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POISSON CAPACITIES

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Abstract. Poisson capacities appeared, first, in the theory of removable boundary singularities for solutions of semilinear elliptic differential equations. It seems that these capacities should play a key role for solving the problem: to describe all positive solutions of the equation $Lu = u^\alpha$ with $\alpha > 1$ in a smooth domain. We establish two inequalities for the Poisson capacities which might be useful for this task.

1. Introduction

1.1. Motivation. A Poisson capacity $\mathrm{CP}_\alpha(\Gamma)$ was introduced in [DK96] as a tool for a study of semilinear elliptic differential equations $Lu = u^\alpha$. It was proved that, if D is a bounded smooth domain, then a closed subset Γ of ∂D is a removable singularity for positive solutions of $Lu = u^\alpha$ in D if and only if $\mathrm{CP}_\alpha(\Gamma) = 0$. First, this was conjectured in [Dyn94]. In the case $\alpha = 2$, the conjecture was proved by Le Gall [Le95] who used a different definition of capacity (not applicable to $\alpha \neq 2$). Le Gall's capacity Cap^∂ has the same class of null sets as CP_α.

The present paper is motivated by a recent work by Mselati [Ms02] who proved that positive solutions of $\Delta u = u^2$ are characterized uniquely by their fine boundary traces introduced in [DK98].[1] A key part of Mselati's investigation is establishing bounds for certain solutions (associated with Γ) in terms of $\mathrm{Cap}^\partial(\Gamma)$. A challenging problem is to get a similar result in the case $\alpha \neq 2$ by using Poisson capacities. This note may be considered as a step in this direction.

1.2. Capacity Cap. Fix a constant $\alpha > 1$, a bounded smooth domain D of class $C^{2,\lambda}$ in $\mathbb{R}^d$ and a second order uniformly elliptic differential operator L in D.

Denote by $\mathcal{M}(\Gamma)$ the set of all finite measures on Γ and put $\mu \in \mathcal{P}(\Gamma)$ if, in addition, $\mu(\Gamma) = 1$. There exists a positive function $k(x, y)$ (called the Poisson

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[1]This result was obtained in [DK98] for the so-called σ-moderate solutions. Mselati proved that all solutions are σ-moderate.

kernel of L in D) such that the formula

$$h_\mu(x) = \int_{\partial D} k(x,y)\mu(dy) \tag{1.1}$$

establishes a 1-1 correspondence between $\mathcal{M}(\partial D)$ and the set of all positive solutions h of the equation $Lh = 0$ in D. We define a Poisson capacity of a Borel set $\Gamma \subset \partial D$ by the formula

$$\operatorname{Cap}(\Gamma) = \sup_{\nu \in \mathcal{P}(\Gamma)} H(\nu)^{-1}. \tag{1.2}$$

where

$$H(\mu) = \int_D \operatorname{dist}(y, \partial D) h_\mu(y)^\alpha dy. \tag{1.3}$$

[$\operatorname{Cap}(\Gamma) = \mathrm{CP}_\alpha(\Gamma)^\alpha$ where CP_α is the capacity used in [Dy02]. A subscript α can be dropped since α is fixed.]

We denote by C constants depending only on D, L and α (their values can vary even within one line). We indicate explicitely the dependence on any additional parameter. For instance, we write C_κ for a constants depending on a parameter κ (besides a possible dependence on D, L, α). An upper bound of $\operatorname{Cap}(\Gamma)$ is given by:

Theorem 1.1. *For all* Γ,

$$\operatorname{Cap}(\Gamma) \le C \operatorname{diam}(\Gamma)^{\gamma_+} \tag{1.4}$$

where

$$\operatorname{diam}(\Gamma) = \sup_{x,y \in \Gamma} |x - y|, \tag{1.5}$$

$$\gamma = d\alpha - d - \alpha - 1 \quad \textit{and } \gamma_+ = \gamma \vee 0. \tag{1.6}$$

1.3. Capacities $\mathbf{Cap}_x$. Put

$$H_x(\mu) = \int_D g(x,y) h_\mu(y)^\alpha dy \tag{1.7}$$

where $g(x,y)$ is the Green function of L in D. To every $x \in D$ there corresponds a capacity on ∂D defined by the formula

$$\operatorname{Cap}_x(\Gamma) = \sup_{\nu \in \mathcal{P}(\Gamma)} H_x(\nu)^{-1} \tag{1.8}$$

The second theorem establishes a lower bound for Cap_x in terms of Cap.

