Mathematical Research Letters **9**, 757–769 (2002)

POLYNOMIAL REPRESENTATIVES OF SCHUBERT CLASSES IN $QH^*(G/B)$

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Abstract. We show how the quantum Chevalley formula for G/B, as stated by Peterson and proved rigorously by Fulton and Woodward, combined with ideas of Fomin, S. I. Gelfand, and Postnikov, leads to a formula which describes polynomial representatives of the Schubert cohomology classes in the canonical presentation of $QH^*(G/B)$ in terms of generators and relations. We generalize in this way results of [FGP].

§1 Introduction

A theorem of Borel [B] describes the cohomology[1] ring of the generalized complex flag manifold G/B as the "co-invariant algebra" of the Weyl group of G, which is essentially a quotient of a certain polynomial ring. The Schubert cohomology classes (i.e. Poincaré duals of Schubert varieties) are a basis of $H^*(G/B)$. In order to determine the structure constants of the cup-multiplication on $H^*(G/B)$ with respect to this basis, we need to describe the Schubert cohomology classes in Borel's presentation. According to Bernstein, I. M. Gelfand, and S. I. Gelfand [BGG], we obtain polynomial representatives of Schubert classes in Borel's ring by starting with a representative of the top cohomology and then applying successively *divided difference operators* associated to the simple roots of G. More details concerning the Bernstein-Gelfand-Gelfand construction can be found in section 2 of our paper.

When dealing with the (small) quantum cohomology ring $QH^*(G/B)$ we face a similar situation. There exists a canonical presentation of that ring, again as a quotient of a polynomial algebra, where the variables are the same as in the classical case, plus the "quantum variables" $q_1, \dots, q_l$. As about the ideal of relations, it is generated by the "quantum deformations" of the relations from Borel's presentation of $H^*(G/B)$ (for more details, see section 3). The Schubert classes are a basis of $QH^*(G/B)$ as a $\mathbb{R}[q_1, \dots, q_l]$-module. A natural aim (see the next paragraph) is to describe them in the previous presentation of $QH^*(G/B)$. Our main result gives a method for obtaining such polynomial representatives. It can be described briefly as follows: we start with an *arbitrary*

Received May 28, 2002.
Revised version received August 6, 2002.

[1] The coefficient ring for cohomology will always be $\mathbb{R}$.

polynomial representing the Schubert class σ_w in Borel's description (e.g. by using the B-G-G construction); this is transformed into a polynomial representing σ_w in the canonical description of $QH^*(G/B)$ after successive applications of divided difference operators, multiplications by q_j's and integer numbers and additions. The precise formula is stated in Theorem 3.6 (also see Lemma 3.4 and relation (5) in order to understand the notations). For $G = SL(n, \mathbb{C})$, the same result was proved by Fomin, S. Gelfand, and Postnikov [FGP]. The main ingredient of our proof is a result of D. Peterson [P] (we call it the "quantum Chevalley formula," since Chevalley obtained a similar result for the cup-multiplication on $H^*(G/B)$) which describes the quantum multiplication by degree 2 Schubert classes.

Finally, a few words should be said about the importance of our result. The standard presentation of $QH^*(G/B)$ mentioned above is explicitly determined by Kim [K] (see also [M]). Our description could be relevant for finding the structure constants of the quantum multiplication with respect to the basis consisting of Schubert classes, which would lead immediately to the Gromov-Witten invariants of G/B. The efficiency of this strategy depends very much on the input: we dispose of the *choice* of polynomial representatives of Schubert classes in Borel's ring and this has to be made judiciously (see again [FGP], as well as Billey and Haiman [BH] and Fomin and Kirillov [FK]).

§2 The Bernstein-Gelfand-Gelfand construction

The main object of study of this paper is the generalized complex flag manifold G/B, where G is a connected, simply connected, semisimple, complex Lie group and $B \subset G$ a Borel subgroup. Let $\mathfrak{t}$ be the Lie algebra of a maximal torus of a compact real form of G and $\Phi \subset \mathfrak{t}^*$ the corresponding set of roots. The negative of the Killing form restricted to $\mathfrak{t}$ gives an inner product $\langle\ ,\ \rangle$. To any root α corresponds the coroot

$$\alpha^\vee := \frac{2\alpha}{\langle \alpha, \alpha \rangle}$$

which *is an element of* $\mathfrak{t}$, by using the identification of $\mathfrak{t}$ and $\mathfrak{t}^*$ induced by $\langle\ ,\ \rangle$. If $\{\alpha_1, \dots, \alpha_l\}$ is a system of simple roots then $\{\alpha_1^\vee, \dots, \alpha_l^\vee\}$ is a system of simple coroots. Consider $\{\lambda_1, \dots, \lambda_l\} \subset \mathfrak{t}^*$ the corresponding system of fundamental weights, which are defined by $\lambda_i(\alpha_j^\vee) = \delta_{ij}$. To any positive root α we assign the reflection s_α of $(\mathfrak{t}, \langle\ ,\ \rangle)$ about the hyperplane $\ker \alpha$. The Weyl group W is generated by all reflections s_α, $\alpha \in \Phi^+$: it is actually generated by a smaller set, namely by the simple reflections $s_1 = s_{\alpha_1}, \dots, s_l = s_{\alpha_l}$. To any $w \in W$ corresponds a length, $l(w)$, which is the smallest number of factors in a decomposition of w as a product of simple reflections.

