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# MAXIMAL TORI IN THE SYMPLECTOMORPHISM GROUPS OF HIRZEBRUCH SURFACES

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Abstract. We count the conjugacy classes of maximal tori in the groups of symplectomorphisms of $S^2 \times S^2$ and of the blow-up of $\mathbb{CP}^2$ at a point.

Consider the group $\mathrm{Ham}(M,\omega)$ of Hamiltonian symplectomorphisms of a symplectic manifold.[1] A *k-dimensional torus* in $\mathrm{Ham}(M,\omega)$ is a subgroup which is isomorphic to $(S^1)^k$. A *maximal torus* is one which is not contained in any strictly larger torus.

An action of $(S^1)^k$ on $(M,\omega)$ is called *Hamiltonian* if it admits a moment map, i.e., a map $\Phi\colon M \to \mathbb{R}^k$ such that $d\Phi_i = -\iota(\xi_i)\omega$ for all $i = 1, \dots, k$, where $\xi_1, \dots, \xi_k$ are the vector fields on $M$ that generate the action.

A Hamiltonian $(S^1)^k$-action defines a homomorphism from $(S^1)^k$ to $\mathrm{Ham}(M,\omega)$. The action is effective if and only if this homomorphism is one to one. Its image is then a $k$-dimensional torus in $\mathrm{Ham}(M,\omega)$. Every $k$-dimensional torus in $\mathrm{Ham}(M,\omega)$ is obtained in this way, and two Hamiltonian actions give the same torus if and only if they differ by a reparametrization of $(S^1)^k$. Tori in $\mathrm{Ham}(M,\omega)$ have dimension at most $\frac{1}{2}\dim M$. A Hamiltonian action of a $(\frac{1}{2}\dim M)$-dimensional torus is called *toric*.

**Theorem 1.** *Let $(M,\omega)$ be a compact symplectic four-manifold. Suppose that $\dim H^2(M,\mathbb{R}) \leq 3$ and $\dim H^1(M,\mathbb{R}) = 0$. Then every Hamiltonian circle action on $(M,\omega)$ extends to a toric action.*

**Remark.** Many symplectic four-manifolds do not admit Hamiltonian circle actions. The theorem does not say anything about such manifolds.

Two tori, $T_1$ and $T_2$, in $\mathrm{Ham}(M,\omega)$ are *conjugate* if there exists an element $g \in \mathrm{Ham}(M,\omega)$ such that $gT_1g^{-1} = T_2$. Two torus actions, viewed as homomorphisms $(S^1)^k \to \mathrm{Ham}(M,\omega)$, give conjugate tori in $\mathrm{Ham}(M,\omega)$ if and only if they differ by an equivariant symplectomorphism composed with a reparametrization of $(S^1)^k$.

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[1] Recall, a Hamiltonian symplectomorphism is one which can be connected to the identity by a path $\psi_t$ such that $\frac{d}{dt}\psi_t = X_t \circ \psi_t$ and $\iota(X_t)\omega = dH_t$ for a smooth $H_t\colon M \to \mathbb{R}$.

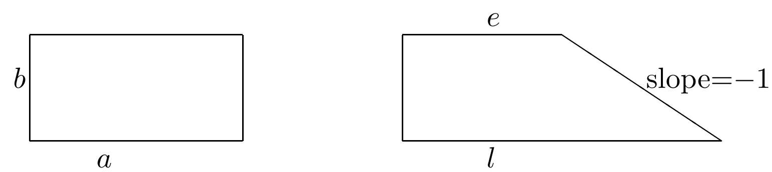


Figure 1. Moment map images for standard torus actions on $(S^2 \times S^2, \omega_{a,b})$ and $(\widetilde{\mathbb{CP}^2}, \widetilde{\omega}_{l,e})$.

On $S^2 \times S^2$, let $\omega_{a,b} = a\omega_1 + b\omega_2$, where $\omega_1$ and $\omega_2$ are the pullbacks through the two projection maps of the rotation invariant area form on $S^2$ with total area $2\pi$. The standard $(S^1)^2$-action has the moment map image shown in Figure 1 on the left.

Let $\widetilde{\mathbb{CP}^2}$ be the blow-up of $\mathbb{CP}^2$ at a point. In it, let $E$ be the exceptional divisor and let $L$ be a $\mathbb{CP}^1$ which is disjoint from $E$. For $l > e > 0$, let $\widetilde{\omega}_{l,e}$ be a symplectic form such that the symplectic areas of $L$ and $E$ are $2\pi l$ and $2\pi e$, respectively. (One can construct $(\widetilde{\mathbb{CP}^2}, \widetilde{\omega}_{l,e})$ explicitly, e.g., as in [Ka, §6.3]. By [McD2], it is unique up to symplectomorphism.) The standard $(S^1)^2$-action (induced from the action on $\mathbb{CP}^2$) has the moment map image shown in Figure 1 on the right.

