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INTERPOLATION BY PROPER HOLOMORPHIC EMBEDDINGS OF THE DISC INTO $\mathbb{C}^2$

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Dedicated to the memory of my mother

1. The result

Let Δ be the open unit disc in $\mathbb{C}$. A map $f\colon\ \Delta \to \mathbb{C}^2$ is called *a proper holomorphic embedding* if it is a holomorphic immersion which is one to one and such that the preimage of every compact set is compact. If $f\colon\ \Delta \to \mathbb{C}^2$ is a proper holomorphic embedding then $f(\Delta)$ is a closed submanifold of $\mathbb{C}^2$ which is, via f, biholomorphically equivalent to Δ.

It is not trivial to prove that there are proper holomorphic embeddings from Δ to $\mathbb{C}^2$ [St, A, GS]. It is known that given a discrete set $E \subset \mathbb{C}^2$ there is a proper holomorphic embedding $f\colon\ \Delta \to \mathbb{C}^2$ such that $E \subset f(\Delta)$ [FGS]. In the present paper we prove a stronger result:

Theorem 1.1 *Given a discrete set $S \subset \Delta$ and a proper injection $\varphi\colon\ S \to \mathbb{C}^2$ there is a proper holomorphic embedding $f\colon\ \Delta \to \mathbb{C}^2$ that extends φ.*

In other words, given an injective sequence $\{\zeta_j\} \subset \Delta$ such that $|\zeta_j| \to 1$ and an injective sequence $\{w_j\} \subset \mathbb{C}^2$ such that $|w_j| \to +\infty$ there is a proper holomorphic embedding $f\colon\ \Delta \to \mathbb{C}^2$ such that $f(\zeta_j) = w_j$ $(j \in \mathbb{N})$.

The proof of the Carleman approximation theorem of Buzzard and Forstnerič [BFo] can be adapted to prove such a result for proper holomorphic embeddings $f\colon\ \mathbb{C} \to \mathbb{C}^2$. In the proof there one uses the fact that $\mathbb{C}$ admits particularly simple embeddings into $\mathbb{C}^2$ of the form $\zeta \to (\zeta, a(\zeta))$ where a is an entire function. There are no such embeddings for Δ so a different proof is necessary in our case. In the induction step of our proof we use simultaneous composition by automorphisms on the left and on the right, a novelty introduced by Buzzard and Forstnerič.

2. The scheme of the proof

Suppose that $S \subset \Delta$ is a discrete set and let $\varphi\colon\ S \to \mathbb{C}^2$ be a proper injection. With no loss of generality assume that S is infinite.

Denote by $\mathbb{B}$ the open unit ball in $\mathbb{C}^2$. We shall construct inductively a sequence K_n of compact subsets of Δ, such that bK_n is a smooth Jordan curve

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for each $n \in \mathbb{N}$ and such that $K_n \subset\subset K_{n+1}$ $(n \in \mathbb{N})$, $\cup_{n=1}^{\infty} K_n = \Delta$, an increasing sequence r_n of positive numbers converging to $+\infty$, a decreasing sequence ε_n of positive numbers and a sequence f_n of holomorphic maps from Δ to $\mathbb{C}^2$ which are one to one and regular and such that the following hold:

(i) $\varphi((\Delta \setminus K_n) \cap S) \subset \mathbb{C}^2 \setminus r_n\overline{\mathbb{B}}$
(ii) $f_n(\Delta \setminus \mathrm{Int}K_n) \subset \mathbb{C}^2 \setminus r_n\overline{\mathbb{B}}$
(iii) $f_{n+1}(\Delta \setminus K_n) \subset \mathbb{C}^2 \setminus r_{n-1}\overline{\mathbb{B}}$
(iv) $f_n|K_n \cap S = \varphi|K_n \cap S$
(v) $|f_{n+1} - f_n| < \varepsilon_n/2^n$ on K_n
(vi) If h is a holomorphic map on $\mathrm{Int}K_n$ that satisfies $|h - f_n| < \varepsilon_n$ on $\mathrm{Int}K_n$, then h is one to one and regular on K_{n-1}
(vii) $(1 - 1/n)\Delta \subset K_n$