There are two different ways to describe $H^*(G/B)$: On the one hand, we can take $B^- \subset G$ the Borel subgroup opposite to B and assign to each $w \in W$ the Schubert variety $\bar{C}_w = \overline{B^-.w}$, which has real codimension $2l(w)$; its Poincaré dual σ_w is an element of $H^{2l(w)}(G/B)$; the set σ_w, $w \in W$ is a basis of $H^*(G/B)$.

On the other hand, let us consider the symmetric algebra $S(\mathfrak{t}^*)$, which consists of polynomial functions on $\mathfrak{t}$. A theorem of Borel says that the ring homomorphism $S(\mathfrak{t}^*) \to H^*(G/B)$ induced by $\lambda_i \mapsto \sigma_{s_i}$, $1 \le i \le l$, is surjective; moreover it induces the ring isomorphism

$$H^*(G/B) \simeq \mathbb{R}[\lambda]/I_W, \tag{1}$$

where I_W is the ideal of $S(\mathfrak{t}^*) = \mathbb{R}[\lambda_1, \dots, \lambda_l] = \mathbb{R}[\lambda]$ generated by the W-invariant polynomials of strictly positive degree.

One is looking for a Giambelli type formula, which connects these two descriptions by assigning to each Schubert cycle σ_w a polynomial representative in the quotient ring $\mathbb{R}[\lambda]/I_W$. We are going to sketch the construction of such polynomials, as performed by Bernstein, I. M. Gelfand, and S. I. Gelfand in [BGG]. It relies on the following facts:

- $H^*(G/B)$ and $\mathbb{R}[\lambda]/I_W$ are generated as rings by σ_{s_i}, respectively λ_i, $1 \le i \le l$,
- we have a formula of Chevalley which gives the matrix of the cup multiplication by σ_{s_i} on $H^*(G/B)$ with respect to the basis $\{\sigma_w : w \in W\}$,
- there is another, "very similar", formula, which involves the divided difference operators Δ_w, $w \in W$ (see below) on the polynomial ring $\mathbb{R}[\lambda]$.

The following result was proved by Chevalley [Ch] (see also Fulton and Woodward [FW]).

Lemma 2.1. (Chevalley's formula). *For any $1 \le i \le l$ and any $w \in W$ we have*

$$\sigma_{s_i}\sigma_w = \sum_{\alpha\in\Phi^+, l(ws_\alpha)=l(w)+1} \lambda_i(\alpha^\vee)\sigma_{ws_\alpha}.$$

To each positive root α we assign the *divided difference operator* Δ_α on the ring $\mathbb{R}[\lambda]$ (the latter being just the symmetric ring $S(\mathfrak{t}^*)$, it admits a natural action of the Weyl group W):

$$\Delta_\alpha(f) = \frac{f - s_\alpha f}{\alpha}$$

If w is an arbitrary element of W, take $w = s_{i_1} \dots s_{i_k}$ a reduced expression and then set

$$\Delta_w = \Delta_{\alpha_{i_1}} \circ \cdots \circ \Delta_{\alpha_{i_k}}.$$

One can show (see for instance [Hi]) that the definition does not depend on the choice of the reduced expression. The operators obtained in this way have the following property:

$$\Delta_w \circ \Delta_{w'} = \begin{cases} \Delta_{ww'}, & \text{if } l(ww') = l(w) + l(w') \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

The importance of those operators for our present context is revealed by the similarity of the following formula with Lemma 2.1:

Lemma 2.2. (Hiller [Hi]) *If* λ_i^* *denotes the operator of multiplication by* λ_i *on* $\mathbb{R}[\lambda]$*, then for any* $w \in W$ *we have*

$$\Delta_w \lambda_i^* - w\lambda_i^* w^{-1}\Delta_w = \sum_{\beta\in\Phi^+, l(ws_\beta)=l(w)-1} \lambda_i(\beta^\vee)\Delta_{ws_\beta}.$$

Let w_0 be the longest element of W. The polynomial

$$c_{w_0} := \frac{1}{|W|}\prod_{\alpha\in\Phi^+}\alpha$$

is homogeneous, of degree $l(w_0)$ and has the property that $\Delta_{w_0}c_{w_0} = 1$. But $l(w_0)$ is at the same time the complex dimension of G/B, and it can be easily shown that the class of c_{w_0} in $\mathbb{R}[\lambda]/I_W$ generates the top cohomology of G/B. To any $w \in W$ we assign $c_w := \Delta_{w^{-1}w_0}c_{w_0}$ which is a homogeneous polynomial of degree $l(w)$ satisfying

$$\Delta_v c_w = \begin{cases} c_{wv^{-1}}, & \text{if } l(wv^{-1}) = l(w) - l(v) \\ 0, & \text{otherwise} \end{cases}$$

