

ON THE CHERN NUMBERS OF GENERALISED KUMMER VARIETIES

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ABSTRACT. Let $A^{[n]}$ denote the $2(n-1)$ -dimensional generalised Kummer variety constructed from the abelian surface A . Further, let X be an arbitrary smooth projective surface with $\int_X c_1(X)^2 \neq 0$, and $X^{[k]}$ the Hilbert scheme of zero-dimensional subschemes of X of length k . We give a formula which expresses the value of any complex genus on $A^{[n]}$ in terms of Chern numbers of the varieties $X^{[k]}$.

In [4] and [5] it is shown how to use Bott's residue formula to effectively calculate the Chern numbers of the Hilbert schemes $(\mathbb{P}^2)^{[k]}$ of points on the projective plane. Since $\int_{\mathbb{P}^2} c_1(\mathbb{P}^2)^2 = 9 \neq 0$, we can use these numbers and our formula to calculate the Chern numbers of the generalised Kummer varieties.

A table with all Chern numbers of the generalised Kummer varieties $A^{[n]}$ for $n \leq 8$ is included.

1. Introduction

The two main series of irreducible holomorphic symplectic complex manifolds are Hilbert schemes of points on a K3 surface and generalised Kummer varieties invented by Beauville [1]. One can ask for their complex cobordism class tensored with \mathbb{Q} , which is given by the values of all their Chern numbers.

In [6] Ellingsrud, Göttsche and Lehn proved that the complex cobordism class of a Hilbert scheme $X^{[n]}$ of zero-dimensional subschemes of length n on a smooth projective surface X over the complex numbers depends only on the cobordism class of the surface X , i.e. on $c_1(X)^2$ and $c_2(X)$ (here and later on, top intersections on surfaces are to be understood as intersection numbers). They showed how this result can be used to calculate the Chern numbers of any such Hilbert scheme $X^{[n]}$ if one knows the Chern numbers of the varieties $(\mathbb{P}^2)^{[k]}$ and $(\mathbb{P}^1 \times \mathbb{P}^1)^{[k]}$, which in turn can be calculated by means of Bott's residue formula.

Therefore, the Chern numbers of the Hilbert schemes of points on a K3 surface can be efficiently calculated though no explicit formula is known. These numbers can, for example, be used to check the conjecture of [3] about the elliptic genus of the Hilbert schemes $X^{[k]}$ where X is a K3 surface (see [6]).¹ In this note,

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¹Note added in proof: Borisov and Libgober verified the conjecture of [12] a short time ago, see preprint [arXiv:math.AG/0206241](https://arxiv.org/abs/math/0206241)

we want to give a method for computing the Chern numbers of the $(n - 1)$ -dimensional generalised Kummer variety $A^{[[n]]}$ for an abelian surface A and for general n as this has not appeared in the literature so far.

The χ_y -genus of $A^{[[n]]}$ has been calculated by Göttsche and Soergel [8]. Expressing this genus in terms of Chern numbers by using the Hirzebruch-Riemann-Roch formula gives us enough information to deduce the Chern numbers of $A^{[[n]]}$ for $n \leq 4$.

Using the theory of Rozansky-Witten invariants, Sawon [14] produced a further relation that allowed him to compute all the Chern numbers for $n \leq 5$. The Chern numbers of $A^{[[6]]}$ were calculated by M. Britze and the author in [2]. J. Sawon informed us that he also had computed these numbers. However, all these methods are not sufficient to compute the Chern numbers for $n > 6$.

We will give a closed formula describing the value of any complex genus on a generalised Kummer variety in terms of genera of the Hilbert schemes of points on a fixed surface X with $c_1(X)^2 \neq 0$. Since these are computable via Bott's residue formula for X being the projective plane \mathbb{P}^2 , we can compute the Chern numbers of $A^{[[n]]}$ for any n . We have done this for $n \leq 11$ (see appendix A, where the values can be found for $n \leq 8$).

2. The generalised Kummer varieties

Let X be a smooth projective surface over the field of complex numbers. For every nonnegative integer we denote by $X^{[n]}$ the Hilbert scheme of zero-dimensional subschemes of X of length n . By a result of Fogarty [7], this scheme is smooth and projective of dimension $2n$. It can be viewed as a resolution $\rho : X^{[n]} \rightarrow X^{(n)}$ of the n -fold symmetric product $X^{(n)} := X^n / \mathfrak{S}_n$ of X . The morphism ρ , sending closed points, i.e. subschemes of X , to their support counting multiplicities, is called the Hilbert-Chow morphism.

