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EVERY MORPHISM IS THE RESTRICTION OF A TORIC ONE

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Abstract. We show that every morphism of divisorial prevarieties is the restriction of a toric morphism. This extends an embedding theorem of Włodarczyk to the setting of morphisms.

Introduction

A prevariety X over an algebraically closed field $\mathbb{K}$ is called *divisorial* if every $x \in X$ has an affine open neighbourhood $X \setminus Z(f)$ obtained by removing the zero set $Z(f)$ of a global section f of some line bundle on X, compare [2]. Nonseparated divisorial prevarieties occur for example in quotient constructions, where they behave more natural than their separated counterparts, see [4].

For divisorial prevarieties, Włodarczyk's Embedding Theorem [6, Theorem C] can be improved as follows: Every divisorial prevariety admits a closed embedding into a smooth toric prevariety with affine diagonal morphism, see [5, Theorem 3.2 and Remark 3.3]. In the present note, we extend this statement to morphisms.

The "ambient morphisms" will be toric morphisms $\psi\colon Z \to Z'$, i.e., morphisms of toric prevarieties Z and Z' that restrict to a homomorphism $T \to T'$ of the respective big tori $T \subset Z$ and $T' \subset Z'$ and satisfy $\psi(t\cdot z) = \psi(t)\cdot\psi(z)$. The description of toric morphisms by combinatorial data is a powerful tool for explicit studies, see [3] and [1].

For the sake of a rounded picture, we formulate our result in terms of a refined concept of divisoriality: Given $k > 0$, we say that a not necessarily irreducible prevariety X is *k-divisorial* if for any collection $x_1, \ldots, x_k \in X$ there is a line bundle L on X and a global section $f\colon X \to L$ such that $X \setminus Z(f)$ is an affine neighbourhood of the points $x_1, \ldots, x_k$, compare also [5]. We prove:

Theorem. *Let $\varphi\colon X \to Y$ be a morphism of k-divisorial prevarieties. Then there exist k-divisorial smooth toric prevarieties Z_X, Z_Y and a commutative*

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diagram

$$\begin{array}{ccc} X & \xrightarrow{\ \varphi\ } & Y \\ \big\downarrow & & \big\downarrow \\ Z_X & \xrightarrow{\ \psi\ } & Z_Y \end{array}$$

where the downwards arrows are closed embeddings and $\psi\colon Z_X \to Z_Y$ *is a toric morphism.*

Note that the theorem applies in particular to all $\mathbb{Q}$-factorial prevarieties with affine diagonal morphism. For $k > 1$, all prevarieties occuring in the statement are in fact separated, and in this case the theorem can also be derived from the results of [5, Section 5] using the graph of the morphism $\varphi\colon X \to Y$.

Proof of the Theorem

Throughout the proof of the main result we shall make use of the methods developed in [5, Section 2]. The first step is to reduce the study of arbitrary morphisms to the study of morphisms of affine varieties. As this might be of independent interest, we give a separate statement:

Lemma. *Let* X, Y *be* k*-divisorial prevarieties, and let* $\varphi\colon X \to Y$ *be any morphism. Then there exist:*

(i) *algebraic tori* H_X, H_Y, *an affine* H_X*-variety* $\overline{X}$ *and an affine* H_Y*-variety* $\overline{Y}$,

(ii) *open dense invariant subsets* $\widehat{X} \subset \overline{X}$ *and* $\widehat{Y} \subset \overline{Y}$ *where the respective tori act freely with geometric quotients*

$$q_X\colon \widehat{X} \to X, \qquad q_Y\colon \widehat{Y} \to Y,$$

(iii) *a homomorphism* $\widetilde{\varphi}\colon H_X \to H_Y$ *of algebraic tori and a commutative diagram of morphisms*

$$\begin{array}{ccc} \overline{X} & \xrightarrow{\ \ \overline{\varphi}\ \ } & \overline{Y} \\ {\scriptstyle \jmath_X}\big\uparrow & & \big\uparrow{\scriptstyle \jmath_Y} \\ \widehat{X} & \xrightarrow{\ \ \varphi\ \ } & \widehat{Y} \\ {\scriptstyle q_X}\big\downarrow{\scriptstyle /H_X} & & {\scriptstyle /H_Y}\big\downarrow{\scriptstyle q_Y} \\ X & \xrightarrow{\ \ \varphi\ \ } & Y \end{array}$$

where $\overline{\varphi}$ *is equivariant in the sense that* $\overline{\varphi}(t\cdot x) = \widetilde{\varphi}(t)\cdot\overline{\varphi}(x)$ *and the maps* $\jmath_X$, $\jmath_Y$ *denote the open inclusions,*