for any $v \in W$ (see (2)). In particular, if $l(v) = l(w)$, then $\Delta_v(c_w) = \delta_{vw}$. Since Δ_w leaves I_W invariant, it induces an operator on $\mathbb{R}[\lambda]/I_W$ which also satisfies $\Delta_v([c_w]) = \delta_{vw}$, provided that $l(v) = l(w)$. Because $\dim \mathbb{R}[\lambda]/I_W = |W|$, it follows that the classes $[c_w]$, $w \in W$, are a basis of $\mathbb{R}[\lambda]/I_W$. We can easily determine any of the coefficients a_v from

$$\lambda_i[c_w] = \sum_{l(v)=l(w)+1} a_v[c_v],$$

by applying Δ_v on both sides and using Lemma 2.2. It follows that

$$\lambda_i[c_w] = \sum_{\alpha\in\Phi^+, l(ws_\alpha)=l(w)+1} \lambda_i(\alpha^\vee)[c_{ws_\alpha}]. \tag{3}$$

From $\Delta_{s_i}(\lambda_j) = \delta_{ij}$, $1 \le i, j \le l$, we deduce that $c_{s_i} = \lambda_i$. We just have to compare (3) with Lemma 2.1 to conclude:

Theorem 2.3. (Bernstein, I. M. Gelfand, and S. I. Gelfand [BGG]) *Let* $[c_{w_0}]$ *be the image of* σ_{w_0} *by the identification* $H^*(G/B) = \mathbb{R}[\lambda]/I_W$ *indicated above. Then the map* $\sigma_w \mapsto [c_w] := \Delta_{w^{-1}w_0}[c_{w_0}]$ *is a ring isomorphism.*

The polynomial $c_w = \Delta_{w^{-1}w_0}c_{w_0}$ being a representative of the Schubert cycle σ_w in $\mathbb{R}[\lambda]/I_W$, is a solution of the classical (i.e. non-quantum) Giambelli problem for G/B.

§3 Quantization map

Additively, the quantum cohomology $QH^*(G/B)$ of G/B is just $H^*(G/B) \otimes \mathbb{R}[q_1, \dots, q_l]$, where l is the rank of G and $q_1, \dots, q_l$ are some variables. The multiplication $\circ$ is uniquely determined by $\mathbb{R}[q]$-linearity and the general formula

$$\sigma_u \circ \sigma_v = \sum_{d=(d_1,\dots,d_l)\geq 0} q^d \sum_{w \in W} \langle \sigma_u | \sigma_v | \sigma_{w_0 w} \rangle_d \sigma_w,$$

$u, v \in W$, where q^d denotes $q_1^{d_1} \dots q_l^{d_l}$. The coefficient $\langle \sigma_u | \sigma_v | \sigma_{w_0 w} \rangle_d$ is the Gromov-Witten invariant, which counts the number of holomorphic curves $\varphi : \mathbb{C}P^1 \to G/B$ such that $\varphi_*([\mathbb{C}P^1]) = d$ in $H_2(G/B)$ and $\varphi(0)$, $\varphi(1)$ and $\varphi(\infty)$ are in general translates of the Schubert varieties dual to σ_u, σ_v, respectively $\sigma_{w_0 w}$. It turns out that this number can be nonzero and finite only if $l(u)+l(v) = l(w)+2\sum_{i=1}^l d_i$; if it is infinity, we set $\langle \sigma_u | \sigma_v | \sigma_{w_0 w} \rangle_d = 0$. The ring $(QH^*(G/B), \circ)$ is commutative and associative (for more details about quantum cohomology we refer the reader to Fulton and Pandharipande [FP]).

One can show that the quantum cohomology ring of G/B is generated by $H^2(G/B) \otimes \mathbb{R}[q_1, \dots, q_l]$, i.e. by $q_1, \dots, q_l, \lambda_1, \dots, \lambda_l$. To determine the ideal of relations, we only have to take any of the fundamental W-invariant polynomials u_i, $1 \leq i \leq l$ — as generators of the ideal I_W of relations in $H^*(G/B)$ — and find its "quantum deformation" R_i. The latter is a polynomial in $\mathbb{R}[q, \lambda]$, uniquely determined by:

(a) the relation $R_i(q_1, \dots, q_l, \sigma_{s_1}\circ, \dots, \sigma_{s_l}\circ) = 0$ holds in $QH^*(G/B)$,
(b) the component of R_i free of q is u_i.