Let us briefly recall the construction of the generalised Kummer varieties introduced by Beauville [1]. Let A be an abelian surface and $n > 0$. There is an obvious summation morphism $A^{(n)} \rightarrow A$. We denote its composition with the Hilbert-Chow morphism $\rho : A^{[n]} \rightarrow A^{(n)}$ by $\sigma : A^{[n]} \rightarrow A$.

Definition 1. The n^{th} generalised Kummer variety² $A^{[[n]]}$ is the fibre of σ over $0 \in A$.

Beauville showed among other things the following property of the varieties $A^{[[n]]}$:

Proposition 1. *The n^{th} generalised Kummer variety is a smooth projective holomorphic symplectic variety of dimension $2(n - 1)$.*

Proof. [1]. □

²There is no consistent notation in the literature for what we denote by $A^{[[n]]}$. There are some authors who prefer the symbol $K^{n-1}A$, and others use $A^{[[n-1]]}$, indicating that this variety is $2(n - 1)$ -dimensional. However, we believe that for numerical considerations the convention we shall use in this paper is the most convenient.

Let A be an abelian surface again. Since A acts on itself by translation, there is also an induced operation of A on the Hilbert schemes $A^{[n]}$. Let us denote the restriction of this operation to the generalised Kummer variety $A^{[[n]]}$ by $\nu : A \times A^{[[n]]} \rightarrow A^{[n]}$. The following diagram is cartesian:

$$(1) \quad \begin{array}{ccc} A \times A^{[[n]]} & \xrightarrow{\nu} & A^{[n]} \\ \pi_A \downarrow & & \downarrow \sigma \\ A & \xrightarrow[n]{} & A. \end{array}$$

Here, $n : A \rightarrow A, a \mapsto na$ is the (multiplication by n)-morphism. It is a Galois covering of degree n^4 . Therefore, also ν is a Galois covering of degree n^4 .

Next, we want to introduce certain line bundles on the Hilbert schemes and generalised Kummer varieties that are constructed from line bundles on the underlying surface:

Each line bundle L on a smooth projective surface X gives us a line bundle L_n on $X^{[n]}$ in the following way: $L^{\boxtimes n}$ is a \mathfrak{S}_n -invariant line bundle on the n^{th} -product X^n of X . Therefore, we can define the sheaf $L^{(n)} := (\pi_*(L^{\boxtimes n}))^{\mathfrak{S}_n}$ of \mathfrak{S}_n -invariant sections of $\pi_*(L^{\boxtimes n})$ on $X^{(n)}$ where $\pi : X^n \rightarrow X^{(n)}$ is the canonical projection. The pull-back $L_n := \rho^*L^{(n)}$ by the Hilbert-Chow morphism is a line bundle on $X^{[n]}$. Note that $\text{Pic}(X) \rightarrow \text{Pic}(X^{[n]}), L \mapsto L_n$ is a homomorphism of groups.

This construction has already appeared for example in [6] and [2]. If X is an abelian surface, we denote by $L^{[[n]]}$ the restriction of L_n to the generalised Kummer variety $X^{[[n]]} \subseteq X^{[n]}$. By using the seesaw principle (cf. [13]) and $H^1(X^{[[n]]}) = 0$, it can be shown that

$$(2) \quad \nu^*L_n = L^n \boxtimes L^{[[n]]}$$

(cf. [2]).

Besides L_n , one can define certain further sheaves on $X^{[n]}$ that are constructed from sheaves on X . They are called tautological sheaves of $X^{[n]}$ (cf. [11]). The construction is as follows: Since $X^{[n]}$ represents a functor, there is a universal family $\Xi_n \subseteq X^{[n]} \times X$ of zero-dimensional subschemes over $X^{[n]}$. Let us denote by \mathcal{O}_n its structure sheaf. For any locally free sheaf F of rank r on X , we define the sheaf $F^{[n]} := p_*(\mathcal{O}_n \otimes q^*F)$ on $X^{[n]}$, where $p : X^{[n]} \times X \rightarrow X^{[n]}$ and $q : X^{[n]} \times X \rightarrow X$ are the canonical projections. $F^{[n]}$ is again locally free of rank nr .