Suppose for a moment that we have done this. By (v) and (vii) f_n converges, uniformly on compacta in Δ, to a holomorphic map f. By (v), $|f_n - f| \leq \sum_{j=n}^{\infty} |f_{j+1} - f_j| \leq \sum_{j=n}^{\infty} \varepsilon_j/2^j < \varepsilon_n$ on K_n which implies by (vi) that f is regular and one to one on K_{n-1}. As this holds for every n it follows that f is regular and one to one on Δ . By (iv), f extends φ. Let $\zeta \in K_{n+1} \setminus K_n$. By (v), $|f_{j+1}(\zeta) - f_j(\zeta)| < \varepsilon_j/2^j$ $(j \geq n+1)$ which, by (iii) implies that $|f(\zeta)| \geq |f_{n+1}(\zeta)| - \sum_{j=n+1}^{\infty} |f_{j+1}(\zeta) - f_j(\zeta)| \geq r_{n-1} - \sum_{j=n+1}^{\infty} \varepsilon_j/2^j \geq r_{n-1} - \varepsilon_{n+1}$. This holds for every n. Since r_n increase to $+\infty$ and since ε_n are decreasing it follows that the map f is proper. Thus, f has all the required properties.

In the process we shall also construct two sequences S_n, T_n of positive numbers such that $S_{n+1} = S_n$ for even n and $T_{n+1} = T_n$ for odd n. Each map f_n will be of the form $f_n = A_n \circ g_n$ where A_n is a holomorphic automorphism of $\mathbb{C}^2$ and g_n is a one to one and regular holomorphic map from an open neighbourhood U_n of $\overline{\Delta}$ to $\mathbb{C}^2$ which, for even n is transverse to $\{(z,w)\colon\ |z| = S_n\}$ and satisfies $g_n^{-1}(\{|z| = S_n\}) = b\Delta$, and for odd n, is transverse to $\{(z,w)\colon\ |w| = T_n\}$ and satisfies $g_n^{-1}(\{|w| = T_n\}) = b\Delta$.

With no loss of generality assume that $0 \notin S$. To begin the induction, let $f_1(\zeta) = (0, \zeta)$ and let r_1, $0 < r_1 < 1/2$ be such that $2r_1\overline{\Delta}$ contains no point of S. Put $K_0 = r_1\overline{\Delta}$, $K_1 = 2r_1\overline{\Delta}$. Then (i), (ii) and (vii) are satisfied for $n = 1$ and (iv) is vacuously satisfied for $n = 1$. Put $S_1 = T_1 = 1$ and $A_1 = \mathrm{Id}$ so that $f_1 = A_1 \circ g_1$ where $g_1(\zeta) = (0, \zeta)$ and $U_1 = \mathbb{C}$. Clearly g_1 is transverse to $\{|w| = T_1\}$ and $g_1^{-1}(\{|w| = T_1\}) = b\Delta$. Put $r_0 = r_1/2$. Then $A_1(\{|w| > T_1/2\})$ misses $2r_0\mathbb{B}$. Put $\varepsilon_0 = \min\{1, r_1/2\}$.

Given $f_n = A_n \circ g_n$ we shall have $f_{n+1} = A_{n+1} \circ g_{n+1}$ with $A_{n+1} = \Psi_{n+1} \circ \Theta_{n+1} \circ A_n$ where Θ_{n+1} and Ψ_{n+1} are holomorphic automorphisms of $\mathbb{C}^2$ and with $g_{n+1} = G_{n+1} \circ g_n \circ p_{n+1}$ where p_{n+1} is a conformal map from a neighbourhood U_{n+1} of $\overline{\Delta}$ to $p_{n+1}(U_{n+1}) \subset \mathbb{C}$ which is a slight perturbation of the identity on $\overline{\Delta}$ and G_{n+1} is an automorphism of $\mathbb{C}^2$ of the form

$$G_{n+1}(z,w) = \Big(z + S_{n+1}\Big(\frac{w}{T_n}\Big)^{M_{n+1}},\ w\Big) \ \text{ if } n \text{ is odd,} \tag{2.1$'$}$$

$$G_{n+1}(z,w) = \Big(z,\ w + T_{n+1}\Big(\frac{z}{S_n}\Big)^{M_{n+1}}\Big) \text{ if } n \text{ is even.} \tag{2.1''}$$

