; Theorem 1.1 does this for curves. (Crew's proof in [Cr] uses p-adic functional analytic techniques; the general finiteness proof in [Ke3] uses a devissage to the curve case.)

2. Definitions and notations

We set up notation following [Ke3]; this makes it a bit more convenient to work with global objects than does the notation of [Ke2]. We retain the convention of [Ke2] that all matrices are $n \times n$ matrices and I denotes the identity matrix.

Let k be a field of characteristic $p > 0$, and let $C(k)$ be a Cohen ring for k, that is, a complete discrete valuation ring with residue field k and maximal ideal generated by p. (See [Bo] for proof of existence and basic properties of Cohen rings.) Let K be a finite totally ramified extension of $\operatorname{Frac} C(k)$, let $\mathcal{O}$ be the integral closure of $C(k)$ in K, and let v_p denote the p-adic valuation on K. Assume that there exists a ring endomorphism σ_0 on $\mathcal{O}$ lifting the p-power Frobenius on k. Let $q = p^a$ be a power of p, and put $\sigma = \sigma_0^a$.

The ring $\mathcal{R}_r$ consists of bidirectional power series $\sum_{i\in\mathbb{Z}} c_i u^i$, with $c_i \in K$, such that

$$\lim_{i\to\pm\infty} s v_p(c_i) + i = \infty \qquad (0 < s \leq r);$$

for each s, the function $w_s(\sum_i c_i u^i) = \min_i\{s v_p(c_i) + i\}$ is a nonarchimedean valuation on $\mathcal{R}_r$. The ring $\mathcal{R}$ (the "Robba ring") is the union of the $\mathcal{R}_r$ over all $r > 0$. Its subring $\mathcal{R}^{\mathrm{int}}$ consists of those series with $c_i \in \mathcal{O}$ for all i; this subring is a (noncomplete) discrete valuation ring, unramified over $\mathcal{O}$, with residue field the field $k((t))$ of formal Laurent series in k. (In [Ke2], the rings $\mathcal{R}$ and $\mathcal{R}^{\mathrm{int}}$ are called $\Gamma_{\mathrm{an,con}}$ and Γ_{con}, respectively.) By adding the superscript $+$ or $-$ to $\mathcal{R}$ or $\mathcal{R}^{\mathrm{int}}$, we will mean the subring with only nonnegative or nonpositive powers of u, respectively.

For $L/k((t))$ finite separable, there is a natural discrete valuation ring $\mathcal{R}^{\mathrm{int}}_L$, integral and unramified over $\mathcal{R}^{\mathrm{int}}$, with residue field L. Namely, take any monic polynomial $P(x)$ over $\mathcal{R}^{\mathrm{int}}$ whose reduction $\overline{P}$ satisfies $K \cong k((t))[x]/(\overline{P}(x))$, and put $\mathcal{R}^{\mathrm{int}}_L \cong \mathcal{R}^{\mathrm{int}}[x]/(P(x))$.

The Monsky-Washnitzer algebra of rank n is defined as

$$W_n = \left\{ \sum_I c_I x^I : c_I \in \mathcal{O}, \quad \liminf_I \frac{v_p(c_I)}{\sum I} > 0 \right\},$$

where $I = (i_1, \dots, i_n)$ represents an n-tuple of nonnegative integers, $x^I = x_1^{i_1} \cdots x_n^{i_n}$ and $\sum I = i_1 + \cdots + i_n$. An *integral dagger algebra* is any quotient A^{int} of a Monsky-Washnitzer algebra which is flat over $\mathcal{O}$ and for which $\mathrm{Spec}(A^{\mathrm{int}} \otimes_{\mathcal{O}} k)$ is smooth over $\mathrm{Spec}(k)$. (Given $A^{\mathrm{int}} \otimes_{\mathcal{O}} k$, one can always find a corresponding A^{int}; see [vdP].) A *dagger algebra* A is an algebra of the form $A^{\mathrm{int}} \otimes_{\mathcal{O}} K$ for some integral dagger algebra A^{int} (uniquely determined by A and the p-adic valuation on A).

Given a dagger algebra A with $A^{\mathrm{int}} \cong W_n/\mathfrak{a}$ and $f \in A$ not a zero divisor, the *localization* A' of A at f is the dagger algebra with

$$(A')^{\mathrm{int}} \cong W_{n+1}/(W_{n+1}\mathfrak{a} + (f x_{n+1} - 1)W_{n+1});$$

this is a dagger algebra in which f is invertible.