If I_W^q denotes the ideal of $\mathbb{R}[q, \lambda]$ generated by $R_1, \dots, R_l$, then we have the ring isomorphism

$$QH^*(G/B) \simeq \mathbb{R}[q, \lambda]/I_W^q. \tag{4}$$

The challenge is now to solve the "quantum Giambelli problem": via the isomorphism (4), find a polynomial representative in $\mathbb{R}[q, \lambda]/I_W^q$ for each Schubert class σ_w, $w \in W$. We can actually use Theorem 2.3 in order to rephrase the problem as follows: Describe (the image of $[c_w]$ via) the map

$$\mathbb{R}[q, \lambda]/(I_W \otimes \mathbb{R}[q]) = \mathbb{R}[\lambda]/I_W \otimes \mathbb{R}[q] \xrightarrow{\cong} H^*(G/B) \otimes \mathbb{R}[q] = QH^*(G/B) \xrightarrow{\cong} \mathbb{R}[q, \lambda]/I_W^q.$$

Note that the latter is an isomorphism of $\mathbb{R}[q]$-modules, but *not* of algebras; following [FGP], we call it the *quantization map*. So the main goal of our paper is to give a presentation of the quantization map. For $G = SL(n, \mathbb{C})$, the problem has been solved by Fomin, Gelfand, and Postnikov [FGP]. We are going to extend their result to an arbitrary semisimple Lie group G.

As in the non-quantum case, we will essentially rely on the Chevalley formula, this time in its quantum version: the formula was obtained by D. Peterson in [P] (for more details, see section 10 of Fulton and Woodward [FW]). If $\alpha^\vee$ is a positive coroot, we consider its *height*

$$|\alpha^\vee| = m_1 + \ldots + m_l,$$

where the positive integers $m_1, \ldots, m_l$ are given by $\alpha^\vee = m_1\alpha_1^\vee + \ldots + m_l\alpha_l^\vee$. We also put

$$q^{\alpha^\vee} = q_1^{m_1} \ldots q_l^{m_l}.$$

Theorem 3.1. (Quantum Chevalley Formula; Peterson [P], Fulton and Woodward [FW]) *In* $(QH^*(G/B), \circ)$ *one has*

$$\sigma_{s_i} \circ \sigma_w = \sigma_{s_i}\sigma_w + \sum_{l(ws_\alpha)=l(w)-2|\alpha^\vee|+1} \lambda_i(\alpha^\vee) q^{\alpha^\vee} \sigma_{ws_\alpha}.$$

The following inequality can be found in Peterson's notes [P], as well as in Brenti, Fomin, and Postnikov [BFP]. For the sake of completeness, we will give our own proof of it.

Lemma 3.2. *For any positive root* α *we have* $l(s_\alpha) \le 2|\alpha^\vee| - 1$.

Proof. We prove the lemma by induction on $l(s_\alpha)$. If $l(s_\alpha) = 1$, then α, as well as $\alpha^\vee$, is simple, so $|\alpha^\vee| = 1$. Let now α be a positive, non-simple root. There exists a simple root β such that $\alpha(\beta^\vee) > 0$ (otherwise we would be led to $\alpha(\alpha^\vee) \le 0$). Consequently, $\beta(\alpha^\vee)$ is a strictly positive number, too, hence

$$s_\alpha(\beta) = \beta - \beta(\alpha^\vee)\alpha$$

must be a negative root. Also

$$s_\beta s_\alpha(\beta) = (\alpha(\beta^\vee)\beta(\alpha^\vee) - 1)\beta - \beta(\alpha^\vee)\alpha$$

is a negative root. By Lemma 3.3, chapter 1 of [Hi], we have $l(s_\beta s_\alpha s_\beta) = l(s_\alpha) - 2$. Because

$$s_\beta(\alpha)^\vee = s_\beta(\alpha^\vee) = \alpha^\vee - \beta(\alpha^\vee)\beta^\vee,$$

we have $|s_\beta(\alpha)^\vee| = |\alpha^\vee| - \beta(\alpha^\vee)$. By the induction hypothesis we conclude:

$$l(s_\alpha) = l(s_\beta s_\alpha s_\beta) + 2 \le 2|s_\beta(\alpha)^\vee| - 1 + 2 = 2|\alpha^\vee| - 1 + 2(1 - \beta(\alpha^\vee)) \le 2|\alpha^\vee| - 1.$$

□

Denote by $\tilde{\Phi}^+$ the set of all positive roots α with the property $l(s_\alpha) = 2|\alpha^\vee| - 1$. The following operators

$$\Lambda_i = \lambda_i + \sum_{\alpha \in \tilde{\Phi}^+} \lambda_i(\alpha^\vee) q^{\alpha^\vee} \Delta_{s_\alpha} \quad (5)$$

on $\mathbb{R}[q,\lambda]$, $1 \le i \le l$ have been considered by Peterson in [P]. His key observation is that we have

$$\Lambda_i[c_w] = \lambda_i[c_w] + \sum_{l(ws_\alpha)=l(w)-2|\alpha^\vee|+1} \lambda_i(\alpha^\vee) q^{\alpha^\vee}[c_{ws_\alpha}], \tag{6}$$

the right hand side being, by the quantum Chevalley formula, just $\lambda_i \circ [c_w]$. In order to justify (6), we only have to say that if $w \in W$ and α is a positive root with $l(ws_\alpha) = l(w) - 2|\alpha^\vee| + 1$, then, by Lemma 3.2, α must be in $\tilde{\Phi}^+$.