The connection between L_n and $L^{[n]}$ is as follows: As noted in section 5 of [6], we have

$$(3) \quad \det(F^{[n]}) = \det(F)_n \otimes (\det \mathcal{O}_X^{[n]})^{\text{rk } F}$$

for any locally free sheaf F of constant rank on X . Applying this to $F = L$ leads to

$$(4) \quad L_n = \det(L^{[n]}) \otimes \det(\mathcal{O}_X^{[n]})^{-1}.$$

3. Complex genera in general

Let $\Omega := \Omega^U \otimes \mathbb{Q}$ denote the (rational) complex cobordism ring. By a result of Milnor ([12]), it is generated by the cobordism classes $[X]$ of all complex manifolds X , and two complex manifolds X and Y lie in the same cobordism class if and only if they have the same Chern numbers, i.e. the Chern numbers determine the cobordism class and vice versa. Recall that the sum in the ring is induced by the disjoint union of manifolds, and the product by the cartesian product of manifolds.

A *complex genus* ϕ is a ring homomorphism $\phi : \Omega \rightarrow R$ into any \mathbb{Q} -algebra R . By Hirzebruch's theory of genera and multiplicative sequences ([9]), the R -valued complex genera are in one-to-one correspondence with the formal power series $f_\phi \in R[[x]]$ over R with constant coefficient 1. The correspondence is given as follows:

$$(5) \quad \phi(X) = \int_X \prod_{i=1}^n f_\phi(\gamma_i)$$

for all complex manifolds X , where n is the dimension of X , and $\gamma_1, \dots, \gamma_n$ are the Chern roots of its tangent bundle.

Since the cobordism class is known if one knows the value of all genera (it suffices to know the value of *the universal genus* $\text{id}_\Omega : \Omega \rightarrow \Omega$), the knowledge of the values of all genera implies the knowledge of all Chern numbers.

Now, let us slightly generalise the notion of a genus.

Definition 2. Let ϕ be a complex genus. For a complex manifold X together with a line bundle L on X we define

$$(6) \quad \phi(X, L) := \int_X e^{c_1(L)} \prod_{i=1}^n f_\phi(\gamma_i)$$

as *the genus ϕ of the pair (X, L)* .

Remark 1. Obviously, $\phi(X, \mathcal{O}_X) = \phi(X)$.

Example 1. If $\text{td}(X)$ denotes the Todd genus of X , and $\chi(X, L)$ the holomorphic Euler characteristic of the line bundle L on X , we have by the Hirzebruch-Riemann-Roch theorem that

$$(7) \quad \text{td}(X, L) = \chi(X, L).$$

The genera of pairs (X, L) have the following properties, which follow directly from the appropriate properties of Chern classes/roots.

Proposition 2. *Let $\phi : \Omega \rightarrow R$ be any complex genus with values in R . We have*

1. $\phi(X \times Y, L \boxtimes M) = \phi(X, L) \cdot \phi(Y, M)$ for two complex manifolds X and Y together with a line bundle L resp. M .
2. $\phi(X, \nu^*L) = \deg(\nu) \cdot \phi(Y, L)$ for any Galois covering $\nu : X \rightarrow Y$ and any line bundle L on Y .

Given a genus, we can deform it in the following sense to get a new genus:

Definition 3. Let ϕ be a complex genus with values in the ring R . By ϕ_t we denote the genus with values in $R[t]$ given by

$$(8) \quad \phi_t(X) := \int_X \prod_{i=1}^n (f_\phi(\gamma_i) e^{t\gamma_i})$$

for any complex manifold X .

Remark 2. For any integer n , we have $\phi_n(X) = \phi(X, K_X^{-n})$, where K_X is the canonical line bundle on X .

Example 2. Let χ_y be Hirzebruch’s χ_y -genus, and χ_{yz} the twisted χ_y -genus (see [15]). In our notation, $\chi_{yz} = (\chi_y)_z$.

4. Complex genera of Hilbert schemes of points on surfaces

In this section, we want to cite some of the results of [6] and give some corollaries which will be used later on.

