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A REMARK ON MAXIMAL OPERATORS ALONG DIRECTIONS IN $\mathbb{R}^2$

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Abstract. In this paper we give a simple proof of a long-standing conjecture, recently proved by N. Katz, on the weak-type norm of a maximal operator associated with an arbitrary collection of directions in the plane. The proof relies upon a geometric argument and on induction with respect to the number of directions. Applications are given to estimate the behavior of several types of maximal operators.

1. Introduction

Let Ω be a subset of $[0, \frac{\pi}{4})$. Associated to it, we define the maximal operator M_Ω acting on locally integrable functions f on $\mathbb{R}^2$ by

$$M_\Omega f(x) = \sup_{x \in R \in \mathcal{B}_\Omega} \frac{1}{|R|} \int_R |f(y)|\, dy,$$

where $\mathcal{B}_\Omega$ denotes the basis of all rectangles with longest side forming an angle α with the x-axis, for some $\alpha \in \Omega$.

For finite sets Ω, it was conjectured that the L^2 norm of M_Ω had logarithmic growth with respect to the cardinality, N, of Ω. More precisely, that M_Ω satisfied the following estimate

$$\|M_\Omega f\|_{L^2(\mathbb{R}^2)} \leq C (\log N)^\beta \|f\|_{L^2(\mathbb{R}^2)}, \tag{1}$$

for some exponent β and with C independent of Ω and f.

This inequality was first proved in the seventies for some special sets Ω, such as uniformly distributed directions (see [9]). Also, for lacunary sequences the result holds with $\beta = 0$ (see [4] and [7]) and therefore one can take an infinite sequence in this case. In 1995, Barrionuevo [3] obtained the following result:

$$\|M_\Omega f\|_{L^2(\mathbb{R}^2)} \leq C N^{2/\sqrt{\log N}} \|f\|_{L^2(\mathbb{R}^2)},$$

again for card$\Omega = N < \infty$ and C independent of Ω and f.

Conjecture (1) for general finite sets Ω was finally proved by Katz in 1999, [6], (see also [5]), with the best possible exponent, that is, with $\beta = 1$.

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The main result of this paper (Theorem 1 below) establishes an estimate for the weak type (2,2) of an operator in a related problem which gives, as a consequence, a simple proof of that conjecture. With our arguments we do not obtain the sharp exponent of the logarithm given in Katz' proof. Theorem 1 however does provide a wide range of applications which we explore in the last section. The proof of Theorem 1 uses some geometric arguments, in the same spirit of the work of Strömberg, [9], and Córdoba and Fefferman, [4].

In order to state our result, we introduce first some notation. Given the set $\Omega \subset [0, \frac{\pi}{4})$ we consider a subset $\Omega_0 = \{\theta_l : l = 1, 2, \dots\} \subset \Omega$ with $\frac{\pi}{4} > \theta_1 > \theta_2 > \dots > \theta_l > \dots$. There is no restriction on whether Ω_0 is finite or not. Let us define for $l = 1, 2, \dots$, $\Omega_l = \{\alpha \in \Omega : \theta_l \leq \alpha < \theta_{l-1}\}$, where we have set $\theta_0 = \frac{\pi}{4}$. We shall assume that Ω_0 is chosen so that $\Omega = \cup_{l \geq 1} \Omega_l$. To each set $\Omega_l,\ l = 0, 1, 2, \dots$ we associate the corresponding basis $\mathcal{B}_l \subset \mathcal{B}_\Omega$. We define the maximal operators associated to each Ω_l as

$$M_{\Omega_l} f(x) = \sup_{x \in R \in \mathcal{B}_l} \frac{1}{|R|} \int_R |f(y)|\, dy, \quad l = 0, 1, 2, \dots$$

In the next section we shall prove the following

Theorem 1. *There exist constants C_1, C_2 independent of Ω such that*

$$\|M_\Omega\|^2_{L^2 \to L^{2,\infty}} \leq C_1 \sup_{1 \leq l} \|M_{\Omega_l}\|^2_{L^2 \to L^{2,\infty}} + C_2 \|M_{\Omega_0}\|^2_{L^2 \to L^{2,\infty}}, \tag{2}$$

where $\|T\|_{L^2 \to L^{2,\infty}}$ denotes the "weak type $(2,2)$" norm of the operator T.