(iv) *regular* H_Y*-homogeneous functions* $g_1, \ldots, g_s \in \mathcal{O}(\overline{Y})$ *such that*
   (a) $\overline{Y}_{g_j} = q_Y^{-1}(Y \setminus Z(g_j))$ *holds for every* $j = 1, \ldots, s$,
   (b) *any collection* $y_1, \ldots, y_k \in \widehat{Y}$ *is contained in some* $\overline{Y}_{g_j}$,

(v) *regular* H_X*-homogeneous functions* $f_1, \ldots, f_r \in \mathcal{O}(\overline{X})$ *such that*

(a) $\overline{X}_{f_i} = q_X^{-1}(X \setminus Z(f_i))$ *holds for every* $i = 1, \dots, r$,
(b) *any collection* $x_1, \dots, x_k \in \widehat{X}$ *is contained in some* $\overline{X}_{f_i}$.

Proof. Choose line bundles $E_1, \dots, E_d$ on Y and sections $g_j \colon Y \to E_j$ such that the sets $Y_j := Y \setminus Z(g_j)$ are affine and any collection $y_1, \dots, y_k$ is contained in some Y_j. In doing this, we may assume that the line bundles generate a free abelian group Γ in the sense of [5, Section 2]. For each $E \in \Gamma$, let $\mathcal{B}_E$ denote its sheaf of sections. Consider the associated Γ-graded $\mathcal{O}_Y$-algebra

$$\mathcal{B} := \bigoplus_{E \in \Gamma} \mathcal{B}_E.$$

This algebra gives rise to a prevariety $\widehat{Y} := \mathrm{Spec}(\mathcal{B})$ and a canonical morphism $q_Y \colon \widehat{Y} \to Y$. The morphism q_Y is a geometric quotient for the action of the algebraic torus $H_Y := \mathrm{Spec}(\mathbb{K}[\Gamma])$ on $\widehat{Y}$ defined by the grading of $\mathcal{B}$. Note that $\widehat{Y}$ is quasiaffine; in fact, by [5, Lemma 2.4], there is an H_Y-equivariant affine closure $\overline{Y}$ of $\widehat{Y}$ such that the g_j extend regularly to $\overline{Y}$ and satisfy $\overline{Y}_{g_j} = q_Y^{-1}(Y_j)$.

Now, let Λ be a finitely generated free group of line bundles on X that contains bundles $L_1, \dots, L_l$ and sections $f_i \colon X \to L_i$ such that any collection $x_1, \dots, x_k \in X$ has a common affine neighbourhood $X_i := X \setminus Z(f_i)$. Enlarging Λ and replacing Γ and Λ with suitable subgroups, we can assume that there is a canonical pullback homomorphism

$$\varphi^* \colon \Gamma \to \Lambda, \qquad E \mapsto \varphi^*(E).$$

Let $\mathcal{A}$ be the graded $\mathcal{O}_X$-algebra associated to the group Λ. Consider the corresponding prevariety $\widehat{X} := \mathrm{Spec}(\mathcal{A})$ over X, and denote the canonical map by $q_X \colon \widehat{X} \to X$. Again $\widehat{X}$ is quasiaffine and q_X is a geometric quotient for the action of $H_X := \mathrm{Spec}(\mathbb{K}[\Gamma])$ on X. Note that the pullback homomorphism $\varphi^* \colon \Gamma \to \Lambda$ induces a homomorphism $\widetilde{\varphi} \colon H_X \to H_Y$ of algebraic tori.

Use again [5, Lemma 2.4] to choose an H_X-equivariant affine closure $\overline{X}$ of $\widehat{X}$ such that for some finite system $h_n \in \mathcal{B}(Y)$ of homogeneous generators of the algebra $\mathcal{O}(\overline{Y})$ the pullback sections $\varphi^*(h_n)$ extend regularly to $\overline{X}$. Additionally, ensure that the functions $f_i \in \mathcal{O}(\widehat{X})$ extend regularly to $\overline{X}$ and satisfy $\overline{X}_{f_i} = q_X^{-1}(X_i)$.

So far, we have defined all the data occuring in the items (i), (ii), (iv) and (v). The only missing thing is to establish the commutative diagram of (iii). For this define a φ^*-graded homomorphism $\overline{\varphi}^* \colon \mathcal{O}(\overline{Y}) \to \mathcal{O}(\overline{X})$ by setting on homogeneous elements

$$\mathcal{B}_E(Y) \ni g \mapsto \varphi^*(g) \in \mathcal{A}_{\varphi^*(E)}(X).$$