The values $\alpha < (d+1)/(d-1)$ are called subcritical and the values $\alpha \ge (d+1)/(d-1)$ are called supercrtical.

Theorem 1.2. *Suppose that L is an operator of divergence form* (1.13) *and $d \ge 3$. Put*

$$\varphi(x) = \operatorname{dist}(x, \partial D) \operatorname{dist}(x, \Gamma)^{-d}. \tag{1.9}$$

If α is subcritical, then

$$\operatorname{Cap}_x(\Gamma) \geq C\varphi(x)^{-1}\operatorname{Cap}(\Gamma). \tag{1.10}$$

for all Γ and x.

If α is supercritical, then, for every $\kappa > 0$ there exists a constant C_κ such that

$$\operatorname{Cap}_x(\Gamma) \geq C_\kappa\varphi(x)^{-1}\operatorname{Cap}(\Gamma) \tag{1.11}$$

for all Γ and x subject to the condition

$$\operatorname{dist}(x,\Gamma) \geq \kappa \operatorname{diam}(\Gamma). \tag{1.12}$$

[An analog of formula (1.11) with Cap replaced by $\operatorname{Cap}^\partial$ follows from formula (3.34) in [Ms02] in the case $L = \Delta, \alpha = 2, d \geq 4$ and $\kappa = 4$.]

1.4. On operator L and domain D. We consider an operator

$$Lu(x) = \sum_{i,j=1}^d a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}u(x) + \sum_{i=1}^d b_i(x)\frac{\partial}{\partial x_i}u(x)$$

in a domain D in $\mathbb{R}^d$. Without loss of generality we can put $a_{ij} = a_{ji}$. We assume that

1.4.A.[2] There exists a constant $\beta > 0$ such that

$$\sum a_{ij}(x)t_i t_j \geq \beta \sum t_i^2 \quad \text{for all } x \in D, t_1, \dots, t_d \in \mathbb{R}.$$

1.4.B. All coefficients $a_{ij}(x)$ and $b_i(x)$ are Hölder continuous in $\bar{D}$ with exponent λ and Hölder's coefficient Λ.

We assume that the domain D is smooth (more precicely, D is a domain of class $C^{2,\lambda}$) which means that ∂D can be straightened near every point $x \in \partial D$. To define straightening, we consider a half-space $E_+ = \{x = (x_1, \dots, x_d) : x_d > 0\} = \mathbb{R}^{d-1} \times \mathbb{R}_+$ and its boundary $E_0 = \{x = (x_1, \dots, x_d) : x_d = 0\}$. We assume that, for every $x \in \partial D$, there exists a ball $B(x,\varepsilon) = \{y : |x-y| < \varepsilon\}$ and a diffeomorphism ψ_x of class $C^{2,\lambda}$ from $B(x,\varepsilon)$ onto a domain $\tilde{D} \subset \mathbb{R}^d$ such that $\psi(B(x,\varepsilon) \cap D) \subset E_+$ and $\psi(B(x,\varepsilon) \cap \partial D) \subset E_0$. We say that ψ_x straightens the boundary in $B(x,\varepsilon)$.

In Theorem 1.2 we restrict ourself to operators of divergence form

$$Lu(x) = \sum_{i,j=1}^d \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} u(x). \tag{1.13}$$

[2] The property 1.4.A is called uniform ellipticity and β is called the *ellipticity coefficient* of L.