From the associativity of the quantum product $\circ$ it follows that any two Λ_i and Λ_j commute *as operators on* $(\mathbb{R}[\lambda]/I_W)\otimes\mathbb{R}[q]$. In fact the following stronger result (also stated by Peterson in [P]) holds:

Lemma 3.3. *The operators* $\Lambda_1, \dots, \Lambda_l$ *on* $\mathbb{R}[q,\lambda]$ *commute.*

Proof. Put $w = s_\alpha$ in Lemma 2.2 and obtain:

$$\Delta_{s_\alpha}\lambda_i^* = (\lambda_i^* - \lambda_i(\alpha^\vee)\alpha^*)\Delta_{s_\alpha} + \sum_{\gamma\in\Phi^+, l(s_\alpha s_\gamma)=l(s_\alpha)-1} \lambda_i(\gamma^\vee)\Delta_{s_\alpha s_\gamma}.$$

It follows

$$\begin{aligned}
\Lambda_j\Lambda_i =& (\lambda_j\lambda_i)^* + \sum_{\alpha\in\tilde{\Phi}^+} \lambda_i(\alpha^\vee) q^{\alpha^\vee}\lambda_j^*\Delta_{s_\alpha} \\
&+ \sum_{\alpha\in\tilde{\Phi}^+} \lambda_j(\alpha^\vee) q^{\alpha^\vee}\lambda_i^*\Delta_{s_\alpha} - \sum_{\alpha\in\tilde{\Phi}^+} \lambda_j(\alpha^\vee)\lambda_i(\alpha^\vee) q^{\alpha^\vee}\alpha^*\Delta_{s_\alpha} \\
&+ \sum_{\alpha\in\tilde{\Phi}^+, \gamma\in\Phi^+, l(s_\alpha s_\gamma)=l(s_\alpha)-1} \lambda_j(\alpha^\vee)\lambda_i(\gamma^\vee) q^{\alpha^\vee}\Delta_{s_\alpha s_\gamma} \\
&+ \sum_{\alpha,\beta\in\tilde{\Phi}^+, l(s_\alpha s_\beta)=l(s_\alpha)+l(s_\beta)} \lambda_j(\alpha^\vee)\lambda_i(\beta^\vee) q^{\alpha^\vee+\beta^\vee}\Delta_{s_\alpha s_\beta}.
\end{aligned}$$

Denote by Σ_{ij} the sum of the last two sums: the rest is obviously invariant by interchanging $i \leftrightarrow j$.

Let us return to the Bernstein-Gelfand-Gelfand construction described in the first section: Fix $c_{w_0} \in \mathbb{R}[\lambda]$ such that $[c_{w_0}] = \sigma_{w_0}$ and then set $c_w = \Delta_{w^{-1}w_0} c_{w_0}$, $w \in W$; their classes modulo I_W are a basis of $\mathbb{R}[\lambda]/I_W$. As we said earlier, from the associativity of the quantum product we deduce that $\Lambda_j\Lambda_i[c_w]$ is symmetric in i and j, for any $w \in W$. In particular, $\Sigma_{ij}[c_{w_0}]$ is symmetric in i and j. Because $l(w_0 v) = l(w_0) - l(v)$ for any $v \in W$, we have

$$\begin{aligned}
\Sigma_{ij}[c_{w_0}] =& \sum_{\alpha\in\tilde{\Phi}^+, l(s_\alpha s_\gamma)=l(s_\alpha)-1} \lambda_j(\alpha^\vee)\lambda_i(\gamma^\vee) q^{\alpha^\vee}[c_{w_0 s_\gamma s_\alpha}] \\
&+ \sum_{\alpha,\beta\in\tilde{\Phi}^+, l(s_\alpha s_\beta)=l(s_\alpha)+l(s_\beta)} \lambda_j(\alpha^\vee)\lambda_i(\beta^\vee) q^{\alpha^\vee+\beta^\vee}[c_{w_0 s_\beta s_\alpha}].
\end{aligned}$$

The latter reproduces exactly the expression of Σ_{ij} itself: $\{[c_w] : w \in W\}$ (actually $\{[c_{w_0w^{-1}}] : w \in W\}$) are linearly independent, exactly like the operators $\{\Delta_w : w \in W\}$. So Σ_{ij} is symmetric in i and j and the lemma is proved. □

The next result is a generalization of Lemma 5.3 of [FGP].

Lemma 3.4. *The map $\psi : \mathbb{R}[q, \lambda] \to \mathbb{R}[q, \lambda]$ given by*

$$f \mapsto f(\Lambda_1, \dots, \Lambda_l)(1)$$

is an $\mathbb{R}[q]$-linear isomorphism. If $f \in \mathbb{R}[q, \lambda]$ has degree d with respect to $\lambda_1, \dots, \lambda_l$, then we can express $\psi^{-1}(f)$ as follows

$$\psi^{-1}(f) = \frac{I - (I - \psi)^d}{\psi}(f)$$

$$= \binom{d}{1} f - \binom{d}{2} \psi(f) + \dots + (-1)^{d-2} \binom{d}{d-1} \psi^{d-2}(f) + (-1)^{d-1} \psi^{d-1}(f), \tag{7}$$

where $\binom{d}{1}, \dots, \binom{d}{d-1}$ are the binomial coefficients.