Let X be a smooth projective surface. Following [6], we define

$$(9) \quad H_X := \sum_{n=0}^{\infty} [X^{[n]}] z^n$$

as an (invertible) element in the formal power series ring $\Omega[[z]]$. Analogously we define

$$(10) \quad K := \sum_{n=1}^{\infty} [A^{[n]}] z^n$$

in $\Omega[[z]]$ where A is any abelian surface. The cobordism class does not depend on the choice of A since the generalised Kummer varieties deform with A . We can reformulate our task to determine the Chern numbers of the generalised Kummer varieties by asking: What is the value $\phi(K) \in R[[z]]$ for any complex genus $\phi : \Omega \rightarrow R$?

The following lemma is a generalization of Theorem 4.2 in [6] for line bundles.

Lemma 1. *Let k be a nonnegative integer, $m_1, \dots, m_k \in \mathbb{Z}$, and $\phi : \Omega \rightarrow R$ be a genus. Then there exist uniquely determined universal power series $A_{i,j} \in R[[z]]$, $1 \leq i \leq j \leq k$, and $B_1, \dots, B_k \in R[[z]]$, and $C, D \in R[[z]]$ depending only on ϕ and m_1, \dots, m_k such that for every smooth projective surface X , and line bundles L_1, \dots, L_k on X , we have*

$$(11) \quad \sum_{n=0}^{\infty} \phi \left(X^{[n]}, \det(L_1^{[n]})^{m_1} \otimes \dots \otimes \det(L_k^{[n]})^{m_k} \right) z^n \\ = \exp \left(\sum_{1 \leq i \leq j \leq k} c_1(L_i)c_1(L_j)A_{ij} + \sum_{i=1}^k c_1(L_i)c_1(X)B_i + c_1(X)^2C + c_2(X)D \right).$$

(Remember that top intersection products on surfaces are to be understood as intersection numbers.)

Proof. First note that for $k = 1$ the statement of the lemma is just Theorem 4.2 of [6] for the case of line bundles with Ψ (in the notation of [6]) being the Chern character of the m_1^{th} power of the determinant.

Theorem 4.2 of Ellingsrud, Göttsche and Lehn, and the proof presented by them can be easily generalised for more than one bundle, i.e. for $k > 1$. Therefore, our lemma as a specialization of this generalization is proven. \square

From the lemma we conclude the following:

Proposition 3. *Let $\phi : \Omega \rightarrow R$ be a genus. Then there exist uniquely determined universal power series $A_\phi, B_\phi, C_\phi, D_\phi \in R[[z]]$ depending only on ϕ such that for every smooth projective surface X together with a line bundle L on it, we have*

$$(12) \quad \phi(H_{X,L}) := \sum_{n=0}^{\infty} \phi(X^{[n]}, L_n) z^n$$

$$(13) \quad = \exp(c_1(L)^2 A_\phi + c_1(L)c_1(X)B_\phi + c_1(X)^2 C_\phi + c_2(X)D_\phi).$$

Proof. We use (4). By the previous lemma,

$$\begin{aligned} \phi(H_{X,L}) &= \sum_{n=0}^{\infty} \phi(X^{[n]}, \det(L^{[n]}) \otimes \det(\mathcal{O}_X^{[n]})^{-1}) z^n \\ &= \exp(c_1(L)^2 A_{11} + c_1(L)c_1(\mathcal{O}_X)A_{12} + c_1(\mathcal{O}_X)^2 A_{22} \\ &\quad + c_1(L)c_1(X)B_1 + c_1(\mathcal{O}_X)c_1(X)B_2 + c_1(X)^2 C + c_2(X)D) \end{aligned}$$

for certain power series $A_{i,j}, B_i, C, D$ independent of X and L . Since $c_1(\mathcal{O}_X) = 0$, this proves the proposition with $A_\phi = A_{11}, B_\phi = B_1, C_\phi = C$ and $D_\phi = D$. \square

It is possible to express the power series A_ϕ in terms of genera of Hilbert schemes of points on surfaces:

Proposition 4. *Let $\phi : \Omega \rightarrow R$ be any genus. For every smooth projective surface X ,*

$$(14) \quad c_1(X)^2 A_\phi = \frac{1}{2} \ln \frac{\phi_1(H_X)\phi_{-1}(H_X)}{\phi(H_X)^2}.$$

Proof. In [10], it is proven that the canonical bundle of $X^{[n]}$ is K_n where K denotes the canonical bundle on X . It follows $K_n^{-m} = (K^{-m})_n$.

