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A NOTE ON THE GUROV-RESHETNYAK CONDITION

A. A. Korenovskyy, A. K. Lerner, and A. M. Stokolos

Abstract. An equivalence between the Gurov-Reshetnyak $GR(\varepsilon)$ and Muckenhoupt A_∞ conditions is established. Our proof is extremely simple and works for arbitrary absolutely continuous measures.

Throughout the paper, μ will be a positive measure on $\mathbb{R}^n$ absolutely continuous with respect to Lebesgue measure. Denote

$$\Omega_\mu(f;Q)=\frac{1}{\mu(Q)}\int_Q |f(x)-f_{Q,\mu}|d\mu(x), \quad f_{Q,\mu}=\frac{1}{\mu(Q)}\int_Q f(x)d\mu(x).$$

Definition 1. We say that a nonnegative function f, μ-integrable on a cube Q_0, satisfies the Gurov-Reshetnyak condition $GR_\mu(\varepsilon)$, $0<\varepsilon<2$, if for any cube $Q\subset Q_0$,

$$\Omega_\mu(f;Q)\le \varepsilon f_{Q,\mu}. \tag{1}$$

When μ is Lebesgue measure we drop the subscript μ.

This condition appeared in [6, 7]. It is important in Quasi-Conformal Mappings, PDEs, Reverse Hölder Inequality Theory, etc. (see, e.g., [2, 8]). Since (1) trivially holds for all positive $f\in L_\mu(Q_0)$ if $\varepsilon=2$, only the case $0<\varepsilon<2$ is of interest. It was established in [2, 6, 7, 8, 13] for Lebesgue measure and in [4, 5] for doubling measures that if ε is small enough, namely $0<\varepsilon<c2^{-n}$, the $GR_\mu(\varepsilon)$ implies $f\in L^p_\mu(Q_0)$ with some $p>1$ depending on ε. The machinery used in the articles mentioned above does not work for $\varepsilon>1/8$ even in the one-dimensional case.

The one-dimensional improvement of these results was done in [9]. Namely, for any $0<\varepsilon<2$ it was proved that $GR(\varepsilon)\subset L^p_{loc}$ where $1<p<p(\varepsilon)$; moreover a sharp bound $p(\varepsilon)$ for the exponent was discovered. The main tool in [9] is the Riesz Sunrising Lemma which has no multidimensional version since it involves the structure of open sets on a real line.

In the present article using simple arguments we prove that for any $n\ge 1$, $0<\varepsilon<2$ and arbitrary absolutely continuous measure μ, the Gurov-Reshetnyak condition $GR_\mu(\varepsilon)$ implies the weighted $A_\infty(\mu)$ Muckenhoupt condition. And conversely, $A_\infty(\mu)$ implies $GR_\mu(\varepsilon_0)$ for some $0<\varepsilon_0<2$.

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In the non-weighted (or doubling) case R.R. Coifman and C. Fefferman [3] have found several equivalent descriptions of the A_∞ property. Recently these descriptions have been transfered by J. Orobitg and C. Perez [12] to the non-doubling case. For our purposes it will be convenient to define $A_\infty(\mu)$ by the following way.

Definition 2. We say that a nonnegative function f, μ-integrable on a cube Q_0, satisfies Muckenhoupt condition $A_\infty(\mu)$ if there exist $0<\alpha,\beta<1$ such that for any cube $Q\subset Q_0$,

$$\mu\{x\in Q : f(x)>\beta f_{Q,\mu}\}>\alpha\mu(Q).$$

Our main result is the following.

Theorem 1.

(i) *Suppose that for some* $0<\varepsilon<2$ *nonnegative function* f *satisfies the inequality*

$$\Omega_\mu(f;Q)\le\varepsilon f_{Q,\mu}.$$

Then, for $\varepsilon<\lambda<2$ *we have*

$$\mu\{x\in Q : f(x)>(1-\varepsilon/\lambda)f_{Q,\mu}\}\ge(1-\lambda/2)\mu(Q);$$

(ii) *Suppose that for some* $0<\alpha,\beta<1$ *nonnegative function* f *satisfies the inequality*