**Theorem 2.** *For $a \geq b > 0$, the number of conjugacy classes of maximal tori in* $\mathrm{Ham}(S^2 \times S^2, \omega_{a,b})$ *is* $\lceil a/b \rceil$.

*For $l > e > 0$, the number of conjugacy classes of maximal tori in* $\mathrm{Ham}(\widetilde{\mathbb{CP}^2}, \widetilde{\omega}_{l,e})$ *is* $\lceil \frac{e}{l-e} \rceil$.

**Remark.** Here, $\lceil r \rceil$ denotes the smallest integer greater than or equal to $r$. We will prove Theorem 2 by enumerating the different conjugacy classes by the set of integers $k$ that satisfy $0 \leq k < r$ for $r = a/b$ and $r = \frac{e}{l-e}$, respectively.

**Remark.** The topology of $\mathrm{Ham}(S^2 \times S^2, \omega_{a,b})$ also changes when the ratio $a/b$ crosses an integer. See [Gr, Ab, AM, McD3, An]. Also see the remark at the end of the paper.

**Remark.** There are infinitely many conjugacy classes of tori in the group of contactomorphisms of an overtwisted $S^3$ or lens space. See [L1].

For a count of the conjugacy classes of tori in the group of contactomorphisms of the pre-quantum line bundle over a Hirzebruch surface, see [L2].

We recall standard facts about Hamiltonian circle actions on compact symplectic manifolds: By local arguments, each component of the fixed point set is a submanifold of even dimension and has even index. By Morse-Bott theory, each local extremum for the moment map is a global extremum, and these extrema are attained on connected sets. See [GS2, §32]. An *interior fixed point* is a fixed point which is not a minimum or maximum for the moment map. A *fixed surface* is a two dimensional connected component of the fixed point set.

**Lemma 1.** *Let $M$ be a closed symplectic four-manifold with a Hamiltonian circle action. The dimension of $H^2(M)$ is equal to the number of interior fixed points plus the number of fixed surfaces. If $\dim H^1(M) = 0$, each fixed surface has genus zero.*

*Proof.* We apply a standard Morse theory argument. The moment map is a perfect Morse-Bott function whose critical points are the fixed points for the circle action [GS2, §32]. Therefore, $\dim H^j(M) = \sum \dim H^{j-i_F}(F)$, where we sum over the connected components of the fixed point set, and where $i_F$ is the index of the component $F$. The theorem follows by a simple computation of the summands, which is summarized in the table below.

| | | | Contribution to | | | | |
|---|---|---|---|---|---|---|---|
| $F$ | $\Phi(F)$ | $i_F$ | $H^0(M)$ | $H^1(M)$ | $H^2(M)$ | $H^3(M)$ | $H^4(M)$ |
| fixed | minimal | 0 | 1 | 2 genus($F$) | 1 | | |
| surface | maximal | 2 | | | 1 | 2 genus($F$) | 1 |
| isolated | minimal | 0 | 1 | | | | |
| fixed | interior | 2 | | | 1 | | |
| point | maximal | 4 | | | | | 1 |

□

*Proof of Theorem 1.* By [Ka, Prop. 5.21], a Hamiltonian circle action extends to a toric action if and only if each fixed surface has genus zero and each non-extremal level set for the moment map contains at most two non-free orbits.

By Lemma 1, since $\dim H^1(M) = 0$, each fixed surface has genus zero.

By [Ka, Theorem 5.1], if all the fixed points are isolated, the circle action extends to a toric action. Therefore, let us assume that there exists at least one fixed surface. Suppose that the moment map attains its maximum on this surface; the case of a minimum can be treated similarly. By Lemma 1, since $\dim H^2(M, \mathbb{R}) \leq 3$, there exist at most two interior fixed points.