Proof. The degrees of elements of $\mathbb{R}[q, \lambda]$ we are going to refer to here are taken *only* with respect to $\lambda_1, \dots, \lambda_l$. First, ψ is injective, because if $g \in \mathbb{R}[q, \lambda]$ has the property that $g(\Lambda_1, \dots, \Lambda_l)(1) = 0$, then obviously g must be 0. In order to prove both surjectivity and the formula for ψ^{-1}, we notice that the operator $I - \psi$ lowers the degree of a polynomial by at least one, so if f is a polynomial of degree d, then $(I - \psi)^d(f) = 0$. □

The next result is a direct consequence of the quantum Chevalley formula.

Proposition 3.5. *For any of the generators $R_1, \dots, R_l$ of the ideal I_W^q, $\psi(R_i)$ is*[2] *an $\mathbb{R}[q]$-linear combination of elements of I_W, the free term with respect to $q_1, \dots, q_l$ being u_i. Hence $\psi(I_W^q) = I_W \otimes \mathbb{R}[q]$ and ψ gives rise to a bijection*

$$\psi : \mathbb{R}[q, \lambda]/I_W^q \to \mathbb{R}[q, \lambda]/(I_W \otimes \mathbb{R}[q]).$$

Proof. We just have to use the fact that

$$\lambda_{i_1} \circ \dots \circ \lambda_{i_k} = \Lambda_{i_1} \dots \Lambda_{i_k}(1) \bmod I_W \otimes \mathbb{R}[q]$$

so that

$$\begin{aligned} \psi(R_i) \bmod I_W \otimes \mathbb{R}[q] &= R_i(q_1, \dots, q_l, \Lambda_1, \dots, \Lambda_l)(1) \bmod I_W \otimes \mathbb{R}[q] \\ &= R_i(q_1, \dots, q_l, \lambda_1 \circ, \dots, \lambda_l \circ) \\ &= 0. \end{aligned}$$

□

Our polynomial representatives of Schubert classes in $QH^*(G/B)$ are described by the following theorem, which is the central result of the paper. The proof is governed by the same ideals that have been used in the non-quantum case (see section 2).

[2]In view of Theorem 5.5 of [FGP], we could actually expect to have $\psi(R_i) = u_i$.

Theorem 3.6. *The quantization map* $\mathbb{R}[q,\lambda]/(I_W\otimes\mathbb{R}[q])\to\mathbb{R}[q,\lambda]/I_W^q$ *is just* ψ^{-1}. *More precisely, if* $w\in W$ *has length* $l(w)=l$*, then the class of* c_w *in* $\mathbb{R}[q,\lambda]/(I_W\otimes\mathbb{R}[q])$ *is mapped to the class of*

$$\frac{I-(I-\psi)^l}{\psi}(c_w)=$$
$$\binom{l}{1}c_w-\binom{l}{2}\psi(c_w)+\ldots+(-1)^{l-2}\binom{l}{l-1}\psi^{l-2}(c_w)+(-1)^{l-1}\psi^{l-1}(c_w)$$

in $\mathbb{R}[q,\lambda]/I_W^q$*, where* ψ *has been defined in Lemma 3.4.*

Proof. For any polynomial $f\in\mathbb{R}[q,\lambda]$, we denote by $[f]$, $[f]_q$ its classes modulo $I_W\otimes\mathbb{R}[q]$, respectively modulo I_W^q. By the definition of ψ, the polynomial $\hat{c}_w:=\psi^{-1}(c_w)$ is determined by

$$\hat{c}_w(\Lambda_1,\ldots,\Lambda_l)(1)=c_w.$$

We take into account (6), where $\Lambda_i[c_w]$ is the same as

$$[\Lambda_i(c_w)]=[\Lambda_i(\hat{c}_w(\Lambda_1,\ldots,\Lambda_l)(1))]=\psi([\lambda_i\hat{c}_w]_q).$$

Because $[c_v]=\psi([\hat{c}_v]_q)$ for any $v\in W$ and the map ψ is bijective, it follows that in $\mathbb{R}[q,\lambda]/I_W^q$ we have

(8)
$$\lambda_i[\hat{c}_w]_q=\sum_{l(ws_\alpha)=l(w)+1}\lambda_i(\alpha^\vee)[\hat{c}_{ws_\alpha}]_q+\sum_{l(ws_\alpha)=l(w)-2|\alpha^\vee|+1}\lambda_i(\alpha^\vee)q^{\alpha^\vee}[\hat{c}_{ws_\alpha}]_q.$$