Using this we have by Proposition 3 that

$$\begin{aligned} (15) \quad \ln \phi_m(H_X) &= \ln \sum_{n=0}^{\infty} \phi_m(X^{[n]}) z^n = \ln \sum_{n=0}^{\infty} \phi(X^{[n]}, K_n^{-m}) z^n \\ &= m^2 A_\phi c_1(K)^2 - m B_\phi c_1(K)c_1(X) + C_\phi c_1(X)^2 + D_\phi c_2(X) \\ &= (m^2 A_\phi + m B_\phi + C_\phi)c_1(X)^2 + D_\phi c_2(X) \end{aligned}$$

for all integers m , which proves the proposition. □

5. Complex genera of the generalised Kummer varieties

In this section, we will relate the (generalised) complex genera of Beauville’s generalised Kummer varieties to the complex genera of Hilbert schemes of points on a surface, which we studied in the previous section.

The first step in this direction is the following:

Proposition 5. *Let $\phi : \Omega \rightarrow R$ be a complex genus with values in the \mathbb{Q} -algebra R . For every abelian surface A together with a line bundle L on it, we have*

$$(16) \quad c_1(L)^2 \phi(A^{[n]}, L^{[n]}) = 2n^2 \phi(A^{[n]}, L_n)$$

for all positive integers n .

Proof. We will make use of (2). Recall that ν is a Galois covering of degree n^4 . By Proposition 2 we have

$$(17) \quad \begin{aligned} \phi(A, L^n) \phi(A^{[n]}, L^{[n]}) &= \phi(A \times A^{[n]}, L^n \boxtimes L^{[n]}) \\ &= \phi(A \times A^{[n]}, \nu^* L_n) = n^4 \phi(A^{[n]}, L_n), \end{aligned}$$

which proves the proposition, once we have shown that $\phi(A, L^n) = \frac{n^2}{2} c_1(L)^2$. This follows from the fact that the Chern classes of an abelian surface are trivial:

$$(18) \quad \phi(A, L^n) = \int_A f_\phi(\gamma_1) f_\phi(\gamma_2) e^{c_1(L^n)} = \int_A \frac{c_1(L^n)^2}{2} = \frac{n^2}{2} c_1(L)^2,$$

where we have used that f_ϕ is a power series with constant coefficient 1. □

In [2], M. Britze and the author expressed the (holomorphic) Euler characteristic of the line bundle $L^{[n]}$ in terms of the Euler characteristic of L in order to deduce a formula for the Euler characteristic of an arbitrary line bundle M on $A^{[n]}$ as a polynomial in the Beauville-Bogomolov quadratic form of $c_1(M)$. By using the analogous expression of the Euler characteristic of the line bundle L_n on $A^{[n]}$ (see [6]), we get the mentioned result of [2] as a corollary of the previous proposition:

Corollary 1 ([2]). *The holomorphic Euler characteristic of the line bundle $L^{[n]}$ on $A^{[n]}$ is given by*

$$(19) \quad \chi(A^{[n]}, L^{[n]}) = n \binom{\chi(A, L) + n - 1}{n - 1}.$$

Proof. By Lemma 5.1 of [6] we have

$$(20) \quad \chi(A^{[n]}, L_n) = \binom{\chi(A, L) + n - 1}{n}.$$

Using this, the corollary follows from the proposition applied to the case for ϕ being the Todd genus (remember Example 1). Also note that $\chi(A, L) = \frac{1}{2} c_1(L)^2$ by the Hirzebruch-Riemann-Roch formula. □

If we are interested in the usual genera of the generalised Kummer varieties, i.e. the genera of the pairs $(A^{[[n]]}, \mathcal{O}_{A^{[[n]]}})$, we cannot use Proposition 5 directly since for $L = \mathcal{O}_A$ it just states $0 = 0$.

However, it is still possible to make use of the proposition. We have to look at all generalised Kummer varieties at the same time. Doing so, we get the following main result of this work:

Theorem 1. ³ *Let $\phi : \Omega \rightarrow R$ be a complex genus with values in the \mathbb{Q} -algebra R . For every smooth projective surface X with $\int_X c_1(X)^2 \neq 0$,*

$$(21) \quad \phi(K) = \frac{1}{c_1(X)^2} \left(z \frac{d}{dz} \right)^2 \ln \frac{\phi_1(H_X)\phi_{-1}(H_X)}{\phi(H_X)^2}.$$