What the theorem says is that if we decompose Ω into disjoint blocks, "separated" by the elements of certain collection of directions Ω_0, then the price that one pays to make the inequality

$$\|\sup_l M_{\Omega_l}\|_{L^2 \to L^{2,\infty}} \leq C \sup_l \|M_{\Omega_l}\|_{L^2 \to L^{2,\infty}} \tag{3}$$

true is given by the norm of the operator M_{Ω_0}, associated to Ω_0, in the precise form stated above. This can be seen as a weak-L^2 quasi-orthogonality principle for the family of operators $\{M_{\Omega_j}\}$. In particular, if the "separating" set Ω_0 is given by a lacunary sequence, then (3) is true no matter how the "intermediate" sets $\Omega_l,\ l = 1, 2, \dots$ are chosen. This extends a result by Sjögren and Sjölin [8]. In Section 3 we present this and some other applications of our theorem, such as the new proof of Katz's theorem that we mentioned above.

2. Proof of the main result

Let us start by changing slightly our previous hypotheses. Given a set $\Omega \subset [0, \frac{\pi}{4})$, we consider the basis $\mathcal{B}_\Omega$ of all parallelograms with the shorter sides parallel to the y-axis and the longer sides forming an angle α with the x-axis, for some $\alpha \in \Omega$. With certain abuse of language, we shall call these parallelograms "rectangles".

We introduce now the following notation: given a rectangle $R \in \mathcal{B}_\Omega$, $P_1(R)$ will denote the projection of R on the x-axis. If $P_1(R) = [a_R^1, a_R^2]$, we also define

$P_{2,1}(R) = \{y : (a_R^1, y) \in R\}$ and $P_{2,2}(R) = \{y : (a_R^2, y) \in R\}$. $P_{2,1}(R)$ and $P_{2,2}(R)$ are the projections of the two "vertical sides" of R on the y-axis. Note that $|P_{2,1}(R)| = |P_{2,2}(R)|$ and $|R| = |P_1(R)| \cdot |P_{2,1}(R)|$.

Let Ω_0 be any ordered subset of Ω. By a simple limiting argument, it is clear that in order to prove (2) we may assume with no loss of generality that Ω (and, hence, Ω_0 too) is finite. We shall denote the elements of Ω_0 by $\theta_1, \theta_2, \ldots, \theta_N$, with

$$\frac{\pi}{4} = \theta_0 > \theta_1 > \theta_2 > \ldots > \theta_{N-1} > \theta_N \geq 0.$$

Define $\Omega_l = \{\alpha \in \Omega : \theta_l \leq \alpha < \theta_{l-1}\}$, for $l = 1, \ldots, N$. Then $\Omega = \cup_{l=1}^N \Omega_l$ by assuming simply that $\theta_N = \min \Omega$. To each set Ω_l, $l = 0, 1, \ldots, N$ we associate the corresponding basis $\mathcal{B}_l \subset \mathcal{B}_\Omega$.

We now define the maximal operators

$$M_\Omega f(x) = \sup_{x \in R \in \mathcal{B}_\Omega} \frac{1}{|R|} \int_R |f(y)|\, dy,$$

and

$$M_{\Omega_l} f(x) = \sup_{x \in R \in \mathcal{B}_l} \frac{1}{|R|} \int_R |f(y)|\, dy, \quad l = 0, 1, \ldots, N.$$

To prove (2) we look at the level sets of M_Ω. If $x \in \{M_\Omega f(x) > \lambda\}$, there is a rectangle $R_x \in \mathcal{B}_\Omega$ containing x such that

$$\frac{1}{|R_x|} \int_{R_x} |f(y)|\, dy > \lambda, \tag{4}$$

and therefore

$$\{M_\Omega f(x) > \lambda\} \subset \bigcup_{x \in \{M_\Omega f(x) > \lambda\}} R_x.$$

So if we consider a compact set $K \subset \{M_\Omega f(x) > \lambda\}$, then $K \subset \bigcup_{j=1}^s R_{x_j}$ for some finite family of rectangles $\mathcal{F} = \{R_{x_j}\}_{j=1}^s$ satisfying (4).