A $Z_k$*-sphere* is a 2-sphere inside $M$ on which the circle acts by rotations with speed $k$. A non-free orbit is either a fixed point or belongs to a $Z_k$-sphere. See [Au] or [Ka, Lemma 2.2]. A $Z_k$-sphere intersects each level set in at most one orbit. The north pole of a $Z_k$-sphere is an isolated, hence interior, fixed point (because we assume that the maximal set of the moment map is not isolated). Different $Z_k$-spheres have different north poles. These considerations show that the number of non-free orbits in an non-extremal level set for the moment map is at most the number of interior fixed points. The theorem follows. □

For each non-negative integer $m$, consider the family of trapezoids shown in Figure 2, parametrized by the height $b$ and average width $a > \frac{m}{2}b$. We call these *standard Hirzebruch trapezoids.* More generally, consider their images under the group $\mathrm{AGL}(2, \mathbb{Z})$ of transformations of $\mathbb{R}^2$ of the form $x \mapsto Rx + v$ with $R \in \mathrm{GL}(2, \mathbb{Z})$ and $v \in \mathbb{R}^2$. These we call *Hirzebruch trapezoids.* Hirzebruch trapezoids modulo $\mathrm{AGL}(2, \mathbb{Z})$ are in natural one-to-one bijection with the set

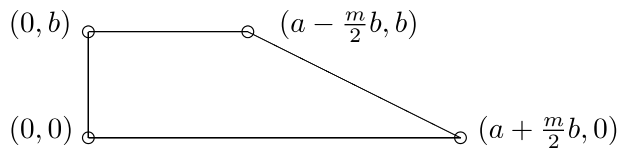


FIGURE 2. A standard Hirzebruch trapezoid

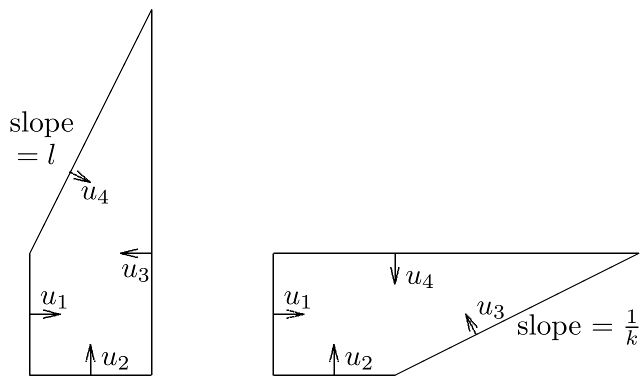


FIGURE 3. Hirzebruch trapezoids

of parameters $(a, b, m)$ with $m$ a non-negative integer, $a$ and $b$ positive real numbers, and $a > \frac{m}{2}b$.

Take a polygon in $\mathbb{R}^2$. Let $u_1, \dots, u_m$ be normal vectors to its edges, in counterclockwise order, pointing inward. The polygon is a *Delzant polygon* if we can choose these normals so that $u_i \in \mathbb{Z}^2$ and $\det(u_i u_{i+1}) = 1$ for all $i$, the indices taken cyclically.[2]

**Lemma 2.** *Every Delzant polygon with four edges is a Hirzebruch trapezoid.*

*Proof.* Fix a Delzant polygon with four edges. Let $u_1$, $u_2$, $u_3$, and $u_4$ be normals to its edges that satisfy the above Delzant condition. Because $\det(u_1 u_2) = 1$, we may assume, without loss of generality, that $u_1 = (1, 0)$ and $u_2 = (0, 1)$. The conditions $\det(u_2 u_3) = 1$ and $\det(u_4 u_1) = 1$ imply that $u_3 = (-1, k)$ and $u_4 = (l, -1)$ for some $k, l \in \mathbb{Z}$. The condition $\det(u_3 u_4) = 1$ then implies $kl = 0$. Each of the cases $k = 0$ and $l = 0$ gives a Hirzebruch trapezoid, as shown in Figure 3. $\square$

The convexity theorem of Atiyah, Guillemin and Sternberg, [At,Gu-St], states that, for a Hamiltonian torus action an a compact symplectic manifold, the image of the moment map is a convex polytope. By Delzant's classification of Hamiltonian toric actions [De],[3] the moment map images for *toric* actions are

[2]More generally, the Delzant condition for a polytope in $\mathbb{R}^n$ is that exactly $n$ facets meet at every vertex and the normals to these facets can be chosen to be generators of $\mathbb{Z}^n$.

[3]Also see [L-T].