3. The induction step, Part 1

Suppose for a moment that we have constructed $f_n = A_n \circ g_n,\ K_n,\ S_n,\ T_n,\ r_n$ and ε_{n-1}. We want to show how to obtain $\varepsilon_n,\ K_{n+1},\ S_{n+1},\ T_{n+1},\ r_{n+1}$ and $f_{n+1} = A_{n+1} \circ g_{n+1}$. Suppose that n is odd so that $g_n\colon\ U_n \to \mathbb{C}^2$ is transverse to $\{(z,w)\colon\ |w| = T_n\}$ and satisfies $g_n^{-1}(\{|w| = T_n\}) = b\Delta$. Put $T_{n+1} = T_n$. Since g_n is transverse to $\{|w| = T_n\}$ and since S is discrete one can, after shrinking U_n if necessary, choose $T_{n1},\ T_{n2},\ T_{n3}$ such that

$$\frac{T_n}{2} < T_{n3} < T_{n2} < T_{n1} < T_n$$

where T_{n3} is so close to T_n that for all $T,\ T_{n3} \le T \le T_n,\ \ g_n$ is transverse to $\{|w| = T\}$ and $g_n^{-1}(\{|w| = T\})$ is a smooth Jordan curve, that

$$K_n \subset g_n^{-1}\{|w| < T_{n3}\},$$

that $g_n^{-1}(\{T_{n3} \le |w| \le T_{n2}\})$ contains no point of S, and that $g_n^{-1}(\{|w| < T_{n1}\})$ contains a point in S that does not belong to K_n. Put

$$P_{n+1} = g_n^{-1}(\{|w| \le T_{n3}\}),\ Q_{n+1} = g_n^{-1}(\{|w| \le T_{n2}\}),\ K_{n+1} = g_n^{-1}(\{|w| \le T_{n1}\}).$$

With no loss of generality assume that T_{n3} has been chosen so close to T_n that (vii) holds with n replaced by $n+1$. We have

$$K_n \subset\subset P_{n+1} \subset\subset Q_{n+1} \subset\subset K_{n+1}.$$

Clearly bK_{n+1} is a smooth Jordan curve.

By (i), $r_n < \min\{|\varphi(w)|\colon\ w \in (\Delta \setminus K_n) \cap S\}$. Thus, one can choose $r_{n+1} > r_n$ such that

$$\min\{|\varphi(w)|\colon w \in (\Delta \setminus K_{n+1}) \cap S\} - 1 < r_{n+1} < \min\{|\varphi(w)|\colon w \in (\Delta \setminus K_{n+1}) \cap S\}. \tag{3.1}$$

Then (i) is satisfied with n replaced by $n+1$. Choose $\varepsilon_n,\ 0 < \varepsilon_n < \varepsilon_{n-1}$, such that

$$\varepsilon_n < r_n - r_{n-1}, \quad \varepsilon_n < r_{n-1}, \tag{3.2}$$

and such that (vi) holds. Since f_n is one to one and regular on Δ this is possible by a lemma of Narasimhan [Na, p. 926].

Choose $R,\ R > 2r_{n+1},\ R > 2r_n + \varepsilon_n,$ so large that $f_n(K_n) + \mathbb{B} \subset R\mathbb{B}$ and that $\varphi(K_{n+1} \cap S) \subset R\mathbb{B}$. We need the following lemma.

Lemma 3.1. *Let $R > 0$ and let $w_1, w_2, \cdots w_n \in R\mathbb{B},\ w_i \ne w_j\ (i \ne j)$. Given $\gamma > 0$ there is a $\delta > 0$ such that whenever $q_1, q_2, \cdots q_n \in \mathbb{C}^2$ satisfy $|q_i - w_i| < \delta,\ 1 \le i \le n$, there is a holomorphic automorphism Ψ of $\mathbb{C}^2$ such that:*

(i) $\Psi(q_i) = w_i\ \ (1 \le i \le n)$
(ii) $|\Psi(w) - w| < \gamma\ \ (w \in R\mathbb{B})$.