1.5. Bounds for $g(x,y)$ and $k(x,y)$. In the proofs of Theorems 1.1 and 1.2 we use the following bounds for the Poisson kernel (see, e.g. [Maz75], Lemma 6 and the Appendix B in [Dy02]):

$$C^{-1}\rho(x)|x-y|^{-d} \le k(x,y) \le C\rho(x)|x-y|^{-d} \tag{1.14}$$

where

$$\rho(x) = \operatorname{dist}(x, \partial D). \tag{1.15}$$

[It follows from (1.14) that, if Cap′ and Cap″ are the Poisson capacities associated with operators L' and L'', then $\operatorname{Cap}'(\Gamma)/\operatorname{Cap}''(\gamma) \le C$ where C is a constant independent of Γ.]

In the proof of Theorem 1.2 we also use bounds for the Green function:

$$g(x,y) \le C|x-y|^{2-d}, \tag{1.16}$$

$$g(x,y) \le C\rho(x)|x-y|^{1-d}, \tag{1.17}$$

$$g(x,y) \le C\rho(x)\rho(y)|x-y|^{-d}. \tag{1.18}$$

established in [GrW] for operators L of divergence form and $d \ge 3$.

2. Upper bound for Cap(Γ)

2.1. Capacity $\widehat{\mathbf{Cap}}$. To prove Theorem 1.1 we introduce a capacity on the set E_0 associated with the kernel

$$\hat{k}(x,y) = r(x)|x-y|^{-d}, x \in E_+, y \in E_0 \tag{2.1}$$

where

$$r(x) = \operatorname{dist}(x, E_0) = x_d. \tag{2.2}$$

Put

$$\mathbb{E} = \{x = (x_1, \dots, x_d) : 0 < x_d < 1\}, \tag{2.3}$$

To every measure $\nu \in \mathcal{P}(E_0)$ there corresponds a function

$$\hat{h}_\nu(x) = \int_{E_0} \hat{k}(x,y)\nu(dy)$$

on E_+. Put

$$\hat{H}(\nu) = \int_{\mathbb{E}} r(x)\hat{h}_\nu(x)^\alpha \, dx.$$

and

$$\hat{H}(\nu, B) = \int_B r(x)\hat{h}(x)^\alpha \, dx$$

for $B \subset E_+$. Note that

$$\hat{k}(x/t, y/t) = t^{d-1}\hat{k}(x,y) \quad \text{for all } t > 0.$$

To every $\nu \in \mathcal{P}(E_0)$ there corresponds a measure ν_t defined by the formula $\nu_t(B) = \nu(tB)$. We have

$$\int_{E_0} f(y)\nu_t(dy) = \int_{E_0} f(y/t)\nu(dy)$$

for every function f and therefore

$$\hat{h}_{\nu_t}(x/t) = \int_{E_0} \hat{k}(x/t, y)\nu_t(dy) = \int_{E_0} \hat{k}(x/t, y/t)\nu(dy) = t^{d-1}\hat{h}_\nu(x). \tag{2.4}$$

Change of variables $x = t\tilde{x}$ and (2.4) yield

$$\hat{H}(\nu_t) = t^\gamma \hat{H}(\nu, t\mathbb{E}).$$

If $t \geq 1$, then $t\mathbb{E} \supset \mathbb{E}$ and we have

$$\hat{H}(\nu_t) \geq t^\gamma \hat{H}(\nu). \tag{2.5}$$

We introduce a capacity on E_0 by the formula

$$\widehat{\mathrm{Cap}}(\Gamma) = \sup_{\nu \in \mathcal{P}(\Gamma)} \hat{H}(\nu)^{-1}. \tag{2.6}$$

Lemma 2.1. *If* $\mathrm{diam}(\Gamma) \leq 1$, *then*

$$\widehat{\mathrm{Cap}}(\Gamma) \leq C_d(\mathrm{diam}(\Gamma))^\gamma. \tag{2.7}$$

The constant C_d depends only on the dimension d. (It is equal to $\widehat{\mathrm{Cap}}(U)$ where $U = \{x \in E_0 : |x| < 1\}$.