As $\mathbb{R}[q]$-algebras, both $QH^*(G/B)$ and $\mathbb{R}[q,\lambda]/I_W^q$ are generated by their degree 2 elements; this is why their structure is uniquely determined by the bases $\{\sigma_w : w\in W\}$, respectively $\{[\hat{c}_w] : w\in W\}$ and the matrices of multiplication by σ_{s_i}, respectively λ_i, $1\le i\le l$. Since $\hat{c}_{s_i}=\lambda_i$, $1\le i\le l$, it follows from Theorem 3.1 and relation (8) that the map

$$QH^*(G/B)\to\mathbb{R}[q,\lambda]/I_W^q \ \text{ given by } \ \sigma_w\mapsto\hat{c}_w, w\in W$$

is an isomorphism of algebras and the proof is finished. □

Example. We will illustrate our main result by giving concrete solutions to the quantum Giambelli problem for G/B, where G is simple of type B_2. This is the first interesting case, different from A_n and for which $\tilde{\Phi}^+\neq\Phi^+$. We will use the following presentation of the root system: if x_1, x_2 are an orthogonal coordinate system of the plane and e_1, e_2 the unit direction vectors of the coordinate axes, then

- the simple roots are $\alpha_1:=x_1$ and $\alpha_2:=x_2-x_1$.
- the positive roots are $\alpha_1,\alpha_2,\alpha_3:=\alpha_1+\alpha_2=x_2$ and $\alpha_4:=2\alpha_1+\alpha_2=x_1+x_2$.

- the positive coroots are $\alpha_1^\vee = 2e_1$, $\alpha_2^\vee = e_2 - e_1$, $\alpha_3^\vee = 2e_2 = \alpha_1^\vee + 2\alpha_2^\vee$ and $\alpha_4^\vee = e_1 + e_2 = \alpha_1^\vee + \alpha_2^\vee$.
- the fundamental weights λ_1, λ_2 are determined by

$$\begin{aligned} x_1 &= 2\lambda_1 - \lambda_2 \\ x_2 &= \lambda_2 \end{aligned}$$

- the simple reflections are $s_1 : (x_1, x_2) \mapsto (-x_1, x_2)$ and $s_2 : (x_1, x_2) \mapsto (x_2, x_1)$. The generators of I_W are obviously $x_1^2 + x_2^2$ and $x_1^2 x_2^2$.
- following [FK], we can obtain polynomial representatives of Schubert classes in $\mathbb{R}[x_1, x_2]/(x_1^2 + x_2^2, x_1^2 x_2^2)$ as indicated in the following table:

w	c_w
$w_0 = s_1 s_2 s_1 s_2$	$(x_1 - x_2)^3 (x_1 + x_2)/16$
$s_2 s_1 s_2$	$-x_2(x_1 - x_2)(x_1 + x_2)/4$
$s_1 s_2 s_1$	$-(x_1 - x_2)^2 (x_1 + x_2)/8$
$s_2 s_1$	$(x_1 + x_2)^2/4$
$s_1 s_2$	$-(x_1 - x_2)(x_1 + x_2)/4$
s_2	x_2
s_1	$(x_1 + x_2)/2$

Note that we have started the B-G-G algorithm with c_{w_0} which differs from $\alpha_1\alpha_2\alpha_3\alpha_4/8$ by a multiple of $x_1^2 + x_2^2$.

Theorem 2.6 will allow us to describe the quantization map without knowing anything about the ideal I_W^q of quantum relations. But for the sake of completeness we will also obtain the two generators of I_W^q, by using the theorem of Kim as presented in our paper [M]. We have to consider the Hamiltonian system which consists of the standard 4-dimensional symplectic manifold $(\mathbb{R}^4, dr_1 \wedge ds_1 + dr_2 \wedge ds_2)$ with the Hamiltonian function

$$E(r, s) = \sum_{i,j=1}^{2} \langle \alpha_i^\vee, \alpha_j^\vee \rangle r_i r_j + \sum_{i=1}^{2} e^{-2s_i} = (2r_1 - r_2)^2 + r_2^2 + e^{-2s_1} + e^{-2s_2}.$$

The first integrals of motion of the system are E and — by inspection — the function

$$F(r, s) = (2r_1 - r_2)^2 r_2^2 + r_2^2 e^{-2s_1} - (2r_1 - r_2) r_2 e^{-2s_2} + 2e^{-2s_1} e^{-2s_2} + \frac{1}{4}(e^{-2s_2})^2.$$

By the main result of [M], the quantum relations are obtained from E, respectively F, by the formal replacements:

$$\begin{aligned} &2r_1 - r_2 \mapsto x_1, r_2 \mapsto x_2 \\ &e^{-2s_1} \mapsto -\langle \alpha_1^\vee, \alpha_1^\vee \rangle q_1 = -4q_1, e^{-2s_2} \mapsto -\langle \alpha_2^\vee, \alpha_2^\vee \rangle q_2 = -2q_2. \end{aligned}$$

In conclusion, I_W^q is the ideal of $\mathbb{R}[q_1, q_2, x_1, x_2]$ generated by

$$x_1^2 + x_2^2 - 4q_1 - 2q_2 = 0 \text{ and } x_1^2x_2^2 - 4q_1x_2^2 + 2q_2x_1x_2 + 16q_1q_2 + q_2^2.$$