Proof. Let L be any line bundle on A . We have

$$(22) \quad \begin{aligned} c_1(L)^2 \sum_{n=1}^{\infty} \phi(A^{[[n]]}, L^{[[n]])} z^n &= 2 \sum_{n=1}^{\infty} n^2 \phi(A^{[n]}, L_n) z^n = 2 \left(z \frac{d}{dz} \right)^2 \phi(H_{A,L}) \\ &= 2 \left(z \frac{d}{dz} \right)^2 \exp(c_1(L)^2 A_\phi + c_1(L)c_1(A)B_\phi + c_1(A)^2 C_\phi + c_2(A)D_\phi) \\ &= 2 \left(z \frac{d}{dz} \right)^2 \exp(c_1(L)^2 A_\phi) = 2 \left(z \frac{d}{dz} \right)^2 c_1(L)^2 A_\phi + O((c_1(L)^2)^2), \end{aligned}$$

which together with Proposition 4 proves the theorem, since there are line bundles on A with $c_1(L) \neq 0$. □

Let $\text{id}_\Omega : \Omega \rightarrow \Omega$ be the universal genus. Like any other genus, we can deform it to a genus $(\text{id}_\Omega)_t : \Omega \rightarrow \Omega[t]$. If $X \in \Omega$ is a rational complex cobordism class, we write

$$(23) \quad X_t := (\text{id}_\Omega)_t(X) \in \Omega[t].$$

With this notation, we can express the cobordism class K by

$$(24) \quad K = \frac{1}{c_1(X)^2} \left(z \frac{d}{dz} \right)^2 \ln \frac{(H_X)_1(H_X)_{-1}}{H_X^2}$$

for all surfaces X with $c_1(X)^2 \neq 0$.

Remark 3. Of course, everything in this chapter still holds true if we replace the abelian surfaces A from which we constructed the generalised Kummer varieties, by an arbitrary complex torus of dimension two.

We used Theorem 1 to produce a table of all Chern numbers of the generalised Kummer varieties up to dimension fourteen. It can be found in Appendix A.

³Note added in proof: the author has recently managed to use this formula to compute the elliptic genera of generalised Kummer varieties, see preprint [arXiv:math.AG/0208077](https://arxiv.org/abs/math/0208077)

Appendix A. The Chern numbers of the generalised Kummer varieties of dimension up to 14

We have used Theorem 1 to compute all Chern numbers of the generalised Kummer varieties of dimension up to fourteen. Our results are as follows:

Chern number	Evaluated on $A^{[*]}$	Chern number	Evaluated on $A^{[*]}$
c_2	24	c_2^7	421414305792
c_2^2	756	$c_2^5 c_4$	149664301056
c_4	108	$c_2^3 c_4^2$	53149827072
c_2^3	30208	$c_2 c_4^3$	18874417152
$c_2 c_4$	6784	$c_2^4 c_6$	24230756352
c_6	448	$c_2^2 c_4 c_6$	8610545664
c_2^4	1470000	$c_4^2 c_6$	3059945472
$c_2^2 c_4$	405000	$c_2 c_6^2$	1397121024
c_4^2	111750	$c_2^3 c_8$	1914077184
$c_2 c_6$	37500	$c_2 c_4 c_8$	681332736
c_8	750	$c_6 c_8$	110853120
c_2^5	84478464	$c_2^2 c_{10}$	71909376
$c_2^3 c_4$	26220672	$c_4 c_{10}$	25700352
$c_2 c_4^2$	8141472	$c_2 c_{12}$	1198080
$c_2^2 c_6$	3141504	c_{14}	7680
$c_4 c_6$	979776		
$c_2 c_8$	142560		
c_{10}	2592		
c_2^6	5603050432		
$c_2^4 c_4$	1881462016		
$c_2^2 c_4^2$	631808744		
c_4^3	212190776		
$c_2^3 c_6$	268796752		
$c_2 c_4 c_6$	90412056		
c_6^2	12976376		
$c_2^2 c_8$	17075912		
$c_4 c_8$	5762400		
$c_2 c_{10}$	441784		
c_{12}	2744		

It is a remarkable fact that all Chern numbers of the varieties $A^{[n]}$ with $n \leq 8$ are positive and divisible by n^3 . (This result is also true for $n \leq 11$.) As the known Chern numbers of Hilbert schemes of points on K3 surfaces are also positive one can wonder if, given a compact Hyperkähler manifold X , all Chern numbers of X are positive.

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