Given a dagger algebra A with $A^{\mathrm{int}} \cong W_n/\mathfrak{a}$ for some ideal $\mathfrak{a}$ of W_n, we define $\Omega^1_{A/K}$ as the free A-module generated by symbols $dx_1, \dots, dx_n$, modulo relations of the form $da = 0$ for $a \in \mathfrak{a} \otimes_{\mathcal{O}} K$. By construction, $\Omega^1_{A/K}$ is equipped with a K-linear derivation $d : A \to \Omega^1_{A/K}$.

3. Log-F-isocrystals and log-(σ, ∇)-modules

Let A be a dagger algebra and $\sigma_0 : A^{\mathrm{int}} \to A^{\mathrm{int}}$ a lift of the p-power Frobenius map extending the given σ_0 on $\mathcal{O}$; again, set $\sigma = \sigma_0^a$. (Such a lift always exists: again, see [vdP].) Given $u \in A$ such that u^σ/u^q is invertible in A, we define the logarithmic module of differentials $\Omega^1_{A/K}[d\log u]$ by adding to $\Omega^1_{A/K}$ a symbol $d\log u$ such that $u(d\log u) = du$; then $d\sigma$ extends to $\Omega^1_{A/K}[d\log u]$ sending du/u to $q\,du/u + d(u^\sigma/u^q)/(u^\sigma/u^q)$. We define a *log-$(\sigma,\nabla)$-module* over A (with respect to u) as a finite locally free A-module M equipped with a σ-linear map F that induces an isomorphism $F : M \otimes_{A,\sigma} A \to M$, and with an A-linear connection $\nabla : M \to M \otimes_A \Omega^1_{A/K}[d\log u]$ which is integrable (i.e., which satisfies $\nabla_1 \circ \nabla = 0$ for $\nabla_1 : M \otimes \Omega^1_{A/K}[d\log u] \to M \otimes \wedge^2_A \Omega^1_{A/K}[d\log u]$ induced by ∇) and which makes the following diagram commute:

$$\begin{array}{ccc} M & \xrightarrow{\nabla} & M \otimes \Omega^1_{A/K}[d\log u] \\ \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F \otimes d\sigma} \\ M & \xrightarrow{\nabla} & M \otimes \Omega^1_{A/K}[d\log u] \end{array}$$

If $u = 1$, we drop the "log" and simply refer to M as a (σ,∇)-module. We analogously define (σ,∇)-modules over $\mathcal{R}$ for $\sigma : \mathcal{R} \to \mathcal{R}$ induced by the a-th composition power of a map $\sigma_0 : \mathcal{R}^{\mathrm{int}} \to \mathcal{R}^{\mathrm{int}}$ of the form

$$\sum_i c_i u^i \mapsto \sum_i c_i^{\sigma_0} (u^{\sigma_0})^i$$

lifting the p-th power map. In this case, we take $\Omega^1_{\mathcal{R}/K}$ to be the free $\mathcal{R}$-module generated by du. (Note: since we are only considering curves, the integrability condition will be superfluous in our situations, as $\wedge^2 \Omega^1$ will vanish.)

Log-(σ, ∇)-modules are Zariski-local avatars of more global objects, namely overconvergent log-F^a-isocrystals. Rather than take space for a full-blown definition of log-F^a-isocrystals here, we summarize the key features of the definition below.

- If X is smooth over k and equipped with the fine log structure associated to some strict normal crossings divisor Z, there is a category of overconvergent log-F^a-isocrystals on the pair (X, Z). (We drop the "log" if Z is empty.)
- An oveconvergent log-F^a-isocrystal on (X, Z) can be specified by giving overconvergent log-F^a-isocrystals on an affine cover of X plus isomorphisms on the pairwise intersections satisfying the cocycle condition. (Loosely put, the category is a Zariski sheaf.)
- Given a dagger algebra A, a Frobenius lift σ, and an element $u \in A^{\mathrm{int}}$ such that u^σ / u^q is invertible in A, the category of log-(σ, ∇)-modules with respect to u is canonically equivalent to the category of overconvergent log-F^a-isocrystals on (X, Z), where $X = \mathrm{Spec}(A^{\mathrm{int}} \otimes_{\mathcal{O}} k)$ and $Z \subseteq X$ is the zero locus of u. In particular, the former does not depend on σ; in fact, there is an explicit formula for transforming a Frobenius structure with respect to a given σ into a Frobenius structure with respect to another (with respect to the same ∇).