From the family $\mathcal{F}$ we select a subfamily $\overline{\mathcal{F}} = \{B_k\}$ in the following way: we take B_1 as the rectangle $R \in \mathcal{F}$ with longest projection on the x-axis. Assuming we have already chosen $B_1, \ldots, B_{n-1}$, we take B_n as the rectangle R of the remaining collection $\mathcal{F} \setminus \{B_k\}_{j=1}^{n-1}$ such that $|P_1(R)|$ is maximal among the rectangles satisfying

$$\sum_{k=1}^{n-1} |R \cap B_k| \leq \frac{1}{2}|R|.$$

It is easy to see that the family $\{B_k\}$ has the following two properties:

$$\sum |B_k| \leq 2|\cup B_k| \tag{5}$$

$$\int \left(\sum \chi_{B_k}\right)^2 \leq 2 \sum |B_k|. \tag{6}$$

In order to estimate the weak type $(2,2)$ norm for M_Ω, we first observe that

$$\sum |B_k| \le \frac{1}{\lambda} \sum \int_{B_k} |f| \le \frac{1}{\lambda} \|f\|_2 \left\| \sum \chi_{B_k} \right\|_2 \le$$

$$\le \frac{\sqrt{2}}{\lambda} \|f\|_2 \left(\sum |B_k| \right)^{1/2},$$

where we have used (6). This implies

$$\left(\sum |B_k| \right)^{1/2} \le \frac{\sqrt{2}}{\lambda} \|f\|_2. \tag{7}$$

If we show that

$$| \cup R_{x_j} \setminus \cup B_k| \le c_0 \sum |B_k|, \tag{8}$$

then, using (7) we get

$$|K| \le |\cup R_{x_j}| \le |\cup B_k| + |\cup R_{x_j} \setminus \cup B_k| \le$$

$$\le (1+c_0) \left(\sum |B_k| \right) \le \frac{2(1+c_0)}{\lambda^2} \|f\|_2^2.$$

Consequently, we would obtain

$$|\{M_\Omega f(x) > \lambda\}| \le \frac{2(1+c_0)}{\lambda^2} \|f\|_2^2,$$

provided c_0 is independent of the compact set K.

It remains to prove then (8). Let R be one of the rectangles in $\mathcal{F} \setminus \overline{\mathcal{F}}$. Then

$$\sum_{B_k : |P_1(B_k)| \ge |P_1(R)|} |R \cap B_k| > \frac{1}{2} |R|.$$

If $R \in \mathcal{B}_l$, we observe that one of the following inequalities must hold:

$$\sum_{B_k \in \mathcal{B}_l : |P_1(B_k)| \ge |P_1(R)|} \frac{|R \cap B_k|}{|R|} > \frac{1}{4}, \tag{9}$$

or

$$\sum_{B_k \notin \mathcal{B}_l : |P_1(B_k)| \ge |P_1(R)|} \frac{|R \cap B_k|}{|R|} > \frac{1}{4}. \tag{10}$$

Let us denote

$$\mathcal{F}_1 = \bigcup_l \left\{ R \in (\mathcal{F} \setminus \overline{\mathcal{F}}) \cap \mathcal{B}_l : \sum_{B_k \in \mathcal{B}_l : |P_1(B_k)| \ge |P_1(R)|} \frac{|R \cap B_k|}{|R|} > \frac{1}{4} \right\},$$

and

$$\mathcal{F}_2 = \bigcup_l \left\{ R \in (\mathcal{F} \setminus \overline{\mathcal{F}}) \cap \mathcal{B}_l : \sum_{B_k \notin \mathcal{B}_l : |P_1(B_k)| \ge |P_1(R)|} \frac{|R \cap B_k|}{|R|} > \frac{1}{4} \right\}.$$

Observe that if $R \in \mathcal{B}_l$ and (9) holds, then $R \subset \{x : M_{\Omega_l}(\sum_{B_k \in \mathcal{B}_l} \chi_{B_k}) > \frac{1}{4}\}$. Hence,

$$| \cup_{\mathcal{F}_1} R| \leq \sum_{l=1}^{N} \left| \left\{ M_{\Omega_l}(\sum_{B_k \in \mathcal{B}_l} \chi_{B_k}) > \frac{1}{4} \right\} \right| \leq 16 \sum_{l=1}^{N} \|M_{\Omega_l}\|^2_{L^2 \to L^{2,\infty}} \| \sum_{B_k \in \mathcal{B}_l} \chi_{B_k} \|_2^2$$

$$\leq 16 \sup_{l=1,\dots,N} \|M_{\Omega_l}\|^2_{L^2 \to L^{2,\infty}} \| \sum_{B_k \in \overline{\mathcal{F}}} \chi_{B_k} \|_2^2. \tag{11}$$