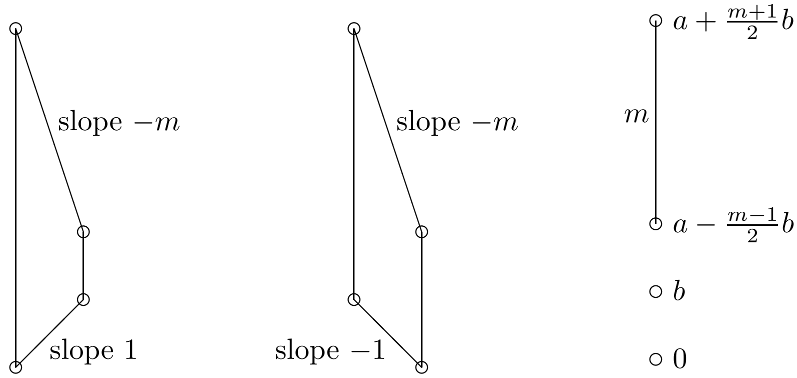


FIGURE 4. Hirzebruch trapezoids with parameters $(m+1, a, b)$ and $(m-1, a, b)$.

precisely the Delzant polytopes, and two toric actions are equivariantly symplectomorphic if and only if their moment map images are translates of each other.

The symplectic four-manifolds that correspond to Hirzebruch trapezoids are *Hirzebruch surfaces.* See [Au] or [Ka, section 6.3]. Specifically, when $m = 0$ or $m = 1$, these are $(S^2 \times S^2, \omega_{a,b})$ and $(\widetilde{\mathbb{CP}^2}, \widetilde{\omega}_{l,e})$, respectively.

**Lemma 3.** *A Hirzebruch surface which corresponds to an integer $m' \geq 2$ is symplectomorphic to the Hirzebruch surface which corresponds to the integer $m' - 2$ with the same parameters $a$ and $b$.*

*Proof.* Take two Hirzebruch trapezoids with the same parameters $a$ and $b$, corresponding to the integers $m'$ and $m' - 2$. After transforming by appropriate elements of $\mathrm{AGL}(2, \mathbb{Z})$, they can be brought to the forms shown in Figure 4, with $m = m' - 1$.

In [Ka, §2.1] we associate a labeled graph to every Hamiltonian $S^1$-space, such that two spaces are isomorphic if and only if their graphs are isomorphic [Ka, Theorem 4.1]. If the $S^1$-action is obtained by restriction of a toric action, the graph can be easily read from the Delzant polygon. See [Ka, §2.2]. The two polygons in Figure 4 give rise to the same graph (shown in Figure 4 on the right). Therefore, the spaces are $S^1$-equivariantly symplectomorphic. □

**Remark.** Lemma 3 (which I learned from S. Tolman) seems to be well known. For instance, the new circle action on $S^2 \times S^2$ obtained by identifying this space with the Hirzebruch surface with parameters $(a, b, m = 2)$ plays a role in [McD1].

**Lemma 4.** *Among the symplectic manifolds $(S^2 \times S^2, \omega_{a,b})$, for $a \geq b > 0$, and $(\widetilde{\mathbb{CP}^2}, \widetilde{\omega}_{l,e})$, for $l > e > 0$, no two are symplectomorphic.*

*Proof.* The manifolds $S^2 \times S^2$ and $\widetilde{\mathbb{CP}^2}$ are not homeomorphic. For instance, the self intersection of the exceptional divisor in $\widetilde{\mathbb{CP}^2}$ is $-1$ whereas in $S^2 \times S^2$ every class in $H_2$ has an even self intersection.

A diffeomorphism of $S^2 \times S^2$ acts on $H^2(S^2 \times S^2) = \mathbb{Z}^2$ by a $2 \times 2$ matrix of integers, with determinant $\pm 1$, which preserves the intersection form $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. There are exactly four matrices with this property; they are

$$\pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \pm \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

These cannot take $\omega_{a,b}$ to a different $\omega_{a',b'}$ with $a \geq b > 0$ and $a' \geq b' > 0$.

A diffeomorphism of $\widetilde{\mathbb{CP}^2}$ acts on $H_2(\widetilde{\mathbb{CP}^2}) = \mathbb{Z}L \oplus \mathbb{Z}E \cong \mathbb{Z}^2$ by a $2 \times 2$ matrix of integers, with determinant $\pm 1$, which preserves the intersection form $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. There are exactly four matrices with this property; they are $\begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix}$. These cannot take $\widetilde{\omega}_{l,e}$ to a different $\widetilde{\omega}_{l',e'}$ with $l > e > 0$ and $l' > e' > 0$. □

*Proof of Theorem 2.* By Theorem 1, we only need to consider two dimensional tori.

The conjugacy classes of 2-tori in $\mathrm{Ham}(M, \omega)$ for *all* possible symplectic four-manifolds $(M, \omega)$ are given by all the Delzant polygons in $\mathbb{R}^2$, modulo $\mathrm{AGL}(2, \mathbb{Z})$-congruence. This follows immediately from Delzant's classification of Hamiltonian toric actions [De] and the fact that a reparametrization of the 2-torus $S^1 \times S^1$ transforms the moment map image by an element of $\mathrm{GL}(2, \mathbb{Z})$.