Then this homomorphism induces a morphism $\overline{\varphi} \colon \overline{X} \to \overline{Y}$ which is $\widetilde{\varphi}$-equivariant in the sense of item (iii). To obtain a restricted morphism $\widehat{\varphi} \colon \widehat{X} \to \widehat{Y}$, we have to verify that $\overline{\varphi}$ maps $\widehat{X}$ indeed to $\widehat{Y}$. This is true because we have

$$\begin{aligned} \widehat{X} &= \bigcup_{j=1}^{s} q_X^{-1}(\varphi^{-1}(Y \setminus Z(g_j))) \\ &= \bigcup_{j=1}^{s} q_X^{-1}(X \setminus Z(\varphi^*(g_j))) \\ &\subset \bigcup_{j=1}^{s} \overline{X}_{\varphi^*(g_j)}. \end{aligned}$$

Since in degree zero, the pullback $\widehat{\varphi}^*$ is nothing but the usual pullback of functions on Y via $\varphi\colon X \to Y$, also the lower part of the diagram in (iii) is commutative. □

Proof of the Theorem. Choose a lifting of the morphism $\varphi\colon X \to Y$ and data as in the Lemma. Complete the collections $f_1, \dots, f_r \in \mathcal{O}(\overline{X})$ and $g_1, \dots, g_s \in \mathcal{O}(\overline{Y})$ by homogeneous functions to obtain closed embeddings

$$\begin{aligned} \imath_X \colon \overline{X} \to \mathbb{K}^{n_X}, &\qquad x \mapsto (f_1(x), \dots, f_r(x), f_{r+1}(x), \dots, f_{n_X}(x)) \\ \imath_Y \colon \overline{Y} \to \mathbb{K}^{n_Y}, &\qquad y \mapsto (g_1(y), \dots, g_s(y), g_{s+1}(y), \dots, g_{n_Y}(y)) \end{aligned}$$

Endow the spaces $\mathbb{K}^{n_X}$ and $\mathbb{K}^{n_Y}$ with the respective diagonal actions of H_X and H_Y such that these embeddings become equivariant. Let $\Phi\colon \mathbb{K}^{n_X} \to \mathbb{K}^{n_Y}$ be a polynomial map that extends $\overline{\varphi}\colon \overline{X} \to \overline{Y}$ and fulfils

$$\Phi(t \cdot z) = \widetilde{\varphi}(t) \cdot \Phi(z) \quad \text{for all } (t, z) \in H_X \times \mathbb{K}^{n_X}.$$

Now, let $n_{X,Y} := n_X + n_Y$, and consider the graph $\Psi\colon \mathbb{K}^{n_X} \to \mathbb{K}^{n_{X,Y}}$ of the map $\Phi\colon \mathbb{K}^{n_X} \to \mathbb{K}^{n_Y}$. Then $\Psi \circ \imath_X$ embeds $\overline{X}$ equivariantly into $\mathbb{K}^{n_{X,Y}}$, where H_X acts on the latter space via

$$t \cdot (x, y) := (t \cdot x, \widetilde{\varphi}(t) \cdot y).$$

In the sequel, we regard $\overline{X}$ as a subvariety of $\mathbb{K}^{n_{X,Y}}$ and $\overline{Y}$ as a subvariety of $\mathbb{K}^{n_Y}$. Thus, denoting by $\mathrm{pr}_Y\colon \mathbb{K}^{n_{X,Y}} \to \mathbb{K}^{n_Y}$ the projection onto the second factor, we are in the following situation:

$$(*) \qquad \begin{array}{ccc} \mathbb{K}^{n_{X,Y}} & \xrightarrow{\mathrm{pr}_Y} & \mathbb{K}^{n_Y} \\ \uparrow & & \uparrow \\ \overline{X} & \xrightarrow{\overline{\varphi}} & \overline{Y} \end{array}$$

Let $W_{\hat{X}} \subset \mathbb{K}^{n_{X,Y}}$ be the minimal open toric subset containing $\widehat{X}$. Analogously let $W_{\hat{Y}} \subset \mathbb{K}^{n_Y}$ be the minimal open toric subset with $\widehat{Y} \subset W_{\hat{Y}}$. Then we claim:

$$\widehat{X} = \overline{X} \cap W_{\hat{X}}, \qquad \widehat{Y} = \overline{Y} \cap W_{\hat{Y}}, \qquad \mathrm{pr}_Y(W_{\hat{X}}) \subset W_{\hat{Y}}.$$

For the first equation note that $\widehat{X}$ is obtained by removing the common zeroes of the first r coordinates of $\mathbb{K}^{n_{X,Y}}$ from $\overline{X}$. Thus $\widehat{X}$ is the intersection of $\overline{X}$ with an open toric subset of $\mathbb{K}^{n_{X,Y}}$. Minimality of W_X gives the desired statement. The same argument works for the second equation.