Now, suppose we know that there exists an universal constant c with the property that, for any $R \in \mathcal{F}_2 \cap \mathcal{B}_l$, there are $\widehat{R_+}$, $\widehat{R_-} \in \mathcal{B}_0$ containing R so that

$$\frac{|B_k \cap R|}{|R|} \leq c \frac{|B_k \cap \widehat{R_+}|}{|\widehat{R_+}|} + c \frac{|B_k \cap \widehat{R_-}|}{|\widehat{R_-}|}, \tag{12}$$

for all $B_k \notin \mathcal{B}_l$. Then, we would have

$$| \cup_{\mathcal{F}_2} R| \leq |\{M_{\Omega_0}(\sum \chi_{B_k}) > \frac{1}{8c}\}| \leq 64 c^2 \|M_{\Omega_0}\|^2_{L^2 \to L^{2,\infty}} \cdot \| \sum \chi_{B_k}\|_2^2 \leq$$

$$\leq 128 c^2 \|M_{\Omega_0}\|^2_{L^2 \to L^{2,\infty}} \, | \cup B_k|. \tag{13}$$

Combining (11) and (13), we get (8) with $c_0 = 32 \sup \|M_{\Omega_l}\|^2 + 128 c^2 \|M_{\Omega_0}\|^2$. So we have to prove (12).

In order to do this, we introduce first some notation.

Definition. Given $U, V \in \mathcal{B}_\Omega$, we say that U crosses V entirely if there exists an interval J such that

$$J \subset P_1(U), P_1(V),$$

and, if S is the strip $S = \{(x,y) : x \in J\}$, $\widetilde{U} = U \cap S$, $\widetilde{V} = V \cap S$, then

$$\widetilde{U} \cap \widetilde{V} \neq \emptyset,$$

$$P_{2,i}(\widetilde{U}) \cap P_{2,i}(\widetilde{V}) = \emptyset \ \ for\ i = 1, 2.$$

Observe that U crosses entirely V if and only if V crosses entirely U.

Lemma 2. *If V_1, V_2 cross entirely U, with $|P_{2,1}(V_1)| = |P_{2,1}(V_2)|$ and* angle$(V_2, U) = \alpha_2 \leq \alpha_1 =$ angle(V_1, U)*, then*

$$|V_1 \cap U| \leq |V_2 \cap U|.$$

Proof. We may assume without loss of generality that U has sides parallel to the axis. Then, if $a = |P_{2,1}(U)|$ and $b = |P_{2,1}(V_j)|$,

$$|V_j \cap U| = \frac{a \cdot b}{\tan \alpha_j},$$

so if $\alpha_2 \leq \alpha_1$, then $|V_1 \cap U| \leq |V_2 \cap U|$. □

Lemma 3. *If V_1, V_2 are parallel, cross entirely U and $|P_1(V_1)| = |P_1(V_2)| = L$, then*

$$\frac{|U \cap V_1|}{|V_1|} = \frac{|U \cap V_2|}{|V_2|}.$$

Proof. Again, we may assume that U has sides parallel to the axis. Set $a = |P_{2,1}(U)|$, $\alpha = \text{angle}(V_j, U)$ and $b_j = |P_{2,1}(V_j)|$ for $j = 1, 2$ (note that α does not depend on j). Then, as in lemma 2, we have

$$\frac{|U \cap V_j|}{|V_j|} = \frac{a \cdot b_j}{\tan\alpha} \cdot \frac{1}{b_j \cdot L} = \frac{a}{L\tan\alpha},$$

which does not depend on j. □

Once we have these two lemmas, let us prove (12). Let $R \in \mathcal{F}_2 \cap \mathcal{B}_l$, $B \in \overline{\mathcal{F}} \backslash \mathcal{B}_l$ such that $B \cap R \neq \emptyset$ and $|P_1(B)| \geq |P_1(R)|$. Let α_R be the angle that R forms with the x-axis, and α_B the angle that B forms with the x-axis. We shall assume that $\alpha_B > \alpha_R$. (The case $\alpha_B < \alpha_R$ can be handled in a similar way.) Then there exists $\theta_k \in \Omega_0$ such that