See [CI, Section 6] for an informal overview of log-F^a-isocrystals; for a much more detailed study, see [S, Chapter 2]. (It might help to consider the situation without logarithmic structures first; see [Be] for an introduction there.)

The main input into this paper is the p-adic local monodromy theorem ("Crew's conjecture"), established separately by André [A], Mebkhout [M], and the author [Ke2]. We say an extension of $k((t))$ is *nearly finite separable* if it is finite separable over $k^{1/p^n}((t))$ for some nonnegative integer n. With that definition, the local monodromy theorem (e.g., in the form of [Ke2, Theorem 6.12]) implies the following.

Proposition 3.1. *Let M be a (σ, ∇)-module over $\mathcal{R}$. Then there exists a nearly finite separable extension $L/k((t))$ so that $M \otimes_{\mathcal{R}^{\mathrm{int}}} \mathcal{R}^{\mathrm{int}}_L$ admits a basis $\mathbf{v}_1, \dots, \mathbf{v}_n$ such that $\nabla \mathbf{v}_i \in \mathrm{SatSpan}(\mathbf{v}_1, \dots, \mathbf{v}_{i-1}) \otimes \Omega^1$ for $i = 1, \dots, n$.*

In fact, if $u \in \mathcal{R}^{\mathrm{int}}_L$ lifts a uniformizer of L and K' is the integral closure of K in $\mathcal{R}^{\mathrm{int}}_L$ (so that $\mathcal{R}^{\mathrm{int}}_L$ is isomorphic to the integral Robba ring with coefficient field K'), then one can ensure that in fact $\nabla \mathbf{v}_i \in (K'\mathbf{v}_1 + \cdots + K'\mathbf{v}_{i-1}) \otimes \frac{du}{u}$; this implies that $F\mathbf{v}_i \in K'\mathbf{v}_1 + \cdots + K'\mathbf{v}_n$ for all i. If such a basis already exists over $\mathcal{R}$, we say M is *unipotent* over $\mathcal{R}$.

4. Matrix factorizations

In this section, let A be a dagger algebra such that $\mathrm{Spec}(A \otimes_{\mathcal{O}} k)$ is a smooth affine geometrically connected curve over $\mathrm{Spec}(k)$; in particular, A is an integral domain. Suppose $t \in A \otimes_{\mathcal{O}} k$ generates a prime ideal with residue field k; choose a lift $u \in A^{\mathrm{int}}$ of t. Then there is a natural embedding $\rho_u : A^{\mathrm{int}} \hookrightarrow \mathcal{R}^{+,\mathrm{int}}$ sending u to the series parameter; this extends to an embedding of any localization of A into $\mathcal{R}$. We identify each localization of A with its image under ρ_u.

Lemma 4.1. *For $r > 0$, let U be an invertible matrix over $\mathcal{R}_r$. Then there exists an invertible matrix V over some localization A' of A such that $w_r(VU - I) > 0$.*

Proof. Let A_1 be the localization of A at u; then A_1 is dense in $\mathcal{R}_r$ under w_r. Thus we can choose a matrix V over A_1 such that $w_r(V - U^{-1}) > -w_r(U)$; then

$$w_r(VU - I) = w_r((V - U^{-1})U) \geq w_r(V - U^{-1}) + w_r(U) > 0.$$

Since $w_r(VU - I) > 0$, we have $w_r(\det(VU) - 1) > 0$, so in particular $\det(VU) \neq 0$. Hence $\det(V) \neq 0$, so we can form the localization A' of A_1 at $\det(V)$. Over A', V becomes an invertible matrix, as desired. □

Proposition 4.2. *Let U be an invertible matrix over $\mathcal{R}$. Then there exist invertible matrices V over some localization A' of A and W over $\mathcal{R}^+$ such that $U = VW$.*