Lemma 3.1 provides a $\theta_n,\ 0<\theta_n<\varepsilon_n/2^{n+2}$, such that

$$(3.3)\begin{cases}\text{whenever } \psi\colon\ K_{n+1}\cap S\to\mathbb{C}^2 \text{ satisfies } |\psi-\varphi|<3\theta_n \text{ on}\\ K_{n+1}\cap S \text{ there is a holomorphic automorphism } \Psi \text{ of } \mathbb{C}^2 \text{ such that}\\ \Psi\circ\psi=\varphi|K_{n+1} \text{ and such that } |\Psi-\mathrm{Id}|<\varepsilon_n/2^{n+1} \text{ on } R\mathbb{B}.\end{cases}$$

By (3.2) we may assume that

$$(3.4)\qquad r_n-3\theta_n>r_{n-1}+\varepsilon_n+\theta_n,\quad 2r_{n-1}-\theta_n>r_{n-1}+\varepsilon_n+\theta_n.$$

4. Proof of Lemma 3.1

Sublemma 4.1 *Suppose that $R>0$ and let $\alpha_1,\cdots\alpha_n\in R\Delta,\ \alpha_i\neq\alpha_j\ (i\neq j)$. There are $\eta>0$ and $L<\infty$ such that whenever $\beta_1,\cdots\beta_n$ satisfy $|\beta_i-\alpha_i|<\eta,\ 1\le i\le n$, then for every $j,\ 1\le j\le n$, there is a polynomial Q_j such that (i) $Q_j(\beta_i)=\delta_{ji}\ (1\le i,j\le n)$ (ii) $|Q_j(\zeta)|\le L\ (\zeta\in 2R\Delta)$.*

Proof. Choose $\eta>0$ so small that $\alpha_i+\eta\Delta\subset R\Delta\ (1\le i\le n)$ and let $|\beta_i-\alpha_i|<\eta\ (1\le i\le n)$. For each $j,\ 1\le j\le n$, the polynomial

$$Q_j(\zeta)=\prod_{k=1,k\neq j}^{n}\frac{\zeta-\beta_k}{\beta_j-\beta_k}$$

satisfies (i). If $|\zeta|<2R$ then

$$|Q_j(\zeta)|\le\frac{(3R)^{n-1}}{\left(\min_{\ j\neq k}|\beta_j-\beta_k|\right)^{n-1}}.$$

Now, let $\gamma=\ \min_{j\neq k}|\alpha_j-\alpha_k|$. Passing to a smaller η we may assume that $0<\eta<\gamma/2$. If $|\alpha_i-\beta_i|<\eta,\ 1\le i\le n$, then $\min_{j\neq k}|\beta_j-\beta_k|\ge\gamma-2\eta>0$ so Q_j satisfies (ii) with $L=[3R/(\gamma-2\eta)]^{n-1}$. This completes the proof. □

Proof of Lemma 3.1. Choose a coordinate system in $\mathbb{C}^2$ such that if $w_i=(w_i^1,w_i^2)$ then $w_i^1\neq w_j^1,\ w_i^2\neq w_j^2$ if $i\neq j,\ 1\le i,j\le n$. By Sublemma 4.1 there are $\eta>0$ and $L<\infty$ such that whenever β_i^1 satisfy $|\beta_i^1-w_i^1|<\eta$ and β^2 satisfy $|\beta_i^2-w_i^2|<\eta,\ 1\le i\le n$, then for each $j,\ 1\le j\le n$, there are polynomials Q_j^1 and Q_j^2 such that $Q_j^1(\beta_j^1)=1,\ Q_j^1(\beta_i^1)=0\ (i\neq j)$, $Q_j^2(\beta_j^2)=1,\ Q_j^2(\beta_i^2)=0\ (i\neq j)$ and $|Q_j^1|<L,\ |Q_j^2|<L$ on $2R\Delta$. Let $|z_j-w_j|<\eta,\ 1\le j\le n$. Our map Φ will be of the form $\Phi=T\circ S$ where T,S are the automorphisms of $\mathbb{C}^2$

$$T(\xi,\zeta)=(\xi,\zeta+Q_1(\xi)),\quad S(\xi,\zeta)=(\xi+Q_2(\zeta),\zeta)$$

such that

$$(4.1)\qquad S(R\Delta\times R\Delta)\subset(2R\Delta)\times(R\Delta),$$

$$(4.2)\qquad |S(\xi,\zeta)-(\xi,\zeta)|<\gamma/2\quad((\xi,\zeta)\in(R\Delta)^2),$$

$$(4.3)\qquad |T(\xi,\zeta)-(\xi,\zeta)|<\gamma/2\quad((\xi,\zeta)\in(2R\Delta)\times(R\Delta)),$$

and

$$S(z_i^1, z_i^2) = (w_i^1, z_i^2), \quad T(w_i^1, z_i^2) = (w_i^1, w_i^2) \ \ (1 \le i \le n). \tag{4.4}$$