Proof. Since $\widehat{\mathrm{Cap}}$ is translation invariant, we can assume that $0 \in \Gamma$. Let $t = \mathrm{diam}(\Gamma)^{-1}$. Since $t\Gamma \subset U$, we have

$$\widehat{\mathrm{Cap}}(t\Gamma) \leq \widehat{\mathrm{Cap}}(U). \tag{2.8}$$

Since $\nu \to \nu_t$ is a 1-1 mapping from $\mathcal{P}(t\Gamma)$ onto $\mathcal{P}(\Gamma)$, we get

$$\widehat{\mathrm{Cap}}(\Gamma) = \sup_{\nu_t \in \mathcal{P}(\Gamma)} \hat{H}(\nu_t)^{-1} = \sup_{\nu \in \mathcal{P}(t\Gamma)} \hat{H}(\nu_t)^{-1}.$$

Therefore, by (2.6) and (2.5),

$$\widehat{\mathrm{Cap}}(\Gamma) \leq t^{-\gamma}\widehat{\mathrm{Cap}}(t\Gamma)$$

and (2.8) implies (2.7). □

2.2. Two lemmas.

Lemma 2.2. *Suppose $D \subset \mathbb{E}$, $0 \in \Gamma \subset \partial D \cap E_0$ and put $A = \partial D \cap E_+$, $B_\lambda = \{x \in \mathbb{E} : |x| < \lambda\}$. If $\mathrm{dist}(\Gamma, A) > 2\lambda$, then $B_\lambda \subset D$ and $r(x) = \rho(x)$ for $x \in B_\lambda$.*

Proof. If $x \in B_\lambda$, then $r(x) \le |x| < \lambda$. If $x \in B_\lambda$ and $y \in A$, then $|x-y| \ge |y| - |x| > \lambda$ because $|y| > \operatorname{dist}(y,\Gamma) > \operatorname{dist}(A,\Gamma) > 2\lambda$. Hence $\operatorname{dist}(x,A) \ge \lambda$ which implies that $B_\lambda \subset D$.

For $x \in D \cup \partial D$, $\rho(x) = \operatorname{dist}(x, D^c), r(x) = \operatorname{dist}(x, E_+^c)$ and therefore $\rho(x) \le r(x)$. Put $\tilde{A} = \partial D \setminus A$. Let $x \in B_\lambda$. Since $\rho(x) = \operatorname{dist}(x,\tilde{A}) \wedge \operatorname{dist}(x,A)$, the equality $r(x) = \rho(x)$ holds because $\operatorname{dist}(x,\tilde{A}) \ge r(x)$ (since $\tilde{A} \subset E_0$) and $\operatorname{dist}(x,A) \ge \lambda > r(x)$. □

Lemma 2.3. *There exists a constant C (depending only on λ) such that*

$$\hat{H}(\nu, B_\lambda) \ge C\hat{H}(\nu) \tag{2.9}$$

for all $\Gamma \ni 0$ *such that* $\operatorname{diam}(\Gamma) < \lambda/2$ *and for all* $\nu \in \mathcal{P}(\Gamma)$.

Proof. If $x \in F_\lambda = E_+ \setminus B_\lambda$ and $y \in \Gamma$, then $|y| \le \operatorname{diam}(\Gamma) < \lambda/2 \le |x|/2$ and therefore $|x-y| > |x| - |y| \ge |x|/2$. This implies

$$\hat{h}_\nu(x) \le r(x) 2^d |x|^{-d}$$

and

$$\hat{H}(\nu, F_\lambda) \le 2^{d\alpha} \int_{F_\lambda} r(x)^{\alpha+1} |x|^{-d\alpha} dx = C_1 < \infty. \tag{2.10}$$

