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$\mathbf{L^p}$ BOUNDS FOR THE FUNCTION OF MARCINKIEWICZ

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1. Introduction

Let Ω denote a homogeneous function of degree 0 on $\mathbf{R}^n$ which is locally integrable and satisfies

$$\int_{\mathbf{S}^{n-1}} \Omega(y)d\sigma(y) = 0, \tag{1.1}$$

where $d\sigma$ represents the normalized Lebesgue measure on the unit sphere $\mathbf{S}^{n-1}$. For $n \geq 2$ and $f \in L^1_{loc}(\mathbf{R}^n)$, the Marcinkiewicz function of f is given by

$$\mu_\Omega(f)(x) = \left(\int_0^\infty \left| \int_{|y|\leq t} \frac{\Omega(y)}{|y|^{n-1}} f(x-y)dy \right|^2 \frac{dt}{t^3} \right)^{1/2}. \tag{1.2}$$

The above operator was introduced by E.M. Stein in [7] as an extension of the notion of Marcinkiewicz function from one dimension to higher dimensions. By using the L^p boundedness of the 1-dimensional Marcinkiewicz function, Stein showed that μ_Ω is bounded on $L^p(\mathbf{R}^n)$ for $1 < p < \infty$ whenever Ω is odd.

For a general kernel function Ω, the L^p boundedness of μ_Ω has been established under various conditions on Ω. For example, Stein proved that μ_Ω is bounded on $L^p(\mathbf{R}^n)$ for $1 < p \leq 2$ if $\Omega \in Lip(\mathbf{S}^{n-1})$. Benedek, Calderón and Panzone proved in [2] that the L^p boundedness of μ_Ω holds for $1 < p < \infty$ under the condition that $\Omega \in C^1(\mathbf{S}^{n-1})$.

In 1972 T. Walsh showed that the L^p boundedness of μ_Ω can still hold even if Ω is quite rough.

Theorem 1 (Walsh [11])**.** *Suppose that $p \in (1, \infty)$, $r = \min\{p, p'\}$, and $\Omega \in L(\log L)^{1/r}(\log\log L)^{2(1-2/r')}(\mathbf{S}^{n-1})$. Then μ_Ω is bounded on $L^p(\mathbf{R}^n)$.*

When $p = 2$, the condition in Theorem 1 is simply $\Omega \in L(\log L)^{1/2}(\mathbf{S}^{n-1})$, which was shown by Walsh to be optimal in the sense that the exponent $1/2$ in $L(\log L)^{1/2}$ cannot be replaced by any smaller numbers.

On the other hand, Walsh did not consider his condition to be in any sense optimal when $p \neq 2$. Indeed, by comparing with the result of Calderón and Zygmund on singular integrals, one is naturally led to the question whether the condition $\Omega \in L(\log L)^{1/2}(\mathbf{S}^{n-1})$ is also sufficient for the L^p boundedness of μ_Ω even when $p \neq 2$. This problem, which was formally proposed by Y. Ding in [4], is resolved by our next theorem.

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Theorem 2. *If $\Omega \in L(\log L)^{1/2}(\mathbf{S}^{n-1})$ and $p \in (1,\infty)$, then μ_Ω is bounded on $L^p(\mathbf{R}^n)$.*

The method employed in this paper is based in part on ideas from [1], [3], [5] and [10], among others. A great deal more can be obtained by applying variations of this scheme to more general integral operators of Marcinkiewicz type. An extensive discussion of these results will appear in a forthcoming paper.

Throughout the rest of the paper the letter C will stand for a constant but not necessarily the same one in each occurrence.

2. The Main Lemma and Proof of Theorem 2

For a suitable family of measures $\tau = \{\tau_t : \quad t \in \mathbf{R}\}$ on $\mathbf{R}^n$, we define the operators Δ_τ and τ^* by

$$\Delta_\tau(f)(x) = \left(\int_{\mathbf{R}} |\tau_t * f(x)|^2 dt\right)^{1/2} \tag{2.1}$$

and

$$\tau^*(f)(x) = \sup_{t\in\mathbf{R}}(|\tau_t| * |f|)(x). \tag{2.2}$$

The following is our main lemma:

Lemma 3. *Let $a \geq 2$, $A > 0$, $\gamma > 0$, $q > 1$ and $C_q > 0$. Suppose that the family of measures $\{\tau_t : \quad t \in \mathbf{R}\}$ satisfies the following:*

(i) $\|\tau_t\| \leq A$ *for* $t \in \mathbf{R}$;

(ii) $|\hat{\tau}_t(\xi)| \leq A[\min\{a^t|\xi|, (a^t|\xi|)^{-1}\}]^{\gamma/\ln a}$ *for* $\xi \in \mathbf{R}^n$ *and* $t \in \mathbf{R}$;

(iii) $\|\tau^*(f)\|_q \leq C_q A\|f\|_q$ *for* $f \in L^q(\mathbf{R}^n)$.

Then, for every p satisfying $|1/p - 1/2| < 1/(2q)$, there exists a positive constant C_p which is independent of a and A such that

$$\|\Delta_\tau(f)\|_p \leq C_p A\|f\|_p \tag{2.3}$$

for $f \in L^p(\mathbf{R}^n)$.

This lemma can be viewed as a continuous analogue of Theorem B in [5]. The novel feature, which keys its application to the current problem, is the uniformness of the bound on the operator norm with respect to the parameter a.