Proof. Choose $r > 0$ so that U is invertible over $\mathcal{R}_r$. By Lemma 4.1, we can find an invertible matrix X over some localization A_1 of A such that $w_r(XU - I) > 0$. By [Ke2, Proposition 6.5], we can write XU as a product YZ with Y invertible over $\mathcal{R}^-$ and Z invertible over $\mathcal{R}^+$. Let A' be the localization of A_1 at u; then $\mathcal{R}^- \subseteq A'$, so $X^{-1}Y$ is invertible over A'. Thus we may take $V = X^{-1}Y$ and $W = Z$. □

5. Semistable reduction

In this section, we prove that an overconvergent F^a-isocrystal $\mathcal{E}$ on X which is unipotent at each point of $\overline{X} - X$, for $\overline{X}$ a smooth compactification of X, has "semistable reduction." This will yield our proof of Theorem 1.1.

We begin with a result that translates unipotence of a (σ, ∇)-module into semistable reduction. The argument is based on the proof of [Ke1, Theorem 5.0.1].

Theorem 5.1. *Let A be a dagger algebra equipped with a Frobenius lift σ. Suppose the image of $u \in A^{\mathrm{int}}$ in $A^{\mathrm{int}} \otimes_{\mathcal{O}} k$ generates a prime ideal and u^σ / u^q is a unit in A. Let A' be the localization of A at u, and let M be a free (σ, ∇)-module over A' which becomes unipotent over $\mathcal{R}$ (where A' is identified with a subring of $\mathcal{R}$ via ρ_u). Then M is isomorphic to a log-(σ, ∇)-module, with respect to u, over some localization A'' of A in which u is not invertible.*

Proof. Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be a basis of M, so that $F\mathbf{e}_j = \sum_i \Phi_{ij}\mathbf{e}_i$ and $\nabla \mathbf{e}_j = \sum_i N_{ij}\mathbf{e}_i \otimes \frac{du}{u}$. Define the differential operator $\theta(f) = u\frac{df}{du}$. By hypothesis, there exists a matrix U over $\mathcal{R}$ such that $U^{-1}\Phi U^\sigma$ and $U^{-1}NU + U^{-1}\theta(U)$ have entries in $\mathcal{O}$. By Proposition 4.2, we can factor U as VW, where V is invertible over some localization A_1 of A' and W is invertible over $\mathcal{R}^+$. Now put $\mathbf{v}_j = \sum_j V_{ij}\mathbf{e}_i$; then $F\mathbf{v}_j = \sum_i \tilde{\Phi}_{ij}\mathbf{v}_i$ and $\nabla \mathbf{v}_j = \sum_i \tilde{N}_{ij}\mathbf{v}_i \otimes \frac{du}{u}$, where

$$\tilde{\Phi} = V^{-1}\Phi V^\sigma = W(U^{-1}\Phi U^\sigma)W^{-\sigma}$$
$$\tilde{N} = V^{-1}NV + V^{-1}\theta(V) = W(U^{-1}NU + U^{-1}\theta(U))W^{-1} - \theta(W)W^{-1}$$

have entries in $A'' = A_1 \cap \mathcal{R}^+$, which is a localization of A (because it contains A and is contained in the localization A_1) in which u is not invertible (because u is not invertible in $\mathcal{R}^+$). □

We now proceed to the proof of our main theorem.

Proof of Theorem 1.1. Let $K(X)$ be the function field of X, and let $\overline{X}$ be a smooth compactification of X. Without loss of generality, enlarge k so that the geometric points of $Z = \overline{X} \setminus X$ are k-rational. (Given the desired result over a finite extension of k, we deduce the result over k by restriction of scalars.) For each geometric point x of Z, choose a function $t_x \in K(X)$ with a simple zero at x, and choose an open affine neighborhood U_x of x in $\overline{X}$ such that $U_x \cap Z = \{x\}$ and $\operatorname{div}(t_x) \cap U_x = \{x\}$. Let A_x be a dagger algebra with $U \cong \operatorname{Spec}(A_x^{\mathrm{int}} \otimes_{\mathcal{O}} k)$, choose a Frobenius lift σ on A_x and choose a lift u_x of t_x in A_x^{int}. Let A'_x be the localization of A_x at u_x; then the restriction of $\mathcal{E}$ to $U \setminus \{x\}$ corresponds to a (σ, ∇)-module M_x over A'_x. After shrinking U_x if needed, we may assume that M_x is free over A'_x.