$$\mu\{x\in Q : f(x)>\beta f_{Q,\mu}\}>\alpha\mu(Q).$$

Then

$$\Omega_\mu(f;Q)\le 2(1-\alpha\beta)f_{Q,\mu}.$$

Proof. Set $E=\{x\in Q : f(x)>(1-\varepsilon/\lambda)f_{Q,\mu}\},\ E^c=Q\setminus E$. Suppose that $\mu(E^c)>0$ (otherwise part (i) is trivial). Then

$$\begin{aligned}\frac{\varepsilon}{\lambda}f_{Q,\mu} &\le \inf_{x\in E^c}\big(f_{Q,\mu}-f(x)\big)\le\frac{1}{\mu(E^c)}\int_{E^c}\big(f_{Q,\mu}-f(x)\big)d\mu(x)\\ &\le \frac{1}{\mu(E^c)}\int\limits_{\{x\in Q: f(x)<f_{Q,\mu}\}}\big(f_{Q,\mu}-f(x)\big)d\mu(x)\\ &= \frac{1}{\mu(E^c)}\frac{\mu(Q)}{2}\Omega_\mu(f;Q)\le\frac{1}{\mu(E^c)}\frac{\mu(Q)}{2}\varepsilon f_{Q,\mu}.\end{aligned}$$

Hence, $\mu(E^c)\le(\lambda/2)\mu(Q)$, as required.

To prove the second part, set $E = \{x \in Q : f(x) > \beta f_{Q,\mu}\}$, $E^c = Q \setminus E$. Then $\mu(E^c) \le (1-\alpha)\mu(Q)$, and we have

$$\begin{aligned}
\Omega_\mu(f;Q) &= \frac{2}{\mu(Q)} \int\limits_{\{x\in Q:\ f(x)<f_{Q,\mu}\}} \big(f_{Q,\mu} - f(x)\big) d\mu(x) \\
&= \frac{2}{\mu(Q)} \int\limits_{\{x\in Q:\beta f_{Q,\mu}<f(x)<f_{Q,\mu}\}} \big(f_{Q,\mu} - f(x)\big) d\mu(x) \\
&+ \frac{2}{\mu(Q)} \int_{E^c} \big(f_{Q,\mu} - f(x)\big) d\mu(x) \\
&\le \frac{2}{\mu(Q)} f_{Q,\mu}\Big((1-\beta)\mu(E) + \mu(E^c)\Big) \\
&= \frac{2}{\mu(Q)} f_{Q,\mu}\Big((1-\beta)\mu(Q) + \beta\mu(E^c)\Big) \\
&\le \frac{2}{\mu(Q)} f_{Q,\mu}\Big((1-\beta)\mu(Q) + \beta(1-\alpha)\mu(Q)\Big) = 2(1-\alpha\beta) f_{Q,\mu}.
\end{aligned}$$

The theorem is proved. □

Corollary. The following characterization of $A_\infty(\mu)$ holds:

$$A_\infty(\mu) = \bigcup_{0<\varepsilon<2} GR_\mu(\varepsilon).$$

Since $A_\infty(\mu)$ condition is equivalent to the weighted reverse Hölder inequality for some $p > 1$ (cf. [12]), i.e.

$$\left(\frac{1}{\mu(Q)} \int_Q \big(f(x)\big)^p d\mu(x)\right)^{1/p} \le c\frac{1}{\mu(Q)} \int_Q f(x) d\mu(x) \quad (Q \subset Q_0), \tag{2}$$

we see that for any $0 < \varepsilon < 2$ a function f satisfying the Gurov-Reshetnyak condition $GR_\mu(\varepsilon)$ belongs to $L^p_\mu(Q_0)$ for some $p > 1$. Observe that such approach (i.e. $GR_\mu(\varepsilon) \Rightarrow A_\infty(\mu) \Rightarrow$ Reverse Hölder) does not give the optimal order of integrability for small ε, though it is known [2, 4] in doubling case that $GR_\mu(\varepsilon) \subset L^{p(\varepsilon)}_\mu(Q_0)$, where $p(\varepsilon) \asymp c_n/\varepsilon, \varepsilon \to 0$, and this order is sharp. However we will show that part (i) of Theorem 1 allows us to obtain the same order for any measure μ, any $0 < \varepsilon < 2$, and $f \in GR_\mu(\varepsilon)$. We will need the following.