On the other hand, if $x \in B_\lambda, y \in \Gamma$, then $|x-y| \le |x| + |y| \le 3\lambda/2$. Therefore $\hat{h}_\nu(x) \ge (3\lambda/2)^{-d} r(x)$ and

$$\hat{H}(\nu, B_\lambda) \ge (3\lambda/2)^{-d\alpha} \int_{B_\lambda} r(x)^{\alpha+1} dx = C_2 > 0. \tag{2.11}$$

It follows from (2.10) and (2.11) that

$$C_1 \hat{H}(\nu, B_\lambda) \ge C_1 C_2 \ge C_2 \hat{H}(\nu, F_\lambda) = C_2[\hat{H}(\nu) - \hat{H}(\nu, B_\lambda)]$$

and (2.9) holds with $C = C_2/(C_1 + C_2)$. □

2.3. Straightening of the boundary.

Proposition 2.1. *Suppose that D is a bounded smooth domain. Then there exist strictly positive constants ε, a, b such that, for every $x \in \partial D$:*

(a) *The boundary can be straightened in $B(x,\varepsilon)$.*

(b) *The corresponding diffeomorphism ψ_x satisfies the conditions*

$$a^{-1}|y_1 - y_2| \le |\psi_x(y_1) - \psi_x(y_2)| \le a|y_1 - y_2| \quad \text{for all } y_1, y_2 \in B(x,\varepsilon); \tag{2.12}$$

$$a^{-1} \operatorname{diam}(A) \le \operatorname{diam}(\psi_x(A)) \le a \operatorname{diam}(A) \quad \text{for all } A \subset B(x,\varepsilon); \tag{2.13}$$

$$a^{-1} \operatorname{dist}(A_1, A_2) \le \operatorname{dist}(\psi_x(A_1), \psi_x(A_2)) \le a \operatorname{dist}(A_1, A_2) \quad \text{for all } A_1, A_2 \subset B(x,\varepsilon). \tag{2.14}$$

$$b^{-1} \le J_x(y) \le b \quad \text{for all } y \in B(x,\varepsilon) \tag{2.15}$$

where $J_x(y)$ is the Jacobian of ψ_x at y.

Diffeomorphisms ψ_x can be chosen to satisfy additional conditions

$$\psi_x(x) = 0 \quad \textit{and } \psi_x(B(x,\varepsilon)) \subset \mathbb{E}. \tag{2.16}$$

Proof. The boundary ∂D can be covered by a finite number of balls $B_i = B(x_i, \varepsilon_i)$. The function $q(x) = \max_i \operatorname{dist}(x, B_i)$ is continuous and strictly positive on ∂D. Therefore $\varepsilon = \frac{1}{2}\min_x q(x) > 0$. For every $x \in D$ we choose a ball B_i which contains x and we put

$$\psi_x(y) = \psi_{x_i}(y) \quad \text{for } y \in B(x,\varepsilon).$$

This is a diffeomorphism straightening ∂D in $B(x,\varepsilon)$.

Since the closure of $B(x,\varepsilon)$ is contained in B_i, ψ_{x_i} is uniformly continuous on $B(x,\varepsilon)$. The inverse mapping is uniformly continuous on the image of $B(x,\varepsilon)$. Hence there exist constants $a_i > 0$ such that

$$a_i^{-1}|y_1 - y_2| \le |\psi_{x_i}(y_1) - \psi_{x_i}(y_2)| \le a_i|y_1 - y_2| \quad \text{for all } y_1, y_2 \in B(x_i, \varepsilon_i/2).$$

The condition (2.12) holds for $a = \max a_i$. The conditions (2.13) and (2.14) follow from (2.12). The condition (2.15) holds because J_{x_i} is continuous and strictly positive on the closure of B_i.