Let ρ_x be the embedding of A_x into $\mathcal{R}^{+,\mathrm{int}}$ sending u_x to the series parameter. By Proposition 3.1, there exists a nearly finite separable extension L_x of the t_x-adic completion of $K(X)$ such that $M_x \otimes_{A'_x} \mathcal{R}^{\mathrm{int}}_{L_x}$ is unipotent. By Krasner's lemma, after replacing k with k^{1/p^n} for some nonnegative integer n, we can choose a finite separable extension L of $K(X)$ whose completion at any point above x contains L_x for each $x \in Z$. After enlarging k again, we may assume that the places of L above x are all k-rational. Let $f : \overline{X_1} \to \overline{X}$ be the cover corresponding to the extension $L/K(X)$, and put $X_1 = f^{-1}(X)$.

For each geometric point y of $Z_1 = \overline{X_1} \setminus X_1$, choose a function $t_y \in K(X_1)$ with a simple zero at y and no multiple zeroes; then t_y gives rise to a map $g_y : \overline{X_1} \to \mathbb{P}^1$. (If k is finite, it may be necessary to enlarge it again to find such t_y.) Put $x = f(y)$, and choose an open affine neighborhood V_y of y in $\overline{X_1}$ such that $V_y \cap Z_1 = \{y\}$, $\operatorname{div}(t_y) \cap V_y = \{y\}$, $f(V_y) \subseteq U_x$, and V_y does not meet the branch locus of g_y. Then there is a dagger algebra B_y with $V_y \cong \operatorname{Spec}(B_y^{\mathrm{int}} \otimes_{\mathcal{O}} k)$ which is a localization of a finite extension of A_x.

Choose a lift u_y of t_y in B_y^{int}. Since V_y is unramified over its image under g_y, B_y is finite and unramified over some localization C_y of its subring $K\langle u_y \rangle^\dagger$ (the p-adic closure of $K[u_y]$ within B_y). Now $K\langle u_y \rangle^\dagger$ admits a p-power Frobenius lift σ_0' sending u_y to u_y^σ; this lift extends to the localization C_y, then to the unramified extension B_y. Put $\sigma' = (\sigma_0')^a$.

Let B_y' be the localization of B_y at $u_x \in A_x \subseteq B_y$, which is the same as the localization at u_y because $\operatorname{div}(t_x) \cap V_y = \operatorname{div}(t_y) \cap V_y = \{y\}$, and put $V_y' = \operatorname{Spec}((B_y')^{\mathrm{int}} \otimes_{\mathcal{O}} k) = V_y \setminus \{y\}$; then the restriction of $f^*\mathcal{E}$ to V_y' corresponds to the (σ', ∇)-module $M_x \otimes_{A_x'} B_y'$. (That is, its connection is the one induced from M_x, but its Frobenius structure is defined with respect to σ' instead of σ.) By construction, this (σ', ∇)-module is unipotent. Moreover, $u_y^{\sigma'}/u_y^q = 1$, so Theorem 5.1 implies that $M_x \otimes_{A_x'} B_y'$ is isomorphic to a log-(σ, ∇)-module over some localization B_y'' of B_y in which u_y is not invertible. If $V_y'' = \operatorname{Spec}((B_y'')^{\mathrm{int}} \otimes_{\mathcal{O}} k)$, then V_y'' is an open affine neighborhood of y in $\overline{X_1}$ on which $f^*\mathcal{E}$ extends to a log-F-isocrystal.

In short, we have an open affine neighborhood of each $y \in Z_1$ in $\overline{X_1}$, on which $f^*\mathcal{E}$ extends to a log-F-isocrystal relative to $\{y\}$. Each neighborhood contains no other points of y, so the pairwise intersections all lie in X_1. Thus we automatically have glueing isomorphisms on the log-F-isocrystals satisfying the cocycle conditions (since $f^*\mathcal{E}$ is defined on X_1), yielding a log-F-isocrystal on $(\overline{X_1}, Z_1)$, as desired. □

6. Finite versus étale

As noted earlier, Crew [Cr, Section 10] conjectured that an overconvergent F-isocrystal on a curve should extend to a log-F-isocrystal after a base extension which is not just finite and generically étale, but actually étale. It is unclear whether this should hold in general; the best we can do at the moment is prove it in the unit-root case, as done below. Note that this proof does not use the full strength of the quasi-unipotence theorem, but only the unit-root case; this case is due to Tsuzuki [T1]. Also note that a unit-root log-F-isocrystal is automatically an F-isocrystal, so there is no "log" in the statement of the theorem.