$$\alpha_B \geq \theta_k > \alpha_R.$$

Let $\widetilde{R}$ be the smallest rectangle in the direction of θ_k containing R. We define $\widehat{R_+}$ as the rectangle concentric with $\widetilde{R}$ and 9 times bigger. We will call $\widehat{R}_{\text{mid}}$ to the middle third of $\widehat{R_+}$, i.e. the rectangle in the direction of θ_k satisfying

$$P_1(\widehat{R}_{\text{mid}}) = P_1(\widehat{R_+}),$$
$$\widetilde{R} \subset \widehat{R}_{\text{mid}},$$
$$|\widetilde{R}| = \frac{1}{3}|\widehat{R}_{\text{mid}}|.$$

We have to show that

$$\frac{|B \cap R|}{|R|} \leq c\frac{|B \cap \widehat{R_+}|}{|\widehat{R_+}|}.$$

Since $\widehat{R_+} \in \mathcal{B}_0$, this gives (12).

To simplify the notation, from now and on we shall write $\widehat{R}$ instead of $\widehat{R_+}$. We define R^∞ as the smallest infinite strip containing R and with the same slope. Let B' be the smallest rectangle containing B with $P_1(B') \supset P_1(\widehat{R})$, and define

$$B^* = B' \cap [P_1(\widehat{R}) \times \mathbb{R}].$$

Observe that $|B^* \cap \widehat{R}| \leq 3|B \cap \widehat{R}|$.

We shall consider two cases:

Case 1. B^* crosses entirely $\widehat{R}$.
Let R^{rot} be a rectangle in the direction of θ_k such that

$$P_1(R^{rot}) = P_1(\widehat{R}),$$

$$R^{rot} \subset \widehat{R}_{mid},$$

$$|P_{2,1}(R^{rot})| = |P_{2,1}(R)|.$$

Then, by Lemma 2,

$$|B^* \cap R| \leq |B^* \cap R^\infty| \leq |B^* \cap R^{rot}|,$$

and

$$\frac{|B^* \cap R|}{|R|} \leq \frac{|B^* \cap R^{rot}|}{|R|} = \frac{3|B^* \cap R^{rot}|}{|R^{rot}|} = \frac{3|B^* \cap \widehat{R}|}{|\widehat{R}|}.$$

In the last equality we have used Lemma 3. So we get

$$\frac{|B \cap R|}{|R|} \leq \frac{|B^* \cap R|}{|R|} \leq \frac{3|B^* \cap \widehat{R}|}{|\widehat{R}|} \leq \frac{9|B \cap \widehat{R}|}{|\widehat{R}|}.$$

Case 2. B^* does not cross entirely $\widehat{R}$.
We may assume that $|P_{2,1}(B^*)| \leq \frac{1}{3}|P_{2,1}(\widehat{R})|$, because otherwise we would have

$$R \subset \widehat{R}_{mid} \subset 25B^* \subset 125B.$$

But if $|P_{2,1}(B^*)| \leq \frac{1}{3}|P_{2,1}(\widehat{R})|$, then we have

$$|B^*| \leq 3|B^* \cap \widehat{R}|.$$

Let B^{*rot} be a rectangle with slope θ_k such that

$$P_1(B^{*rot}) = P_1(\widehat{R}) = P_1(B^*),$$

$$B^{*rot} \subset \widehat{R}_{mid},$$

$$|P_{2,1}(B^{*rot})| = |P_{2,1}(B^*)|.$$

Note that $|B^*| = |B^{*rot}|$. By Lemma 2, we have

$$|B^* \cap R| \leq |B^* \cap R^\infty| \leq |B^{*rot} \cap R^\infty|,$$

and so we get

$$\frac{|B^* \cap R|}{|B^*|} \leq \frac{|B^{*rot} \cap R^\infty|}{|B^{*rot}|}.$$

By Lemma 3, this is equal to

$$\frac{|\widehat{R}_{mid} \cap R^\infty|}{|\widehat{R}_{mid}|} \leq \frac{3|R|}{|\widehat{R}_{mid}|} = \frac{9|R|}{|\widehat{R}|}.$$

This implies

$$\frac{|B^* \cap R|}{|R|} \leq \frac{9|B^*|}{|\widehat{R}|} \leq 27\frac{|B^* \cap \widehat{R}|}{|\widehat{R}|}.$$