To verify the third observation, let $B_1, \ldots, B_m$ denote the orbits of the big torus $\mathbb{T}^{n_{X,Y}}$ of $\mathbb{K}^{n_{X,Y}}$ that intersect $\widehat{X}$. Then W_X is the union of all $\mathbb{T}^{n_{X,Y}}$-orbits that have one of the B_l in its closure. Thus, given $z = (x, y) \in W_X$ we have

$$\lim_{t \to 0} \lambda(t) \cdot t_0 \cdot z = z_0 \in \widehat{X}$$

with some $t_0 \in \mathbb{T}^{n_{X,Y}}$ and some one parameter subgroup $\lambda \colon \mathbb{K}^* \to \mathbb{T}^{n_{X,Y}}$. Now, applying the projection pr_Y to the above equation yields that $y = \mathrm{pr}_Y(z)$ lies in W_Y, and our claim is verified.

Since H_X acts freely on W_X and so does H_Y on W_Y, there exist geometric quotients $W_X \to Z_X$ and $W_Y \to Z_Y$ for these actions. Thereby the quotients spaces Z_X and Z_Y are smooth toric prevarieties having an affine diagonal morphism, see e.g. [5, Lemma 1.6]. Moreover, we have the following commutative diagram of toric morphisms:

$$\begin{array}{ccc} W_X & \xrightarrow{\mathrm{pr}_Y} & W_Y \\ {\scriptstyle /H_X}\downarrow & & \downarrow{\scriptstyle /H_Y} \\ Z_X & \xrightarrow{\psi} & Z_Y \end{array}$$

By construction, $X = \widehat{X}/H_X$ is embedded into Z_X and so is $Y = \widehat{Y}/H_Y$ into Z_Y. By the commutative diagrams $(*)$ and statement (iii) of the Lemma, the toric morphism $\psi \colon Z_X \to Z_Y$ is an extension of $\varphi \colon X \to Y$.

It remains to show that the prevarieties Z_X and Z_Y are in fact k-divisorial. We consider exemplarily Z_Y. Since Z_Y is smooth and has an affine diagonal morphism, it suffices to show that any k points of Z_Y admit a common affine neighbourhood in Z_Y. We do this by verifying the assumptions of [5, Corollary 4.4].

So, let $B_1, \ldots, B_k \subset Z_Y$ be closed orbits of the big torus of Z_Y. By minimality of W_Y, we find for each l a point $y_l \in Y \cap B_l$. We construct a common affine neighbourhood of the points $y_1, \ldots, y_k$ using the method presented in [5, Proof of Theorem 5.3]:

For each l, choose a point $\widehat{y}_l \in q_Y^{-1}(y_l)$. By construction, there is a coordinate z_j such that $\mathbb{K}^{n_Y}_{z_j}$ is a common neighbourhood of $\widehat{y}_1, \ldots, \widehat{y}_k$. Consider the closed H_Y-invariant subsets

$$A := \mathbb{K}^{n_Y}_{z_j} \setminus W_Y, \qquad \widehat{Y}_j := \overline{Y} \cap \mathbb{K}^{n_Y}_{z_j}$$

of $\mathbb{K}^{n_Y}_{z_j}$. As they are disjoint, the categorical quotient $p \colon \mathbb{K}^{n_Y} \to \mathbb{K}^{n_Y} /\!\!/ H_Y$ separates these sets. In particular, no point $p(\widehat{y}_l)$ lies in $p(A)$. Thus we find an H_Y-invariant regular function h on $\mathbb{K}^{n_Y}_{z_j}$ that vanishes along A but on none of the points $\widehat{y}_l$.

Removing the zero set of the function h from $\mathbb{K}^{n_Y}_{z_j}$ yields a common H_Y-invariant affine open neighbourhood $U \subset W_Y$ of $\widehat{y}_1, \ldots, \widehat{y}_k$. Now, the image $V \subset Z_Y$ of U under the quotient map $W_Y \to Z_Y$ is the desired affine neighbourhood of the points $y_1, \ldots, y_k$. □

References

[1] A. A'Campo-Neuen, J. Hausen, *Toric prevarieties and subtorus actions,* Geom. Dedicata **87** (2001), 35–64.

[2] M. Borelli, *Divisorial varieties,* Pacific J. Math. **13** (1963), 375–388.

[3] W. Fulton, *Introduction to toric varieties,* Annals of Mathematics Studies, 131. The William H. Roever Lectures in Geometry. Princeton University Press, Princeton, NJ, 1993.

[4] J. Hausen, *A generalization of Mumford's geometric invariant theory,* Doc. Math. **6** (2001), 571–592 (electronic).

[5] ———, *Equivariant embeddings into smooth toric varieties,* Canad. J. Math. **54** (2002), 554–570.

[6] J. Włodarczyk, *Embeddings in toric varieties and prevarieties,* J. Algebraic Geom. **2** (1993), 705–726.

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