By replacing $\psi_x(y)$ with $c[\psi_x(y) - \psi_x(x)]$ with is a suitable constant c, we get diffeomorphisms subject to (2.16) in addition to (2.12)-(2.15). □

2.4. Proof of Theorem 1.1.

1°. If $\gamma < 0$, then (1.4) holds because $\operatorname{Cap}(\Gamma) \le \operatorname{Cap}(\partial D) = C$. To prove (1.4) for $\gamma \ge 0$, it is sufficient to prove that, for some $\beta > 0$, there is a constant C_1 such that

$$\operatorname{Cap}(\Gamma) \le C_1 \operatorname{diam}(\Gamma)^\gamma \quad \text{if } \operatorname{diam}(\Gamma) \le \beta.$$

Indeed,

$$\operatorname{Cap}(\Gamma) \le C_2 \operatorname{diam}(\Gamma)^\gamma \quad \text{if } \operatorname{diam}(\Gamma) \ge \beta$$

with $C_2 = \operatorname{Cap}(\partial D)\beta^{-\gamma}$.

2°. Let ε, a be the constants defined in Proposition 2.1 and let $\beta = \varepsilon/2 \wedge 1$. Suppose that $\operatorname{diam}(\Gamma) \le \beta$ and let $x \in \Gamma$. Consider a straightening ψ_x of ∂D in $B(x,\varepsilon)$ which satisfies conditions (2.16). Put $B = B(x,\varepsilon), \tilde{B} = B(x, \varepsilon/2)$. There exists a smooth domain U such that $\tilde{B} \cap D \subset U \subset B \cap D$. Note that $\tilde{B} \cap \partial D \subset \partial U \cap \partial D \subset B \cap \partial D$. If $A = \partial U \cap B \cap D$, then $\operatorname{dist}(x, A) \ge \varepsilon/2$ and $\operatorname{dist}(\Gamma, A) \ge \varepsilon/2 - \operatorname{diam}(\Gamma) \ge \varepsilon/2 - \beta$. Denote by U', Γ', A' the images of U, Γ, A under ψ_x. By (2.13), $\operatorname{diam}(\Gamma') \le \lambda_1 = a\beta$ and $\operatorname{dist}(\Gamma', A) \ge \lambda_2 = (\varepsilon/2 - \beta)/a$. If $\beta < \varepsilon(8a^2 + 2)^{-1}$, then $\lambda_1 < \lambda_2/4$ and Lemmas 2.2 and 2.3 are applicable to U', Γ' and $\lambda = \frac{1}{2}(\lambda_1 + \lambda_2/4)$.

3°. By (1.14) and (2.14), for every $y \in U, z \in \Gamma$,

$$\begin{aligned} k(y,z) &\ge C \operatorname{dist}(y, \partial D)|y - z|^{-d} \\ &\ge C \operatorname{dist}(y, \partial U)|y - z|^{-d} \ge \frac{C}{a} \operatorname{dist}(y', \partial U)|y' - z'|^{-d} \end{aligned} \tag{2.17}$$

where $y' = \psi_x(y), z' = \psi_x(z)$. If ν' is the image of $\nu \in \mathcal{P}(\Gamma)$ under ψ_x, then

$$\int_\Gamma f[\psi_x(z)]\nu(dz) = \int_{\Gamma'} f(z')\nu'(dz')$$

for every positive measurable function f. In particular,

$$\int_\Gamma |y' - \psi_x(z)|^{-d}\nu(dz) = \int_{\Gamma'} |y' - z'|^{-d}\nu'(dz'). \tag{2.18}$$

By (2.17) and (2.18),

$$\int_\Gamma k(y,z)\nu(dz) \geq C \operatorname{dist}(y', \partial U') \int_{\Gamma'} |y' - z'|^{-d}\nu'(dz').$$

If $y' \in B_\lambda$, then, by Lemma 2.2, $\operatorname{dist}(y', \partial U') = r(y')$ and we have

$$h_\nu(y) = \int_\Gamma k(y,z)\nu(dz) \geq C \int_{\Gamma'} r(y')|y' - z'|^{-d}\nu'(dz') = C\hat{h}_{\nu'}[\psi_x(y)]. \tag{2.19}$$