Hence

$$\frac{|B \cap R|}{|R|} \le 81 \frac{|B \cap \widehat{R}|}{|\widehat{R}|},$$

and (12) is proved. □

3. Some Applications

We can use Theorem 1 to give a simple proof of the following

Theorem 4. *There exist positive constants C and α such that for any set $\Omega \subset [0, \frac{\pi}{4})$ of cardinality $N > 1$ one has*

$$\|M_\Omega\|_{L^2 \to L^{2,\infty}} \le C(\log N)^\alpha, \tag{14}$$

Proof. The proof is by induction on the number N of directions. It is clear that (14) holds for small N. Let us assume now that it is true for all $k < N$. We choose a subset Ω_0 of cardinality $N^{1/2}$ so that the corresponding subsets Ω_l, $l = 1, 2, \dots, N^{1/2}$ have all cardinality $N^{1/2}$. Note that $N^{1/2}$ may not be an integer, but we may assume this with appropriate initial assumptions on N.

Then, by Theorem 1,

$$\|M_\Omega\|^2_{L^2 \to L^{2,\infty}} \le C_1 \sup_{1 \le l \le N^{1/2}} \|M_{\Omega_l}\|^2_{L^2 \to L^{2,\infty}} + C_2 \|M_{\Omega_0}\|^2_{L^2 \to L^{2,\infty}}.$$

By hypothesis, (14) holds for M_{Ω_0} and M_{Ω_l}, $l = 1, \dots, N^{1/2}$. So we get

$$\|M_\Omega\|^2_{L^2 \to L^{2,\infty}} \le C_1 C^2 (\log(N^{1/2}))^{2\alpha} + C_2 C^2 (\log N^{1/2})^{2\alpha} \le$$

$$\le C^2 \left(C_1 \left(\frac{1}{2}\right)^{2\alpha} + C_2 \left(\frac{1}{2}\right)^{2\alpha} \right) (\log N)^{2\alpha}.$$

Now we choose α appropriately so that $\left(C_1(\frac{1}{2})^{2\alpha} + C_2(\frac{1}{2})^{2\alpha}\right)$ is less than or equal to 1 and the theorem is proved. □

Well known interpolation arguments show that from (14) one has the strong type estimate

$$\|M_\Omega\|_{L^2 \to L^2} \le C(\log N)^{\alpha + \frac{1}{2}}.$$

It is interesting to observe that the exponent α that we obtain in (14) depends only on the constants C_1 and C_2 in Theorem 1. In particular, inequality (2) with $C_1 = 1$ would give the sharp exponent $\alpha = 1/2$.

Our second application is an extension of a result by Sjögren and Sjölin [8]. We follow the notation introduced in Section 2.

Theorem 5. *Let $\Omega_0 \subset [0, \frac{\pi}{4})$ denote the elements of a lacunary sequence $\{\theta_l\}_l$, say $\theta_l \le \frac{1}{2}\theta_{l-1}$ and consider Ω_l, $l = 1, 2, \dots$ arbitrary sets with $\Omega_l \subset [\theta_l, \theta_{l-1})$. Set $\Omega = \cup_{l \ge 0} \Omega_l$. Then the maximal function M_Ω has the property*

$$\|M_\Omega\|_{L^2 \to L^{2,\infty}} \le C \sup_l \|M_{\Omega_l}\|_{L^2 \to L^{2,\infty}}.$$

In particular, if each set Ω_l is given by the elements of a lacunary set as above (i.e., Ω is a double lacunary set), then M_Ω is of weak type (2,2).

The proof follows easily from Theorem 1.

Remark. At the time of submission of this paper for publication, the authors obtained a new result, similar to Theorem 1, but concerning now the strong type 2. Namely, under the hypothesis of Theorem 1 and with the notation given there, they showed that

$$\|M_\Omega\|_{L^2\to L^2} \leq \sup_{1\leq l} \|M_{\Omega_l}\|_{L^2\to L^2} + C_2\,\|M_{\Omega_0}\|_{L^2\to L^2}. \tag{15}$$

This estimate yields a very simple proof of Katz's result, (1), with the sharp exponent, $\beta = 1$. The details will appear in [2].

Also, the first author has recently extended these results to the case $p \neq 2$ in [1].

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