The proof of the following lemma is straightforward.

Lemma 6.1. *Let B be a matrix over $k[[t]]$, for k a perfect field of characteristic $p > 0$, and let τ denote the q-th power map. Then any solution D of either of the matrix equations*

$$D^{-1} B D^{\tau} = I \qquad \text{or} \qquad D^{\tau} - D = B$$

over the integral closure of $k[[t]]$ in $k((t))^{\mathrm{alg}}$ is defined over an unramified extension of $k[[t]]$.

Theorem 6.2. *Let X be a smooth, geometrically connected curve over a perfect field k of characteristic $p > 0$, and let $\mathcal{E}$ be an overconvergent unit-root F^a-isocrystal on X/K. Then there exists a finite étale morphism $f : X_1 \to X$, a smooth compactification $j : X_1 \hookrightarrow \overline{X_1}$ of X_1, and a unit-root F^a-isocrystal $\mathcal{F}$ on $\overline{X_1}$ such that $j^*\mathcal{F} \cong f^*\mathcal{E}$.*

Proof. If X is projective, there is nothing to prove, so we assume X is affine. Let A be a dagger algebra with $X \cong \mathrm{Spec}(A^{\mathrm{int}} \otimes_{\mathcal{O}} k)$, and choose a Frobenius lift σ on A. Then $\mathcal{E}$ corresponds to a (σ, ∇)-module M over A. Choose (not necessarily free) generators $\mathbf{v}_1, \dots, \mathbf{v}_m$ of M, and let $N \subset M$ be the A^{int}-span of $F^i \mathbf{v}_j$ over $i = 0, 1, \dots$ and $j = 1, \dots, m$. Then N is locally free over A^{int}.

Let L be the p-adic completion of the valuation subring of $\mathrm{Frac}\, A$; note that N is free over L and F acts on any basis of N over L via an invertible matrix. Let π be a uniformizer of $\mathcal{O}$, and pick an integer d such that $v_p(\pi^d) > 1/(p-1)$.

Given any basis $\mathbf{e}_1, \dots, \mathbf{e}_n$ of N to start with, define the matrix Φ by $F\mathbf{e}_j = \sum_{ij} \Phi_{ij} \mathbf{e}_i$. We then solve the matrix equation $C_i^{-1} \Phi C_i^{\sigma} \equiv I \pmod{\pi^i}$ for $i = 1, \dots, d$ to obtain a matrix C_d over some finite unramified extension L' of L. Then $N \otimes_{A^{\mathrm{int}}} L'$ admits a basis $\mathbf{w}_1, \dots, \mathbf{w}_n$ for which $F\mathbf{w}_i \equiv \mathbf{w}_i \pmod{\pi^d}$.

If we insist that L' be minimal for the existence of the basis of the desired form, then it is unique; in particular, it does not depend on the choice of the starting basis. Let X_1 be a curve for which $K(X_1) \cong L'/\pi L'$ and let $f : X_1 \to X$ be the induced map. If we choose the initial basis $\mathbf{e}_1, \dots, \mathbf{e}_n$ over a localization A_1^{int} of A^{int} over which N becomes free, we discover that f is étale over any point in $\mathrm{Spec}(A_1^{\mathrm{int}} \otimes_{\mathcal{O}} k)$ by Lemma 6.1. (Namely, C_0 satisfies an equation modulo π of the first type in the lemma, while $C_{i-1}^{-1} C_i = I + \pi^{i-1} D$ for some matrix D satisfying an equation modulo π of the second type.) Since N is locally free over A^{int}, we can arrange for $\mathrm{Spec}(A_1^{\mathrm{int}} \otimes_{\mathcal{O}} k)$ to contain any closed point of X. Thus f is finite étale. Since $v_p(\pi^d) > 1/(p-1)$, we may apply [T1, Theorem 5.1.1] (at least for $a = 1$; see [Ke2, Proposition 6.11] for a reduction to this case) to see that $f^*\mathcal{E}$ admits a basis of elements in the kernel of ∇, on which F acts by a matrix over $\mathcal{O}$. This allows us to extend $f^*\mathcal{E}$ to a compactification $\overline{X_1}$ of X_1, as desired. □

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