Hence, to find the conjugacy classes of 2-tori in the symplectomorphism group of a particular symplectic four-manifold $(M, \omega)$, we must identify which Delzant polygons correspond to a space which is symplectomorphic to $(M, \omega)$, and we must take these polygons modulo $\mathrm{AGL}(2, \mathbb{Z})$-congruence.

The number of edges of a Delzant polygon is equal to 2 plus the second Betti number of the corresponding space. See Lemma 1 and [Ka, section 2.2]. Therefore, when $M = S^2 \times S^2$ or $M = \widetilde{\mathbb{CP}^2}$, we only need to consider Delzant polygons with four edges. By Lemma 2, we only need to consider Hirzebruch trapezoids.

Consider the Hirzebruch trapezoid with parameters $m \geq 0$ and $a \geq b > 0$. Iterate Lemma 3. If $m$ is even, the corresponding Hirzebruch surface is symplectomorphic to $(S^2 \times S^2, \omega_{a,b})$. If $m$ is odd, it is symplectomorphic to $(\widetilde{\mathbb{CP}^2}, \widetilde{\omega}_{l,e})$, where $l = a + \frac{b}{2}$ and $e = a - \frac{b}{2}$, so that the corresponding trapezoid (shown in Figure 1 on the right) still has height $b$ and average width $a$.

Numbers $a \geq b$ can occur as the average width and the height of a Hirzebruch trapezoid with integer parameter $m \geq 0$ if and only if $\frac{a}{b} > \frac{m}{2}$. When $m = 2k$ is even, this becomes the condition $k < \frac{a}{b}$. When $m = 2k + 1$ is odd, it becomes the condition $k < \frac{e}{l-e}$, with $l$ and $e$ as above.

We have found the required number of distinct torus actions on each of the symplectic manifolds $(S^2 \times S^2, \omega_{a,b})$, $a \geq b > 0$, and $(\widetilde{\mathbb{CP}^2}, \widetilde{\omega}_{l,e})$, $l > e > 0$. By

Lemma 4, this accounts for all possible toric actions on each of these symplectic manifolds. □

**Remark.** To each non-negative integer $m$ we have associated a torus action on $S^2 \times S^2$ if $m$ is even and on $\widetilde{\mathbb{CP}^2}$ if $m$ is odd, for appropriate ranges of values of symplectic forms. Each of these torus actions is in fact obtained from an action of a larger, non-abelian, compact Lie group by restricting to its maximal torus. For instance, the standard actions of $\mathrm{SO}(3) \times \mathrm{SO}(3)$ on $S^2 \times S^2$ and of $U(2)$ on $\widetilde{\mathbb{CP}^2}$ restrict to the torus actions that correspond to the integers $m = 0$ and $m = 1$, respectively. More generally, given a non-negative integer $m$, consider the quotient of $S^3 \times S^2$ by the circle action $\lambda\colon (z,p) \mapsto (z\lambda, \lambda^m \cdot p)$, where $S^1$ is the circle group of complex numbers of norm one, $\lambda \in S^1$ acts on $z \in S^3 \subset \mathbb{C}^2$ by scalar multiplication, and $p \mapsto \lambda \cdot p$ is the circle action on $S^2 \subset \mathbb{R}^3$ by rotations. This space admits natural Kähler structures with which it is symplectomorphic (but not biholomorphic) to $S^2 \times S^2$ if $m$ is even and to $\widetilde{\mathbb{CP}^2}$ if $m$ is odd, with appropriate symplectic forms. On this space there is a natural action of the quotient of $U(2) \times S^1$ by the subgroup $\{(aI, a^m) \mid a \in S^1\}$, where $I \in U(2)$ is the identity matrix. Note that this quotient is a central extension of $\mathrm{SO}(3)$ by $S^1$. The torus action that corresponds to the integer $m$ comes from this action.

In this context we recall that the only four dimensional compact symplectic $\mathrm{SO}(3)$ manifolds are $S^2 \times S^2$ and $\mathbb{CP}^2$, by [I].

We also recall that the Hamiltonian symplectomorphism group of $S^2 \times S^2$ with equal areas of the two factors retracts to $\mathrm{SO}(3) \times \mathrm{SO}(3)$ by [Gr] and that the aforementioned compact Lie subgroups of the Hamiltonian symplectomorphism groups of $S^2 \times S^2$ and of $\widetilde{\mathbb{CP}^2}$ in some sense carry an essential part of the topology of these symplectomorphism groups [Ab, AM, McD3].

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