By (2.14), $\operatorname{dist}(y, \partial D) \geq \operatorname{dist}(y, \partial U) \geq C \operatorname{dist}(y', \partial U')$ and therefore (1.3) and (2.19) imply

$$H(\nu) = \int_D \operatorname{dist}(y, \partial D) h_\nu(y)^\alpha dy \geq C \int_U \operatorname{dist}(\psi_x(y), \partial U')\hat{h}_{\nu'}[\psi_x(y)]^\alpha dy \tag{2.20}$$

Note that

$$\int_{U'} f(y')dy' = \int_U f[\psi_x(y)] J_x(y) dy$$

and, if $f \geq 0$, then, by (2.15),

$$\int_{U'} f(y')dy' \leq b \int_U f[\psi_x(y)]dy.$$

By taking $f(y') = \operatorname{dist}(y', \partial U')\hat{h}_{\nu'}(y')^\alpha$, we get from (2.20)

$$H(\nu) \geq C \int_{U'} \operatorname{dist}(y', \partial U')\hat{h}_{\nu'}(y')^\alpha dy'.$$

By Lemma 2.2, $U' \supset B_\lambda$ and $\operatorname{dist}(y', \partial U') = r(y')$ on B_λ. Hence

$$H(\nu) \geq C \int_{B_\lambda} r(y')\hat{h}_{\nu'}(y')^\alpha dy' = C\hat{H}(\nu', B_\lambda).$$

By Lemma 2.3, this implies $H(\nu) \geq C\hat{H}(\nu')$ and $\operatorname{Cap}(\Gamma) \leq C\widehat{\operatorname{Cap}}(\Gamma')$. The bound $\operatorname{Cap}(\Gamma) \leq C \operatorname{diam}(\Gamma)^\gamma$ follows from Lemma 2.1 and (2.14). $\square$

3. Lower bound for $\mathbf{Cap}_x$

3.1. Put

$$\delta(x) = \operatorname{dist}(x, \Gamma), \quad D_1 = \{x \in D : \delta(x) < \rho(x)/2\}, \quad D_2 = D \setminus D_1;$$

$$H_x(\nu, B) = \int_B g(x,y) h_\nu(y)^\alpha dy \quad \text{for } B \subset D \tag{3.1}$$

and let

$$U_x = \{y \in D : |x-y| < \delta(x)/2\}, \qquad V_x = \{y \in D : |x-y| \geq \delta(x)/2\}. \tag{3.2}$$

Theorem 1.2 follows from the following three lemmas.

Lemma 3.1. *For all* Γ*, all* $\nu \in \mathcal{P}(\Gamma)$ *and all* $x \in D$*,*

$$H_x(\nu, V_x) \leq C\varphi(x) H(\nu). \tag{3.3}$$

Proof. By (1.14), for all $\nu \in \mathcal{P}(\Gamma)$ and all $y \in V_x$,

$$h_\nu(y) = \int_\Gamma k(y,z)\nu(dz) \leq C\rho(y) \int_\Gamma |y-z|^{-d} \nu(dz) \leq C\varphi(x)$$

and, by (1.18),

$$H_x(\nu, V_x) \leq C\rho(x) \int_{V_x} \rho(y)|x-y|^{-d} h_\nu(y)^\alpha dy \leq C\varphi(x) H(\nu).$$

□

Lemma 3.2. *For all* Γ*, all* $\nu \in \mathcal{P}(\Gamma)$ *and all* $x \in D_1$*,*

$$H_x(\nu, U_x) \leq C\varphi(x) H(\nu). \tag{3.4}$$

Proof. By the Harnack's inequality, if h is harmonic in D and if $|x-y|/\rho(x) \leq r < 1$, then

$$\frac{1-r}{(1+r)^{d-1}} h_\nu(x) \leq h_\nu(y) \leq \frac{1+r}{(1-r)^{d-1}} h_\nu(x)$$