Proof of Theorem 2. Let $\Omega \in L(\log L)^{1/2}(\mathbf{S}^{n-1})$ and satisfy (1.1). For $k \in \mathbf{N}$ let $E_k = \{y \in \mathbf{S}^{n-1} : \quad 2^{k-1} \leq |\Omega(y)| < 2^k\}$ and

$$\Omega_k(y) = \Omega(y)\chi_{E_k}(y) - \int_{E_k} \Omega d\sigma.$$

Thus

$$\int_{\mathbf{S}^{n-1}} \Omega_k(y) d\sigma(y) = 0 \tag{2.4}$$

for $k \in \mathbf{N}$. Let $\Lambda = \{k \in \mathbf{N} : \quad \sigma(E_k) > 2^{-4k}\}$ and

$$\Omega_0 = \Omega - \sum_{k \in \Lambda} \Omega_k.$$

It then follows that $\Omega_0 \in L^2(\mathbf{S}^{n-1})$ and

$$\int_{\mathbf{S}^{n-1}} \Omega_0(y) d\sigma(y) = 0.$$

For every $k \in \Lambda$ we define the family of measures $\tau^{(k)} = \{\tau_{k,t} : \quad t \in \mathbf{R}\}$ on $\mathbf{R}^n$ by

$$\int_{\mathbf{R}^n} f d\tau_{k,t} = 2^{-kt} \int_{|y| \le 2^{kt}} \frac{\Omega_k(y)}{|y|^{n-1}} f(y) dy.$$

If we set $a_k = 2^k, A_k = 2 \int_{E_k} |\Omega(y)| d\sigma(y)$ and $\gamma = \frac{\ln 2}{6}$, then the following holds for $t \in \mathbf{R}$, $\xi \in \mathbf{R}^n$, and $p > 1$:

$$\text{(2.5)} \qquad \begin{cases} \text{(i)} & \|\tau_{k,t}\| \le A_k, \\ \text{(ii)} & |\hat{\tau}_{k,t}(\xi)| \le A_k (a_k^t |\xi|)^{\gamma / \ln a_k}, \\ \text{(iii)} & |\hat{\tau}_{k,t}(\xi)| \le C A_k (a_k^t |\xi|)^{-\gamma / \ln a_k}, \\ \text{(iv)} & \|(\tau^{(k)})^*\|_{p,p} \le C_p A_k, \end{cases}$$

where C and C_p are independent of k.

While (2.5.i) is obvious, (2.5.ii) follows immediately from (2.4) and (2.5.i). In addition, (2.5.iv) can be obtained in a straightforward manner (see, for example, Page 823 in [6]).

On the other hand, by the proof of Corollary 4.1 on P. 551 of [5],

$$\text{(2.6)} \qquad |\hat{\tau}_{k,t}(\xi)| \le C \|\Omega_k\|_2 (a_k^t |\xi|)^{-1/6}.$$

Thus, by (2.5.i), (2.6) and the inequality $\|\Omega_k\|_2 \le 2^{2k+2} A_k$,

$$|\hat{\tau}_{k,t}(\xi)| \le A_k^{(k-1)/k} [C 2^{2k+2} A_k (a_k^t |\xi|)^{-1/6}]^{1/k}$$

$$\le C A_k (a_k^t |\xi|)^{-\gamma / \ln a_k},$$

which proves (2.5.iii).

By Minkowski's inequality,

$$\text{(2.7)} \qquad \mu_\Omega(f) \le \mu_{\Omega_0}(f) + \sum_{k \in \Lambda} (k \ln 2)^{1/2} \Delta_{\tau^{(k)}}(f).$$

Finally, by (2.5), (2.7), Theorem 1 and Lemma 3, we obtain

$$\|\mu_\Omega(f)\|_p \le C_p \left(1 + \sum_{k \in \Lambda} \sqrt{k} A_k \right) \|f\|_p$$

$$\le C_p (1 + \|\Omega\|_{L(\log L)^{1/2}}) \|f\|_p$$

for $1 < p < \infty$. The proof of Theorem 2 is now complete. □

References

[1] A. Al-Salman, Y. Pan, *Singular integrals with rough kernels in* $L\log L(\mathbf{S}^{n-1})$, J. London Math. Soc. **66** (2002), 153–174.

[2] A. Benedek, A.-P. Calderón, R. Panzone, *Convolution operators on Banach space valued functions,* Proc. Nat. Acad. Sci. U.S.A. **48** (1962), 356–365.

[3] J. Chen, D. Fan, Y. Pan, *A note on a Marcinkiewicz integral operator,* Math. Nachr. **227** (2001), 33–42.

[4] Y. Ding, *On Marcinkiewicz integral,* Proceedings of the Conference "Singular Integrals and Related Topics, III," 28–38, Osaka, Japan, 2001.

[5] J. Duoandikoetxea, J. L. Rubio de Francia, *Maximal and singular integral operators via Fourier transform estimates,* Invent. Math. **84** (1986), 541–561.

[6] D. Fan, Y. Pan, *Singular integral operators with rough kernels supported by subvarieties,* Amer. J. Math. **119** (1997), 799–839.

[7] E. M. Stein, *On the functions of Littlewood-Paley, Lusin, and Marcinkiewicz,* Trans. Amer. Math. Soc. **88** (1958), 430–466.

[8] ——— *Harmonic Analysis: real-variable methods, orthogonality, and oscillatory integrals,* Princeton Mathematical Series, 43. Monographs in Harmonic Analysis, III. Princeton University Press, Princeton, NJ, 1993.

[9] ——— *Singular integrals and differentiability properties of functions,* Princeton Mathematical Series, 30. Princeton University Press, Princeton, NJ, 1970.

[10] E. M. Stein, S. Wainger, *Problems in harmonic analysis related to curvature,* Bull. Amer. Math. Soc. **84** (1978), 1239–1295.

[11] T. Walsh, *On the function of Marcinkiewicz,* Studia Math. **44** (1972), 203–217.

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