Now, we will determine explicitly the image of each Schubert class σ_w, $w \in W$ via the isomorphism

$$QH^*(G/B) \simeq \mathbb{R}[q_1, q_2, x_1, x_2]/I_W^q.$$

The place of the operators Λ_1, Λ_2 is taken by X_1, X_2 where

$$X_i = x_i + x_i(\alpha_1^\vee)q_1\Delta_{s_1} + x_i(\alpha_2^\vee)q_2\Delta_{s_2} + x_i(\alpha_4^\vee)q_1q_2\Delta_{s_1}\Delta_{s_2}\Delta_{s_1}, \qquad i = 1, 2.$$

More precisely, we have

$$X_1 = x_1 + 2q_1\Delta_{s_1} - q_2\Delta_{s_2} + q_1q_2\Delta_{s_1}\Delta_{s_2}\Delta_{s_1}$$

and

$$X_2 = x_2 + q_2\Delta_{s_2} + q_1q_2\Delta_{s_1}\Delta_{s_2}\Delta_{s_1}.$$

Rather than using the formula for ψ^{-1} given by (7), it seems more convenient to determine $\hat{c}_w := \psi^{-1}(c_w) \in \mathbb{R}[q_1, q_2, x_1, x_2]$ by the definition of ψ, i.e. from the condition

$$\hat{c}_w(X_1, X_2)(1) = c_w(x_1, x_2).$$

We will explain the details just for the case $w = w_0$, which is the most illustrative one. The polynomial we are looking for has the form $\hat{c}_{w_0} = c_{w_0} + q_1a_1 + q_2a_2 + b_1q_1^2 + b_2q_2^2 + b_3q_1q_2$, where a_1, a_2 are homogeneous polynomials of degree 2 in x_1, x_2 and b_1, b_2, b_3 are constant. The condition that determines a_1, a_2, b_1, b_2, b_3 is

$$\begin{aligned} & c_{w_0}(X_1, X_2)(1) + q_1a_1(X_1, X_2)(1) + q_2a_2(X_1, X_2)(1) + b_1q_1^2 + b_2q_2^2 + b_3q_1q_2 \\ & \qquad = c_{w_0}(x_1, x_2). \end{aligned} \tag{9}$$

The first step is to compute $c_{w_0}(X_1, X_2)(1)$ and determine a_1 and a_2. Using

$$\Delta_{s_1}(f) = \frac{f(x_1, x_2) - f(-x_1, x_2)}{x_1} \quad \text{and} \quad \Delta_{s_2}(f) = \frac{f(x_1, x_2) - f(x_2, x_1)}{x_2 - x_1},$$

$f \in \mathbb{R}[x_1, x_2]$ we obtain

$$c_{w_0}(X_1, X_2)(1) = c_{w_0}(x_1, x_2) + \frac{1}{8}q_1(3x_1^2 - 4x_1x_2 + x_2^2) + \frac{1}{4}q_2(x_1^2 - x_2^2) + q_1^2 + q_1q_2.$$

Since the coefficients of q_1, respectively q_2 in the left hand side of (9) must vanish, we deduce:

$$a_1 = -\frac{1}{8}(3x_1^2 - 4x_1x_2 + x_2^2), \quad a_2 = -\frac{1}{4}(x_1^2 - x_2^2).$$

The second step is to compute $a_1(X_1, X_2)(1)$ and $a_2(X_1, X_2)(1)$ and determine b_1, b_2 and b_3. We take into account that

$$X_1 - X_2 = x_1 - x_2 + 2q_1\Delta_{s_1} - 2q_2\Delta_{s_2}$$

and find

$$a_1(X_1, X_2)(1) = -\frac{1}{8}(X_1 - X_2)(3x_1 - x_2) = a_1(x_1, x_2) - \frac{3}{2}q_1 - q_2$$
$$a_2(X_1, X_2)(1) = -\frac{1}{4}(X_1 - X_2)(x_1 + x_2) = a_2(x_1, x_2) - q_1.$$

Coming back to (9), we deduce

$$b_1 = \frac{1}{2},\ b_2 = 0,\ b_3 = 1,$$

hence

$$\hat{c}_{w_0} = c_{w_0} - \frac{1}{8}q_1(3x_1^2 - 4x_1x_2 + x_2^2) - \frac{1}{4}q_2(x_1^2 - x_2^2) + \frac{1}{2}q_1^2 + q_1q_2.$$

The other $\hat{c}_w$, $w \in W$, can be obtained by similar computations. They are described in the following table:

w	$\hat{c}_w - c_w$
$s_2s_1s_2$	q_1x_2
$s_1s_2s_1$	$\frac{1}{2}(x_1 - x_2)q_1 + \frac{1}{2}(x_1 + x_2)q_2$
s_2s_1	$-q_1$
s_1s_2	q_1
s_2	0
s_1	0

Acknowledgements

I would like to thank Martin Guest and Takashi Otofuji for the extensive exchange of ideas which led me to Theorem 3.6. I am also grateful to McKenzie Wang and Chris Woodward for suggesting improvements to previous versions of the paper. Finally, I would like to thank the referee for several helpful suggestions.

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