(see, e.g. [GT98], Problem 2.61). If $x \in D_1$, then this inequality holds with $r = 1/4$ for all $y \in U_x$. Therefore

$$H_x(\nu, U_x) \leq C_d h_\nu(x)^\alpha \int_{U_x} g(x,y) dy \tag{3.5}$$

and

$$H(\nu) \geq C_d h_\nu(x)^\alpha \int_{U_x} \rho(y) dy. \tag{3.6}$$

where C_d depends only on d. By (1.17),

$$\int_{U_x} g(x,y) dy \leq C\rho(x) \int_{U_x} |x-y|^{1-d} dy \leq C\delta(x)\rho(x). \tag{3.7}$$

Since $\rho(y) \geq \rho(x)/2$ for $y \in U_x$, we have

$$\int_{U_x} \rho(y) dy \geq \frac{1}{2}\rho(x) \int_{U_x} dy = C_d \rho(x)\delta(x)^d. \tag{3.8}$$

Since $\delta\rho \leq \varphi\rho\delta^d/2$, bound (3.4) follows from (3.5)–(3.8). □

Lemma 3.3. *For all* Γ*, all* $\nu \in \mathcal{P}(\Gamma)$ *and all* $x \in D_2$*,*

$$H_x(\nu, U_x) \leq C\varphi(x)\theta(x)^{-\gamma_+}H(\nu) \tag{3.9}$$

where

$$\theta(x) = \operatorname{dist}(x, \Gamma)/\operatorname{diam}(\Gamma).$$

Proof. If $\operatorname{diam}(\Gamma) = \lambda$, then, by (1.2), and Theorem 1.1,

$$H(\nu)^{-1/\alpha} \leq \operatorname{Cap}(\Gamma) \leq C\lambda^{\gamma+/\alpha}.$$

Hence,

$$H(\nu) \geq C\lambda^{-\gamma+}. \tag{3.10}$$

If $x \in D_2$ and $y \in U_x$, then $\delta(y) \geq \delta(x) - |x - y| > \delta(x)/2$ and $\rho(y) \leq \rho(x) + |x - y| \leq 2\delta(x) + \delta(x)/2 = 5\delta(x)/2$. For all $z \in \Gamma, y \in U_x$, $|y - z| \geq |z - x| - |y - x| \geq \delta(x)/2$ and, by (1.14),

$$k(y, z) \leq C\rho(y)|y - z|^{-d} \leq C\delta(x)^{1-d}.$$

Therefore $h_\nu(y) \leq C\delta(x)^{1-d}$ and, by (1.17),

$$H_x(\nu, U_x) \leq C\rho(x)\delta(x)^{(1-d)\alpha}\int_{U_x}|x - y|^{1-d}dy \leq C\varphi(x)\delta(x)^{-\gamma}. \tag{3.11}$$

If $\gamma < 0$, then $\delta(x)^{-\gamma} \leq \operatorname{diam}(D)^{-\gamma} = C$. If $\gamma \geq 0$, then $\gamma = \gamma_+$. Hence, the bound (3.9) follows from (3.10) and (3.11). □

3.2. Proof of Theorem 1.2. Since $H_x(\nu) = H_x(\nu, V_x) + H_x(\nu, U_x)$, we get from Lemmas 3.1–3.3 that

$$H_x(\nu) \leq C\ (1 \vee \theta(x)^{-\gamma_+})\varphi(x)H(\nu)$$

for all Γ, all $\nu \in \mathcal{P}(\Gamma)$ and all $x \in D$. By (1.2) and (1.8), this implies

$$\operatorname{Cap}_x(\Gamma) \geq C(1 \vee \theta(x)^{\gamma_+})\varphi(x)^{-1}\operatorname{Cap}(\Gamma) \tag{3.12}$$

for all Γ and all $x \in D$. If α is subcritical, then $\gamma < 0$ and (3.12) implies (1.10). If α is supercritical, then $\gamma \geq 0$ and (1.11) holds under the condition (1.12). □

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