Covering Lemma [10]**.** *Let E be a subset of Q_0, and suppose that $\mu(E) \le \rho\mu(Q_0)$, $0 < \rho < 1$. Then there exists a sequence $\{Q_i\}$ of cubes contained in Q_0 such that*

(i) $\mu(Q_i \cap E) = \rho\mu(Q_i)$;
(ii) *the family $\{Q_i\}$ is almost disjoint with constant $B(n)$, that is, every point of Q_0 belongs to at most $B(n)$ cubes Q_i;*
(iii) *$E' \subset \cup_j Q_j$, where E' is the set of μ-density points of E.*

Recall that the non-increasing rearrangement of f on a cube Q_0 with respect to μ is defined by

$$f^*_\mu(t) = \sup_{E\subset Q_0:\mu(E)=t} \inf_{x\in E} |f(x)| \quad (0 < t < \mu(Q_0)).$$

Denote $f^{**}_\mu(t) = t^{-1}\int_0^t f^*_\mu(\tau)d\tau$.

Theorem 2. Let $0 < \varepsilon < 2$, and $f \in GR_\mu(\varepsilon)$. Then for $\varepsilon < \lambda < 2$, $\rho < 1 - \lambda/2$, and $t \le \rho\mu(Q_0)$ we have

$$f^{**}_\mu(t) \le \Big(B(n)\frac{\lambda/\rho + 1}{\lambda - \varepsilon}\varepsilon + 1\Big) f^*_\mu(t). \tag{3}$$

Remark. A well-known argument due to Muckenhoupt [11, Lemma 4] shows that (3) implies the reverse Hölder inequality (2) for all $p < 1 + \big(\frac{\lambda-\varepsilon}{B(n)(\lambda/\rho+1)}\big)\frac{1}{\varepsilon}$.

Proof of Theorem 2. Set $E = \{x \in Q_0 : f(x) > f^*_\mu(t)\}$, and apply the Covering Lemma to E and number ρ. We get cubes $Q_i \subset Q_0$, satisfying (i)-(iii). Since $\rho < 1 - \lambda/2$, we obtain from (i) that for each Q_i,

$$\big(f\chi_{Q_i}\big)^*_\mu\big((1-\lambda/2)\mu(Q_i)\big) \le f^*_\mu(t).$$

Hence, by Theorem 1,

$$f_{Q_i,\mu} \le \frac{\lambda}{\lambda-\varepsilon}\big(f\chi_{Q_i}\big)^*_\mu\big((1-\lambda/2)\mu(Q_i)\big) \le \frac{\lambda}{\lambda-\varepsilon} f^*_\mu(t), \tag{4}$$

and so,

$$\Omega_\mu(f;Q_i) \le \frac{\varepsilon\lambda}{\lambda-\varepsilon} f^*_\mu(t). \tag{5}$$

Further, by (ii),

$$\sum_i \mu(Q_i \cap E) \le B(n)\mu(E) \le B(n)t.$$

Therefore, using a well-known property of rearrangement (see, e.g. [1]) and (4), (5), we obtain

$$\begin{aligned}
t\big(f^{**}_\mu(t) - f^*_\mu(t)\big) &= \int_E \big(f(x) - f^*_\mu(t)\big)d\mu(x) = \sum_i \int_{E\cap Q_i}\big(f(x) - f^*_\mu(t)\big)d\mu(x)\\
&= \sum_i \int_{E\cap Q_i}\big(f(x) - f_{Q_i,\mu}\big)d\mu(x)\\
&+ \sum_i \mu(E\cap Q_i)\big(f_{Q_i,\mu} - f^*_\mu(t)\big)\\
&\le \frac{\varepsilon\lambda}{\lambda-\varepsilon} f^*_\mu(t)\sum_i \mu(Q_i) + \frac{\varepsilon}{\lambda-\varepsilon} f^*_\mu(t)\sum_i \mu(E\cap Q_i)\\
&\le B(n)\frac{\lambda/\rho+1}{\lambda-\varepsilon}\varepsilon t f^*_\mu(t),
\end{aligned}$$

which gives the desired result. □

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Department of Mathematical Analysis, IMEM, National University of Odessa, Dvoryanskaya, 2, 65026 Odessa, Ukraine.

E-mail address: anakor@paco.net

Department of Mathematics and Computer Science, Bar-Ilan University, 52900 Ramat Gan, Israel.

E-mail address: aklerner@netvision.net.il

Department of Mathematics, University of Connecticut, U-9, Storrs, CT 06268, U.S.A.

E-mail address: stokolos@math.uconn.edu