By (4.1)-(4.4) the map Φ satisfies (i) and (ii) in Lemma 3.1. To construct S, put $\beta_j^2 = z_j^2, \ 1 \le j \le n$, and let $Q_j^2, \ 1 \le j \le n$, be as above. In particular, $Q_j^2(z_i^2) = \delta_{ji}, \ 1 \le i,j \le n$. Put

$$Q_2(\zeta) = \sum_{j=1}^{n} (w_j^1 - z_j^1) Q_j^2(\zeta).$$

We have

$$Q_2(z_j^2) = \sum_{i=1}^{n} (w_i^1 - z_i^1) Q_i^2(z_j^2) = w_j^1 - z_j^1$$

and so $S(z_i^1, z_i^2) = (z_i^1 + w_i^1 - z_i^1, z_i^2) = (w_i^1, z_i^2)$. We have

$$|Q_2(\zeta)| \le n \cdot \max_{1 \le j \le n} |w_j^1 - z_j^1| \cdot L, \quad (|\zeta| < R)$$

which implies that

$$|S(\xi,\zeta) - (\xi,\zeta)| = |(Q_2(\zeta), 0)| \le n \cdot L \cdot \max_{1 \le j \le n} |w_j - z_j|, \quad (|\zeta| < R).$$

In particular, if $\eta > 0$ is small enough then $|Q_2(\zeta)| < R, \ (|\zeta| < R)$, so that (4.1) and (4.2) hold. To construct T, put $\beta_j^1 = w_j^1, \ 1 \le j \le n$, and let $Q_j^1, \ 1 \le j \le n$, be as above. Put

$$Q_1(\zeta) = \sum_{j=1}^{n} (w_j^2 - z_j^2) Q_j^1(\zeta).$$

We have $Q_1(w_j^1) = w_j^2 - z_j^2 \ \ (1 \le j \le n)$, so $T(w_i^1, z_i^2) = (w_i^1, z_i^2 + w_i^2 - z_i^2) = (w_i^1, w_i^2), \ (1 \le i \le n)$. Again, $|Q_1(\zeta)| \le n \cdot \max_{1 \le j \le n} |w_j^2 - z_j^2| \cdot L, \ (|\zeta| < 2R)$, which implies that

$$|T(\xi,\zeta) - (\xi,\zeta)| = |(0, Q_1(\xi))| \le n \cdot \max_{1 \le j \le n} |w_j - z_j| \cdot L, \quad (|\xi| < 2R).$$

In particular, if $\delta = \eta$ is small enough then (4.3) holds. The equality (4.4) is clear. This completes the proof. □

Remark. Lemma 3.1 holds for $\mathbb{C}^N, \ N \ge 2$. The proof is an easy elaboration of the proof above.

5. The induction step, Part 2

We need the following:

Lemma 5.1 *Let $r > 0$ and let $\Phi\colon \ \mathbb{C} \to \mathbb{C}^2$ be a proper holomorphic embedding. Let $\Sigma \subset\subset \mathbb{C}$ be a domain bounded by a smooth Jordan curve and assume that $\Phi(b\Sigma) \subset \mathbb{C}^2 \setminus r\,\overline{\mathbb{B}}$. Then the set $(r\,\overline{\mathbb{B}}) \cup \Phi(\overline{\Sigma})$ is polynomially convex.*

Proof. Since Σ is a Jordan domain with smooth boundary it is easy to see that if $K \subset \mathbb{C} \setminus \overline{\Sigma}$ is a compact set, if $a, b \in (\mathbb{C} \setminus \overline{\Sigma}) \setminus K$, and if p is a path in $\mathbb{C} \setminus K$ joining a and b then there is a path $\tilde{p}$ in $(\mathbb{C} \setminus \overline{\Sigma}) \setminus K$ joining a and b. Let $K = \{\zeta \in \mathbb{C} \setminus \overline{\Sigma}\colon\ |\Phi(\zeta)| \leq r\}$. Since $\Phi(b\Sigma) \subset \mathbb{C}^2 \setminus r\,\overline{\mathbb{B}}$ and since $|\Phi(\zeta)| \to +\infty$ as $|\zeta| \to +\infty$, the set K is compact. Suppose for a moment that $(\mathbb{C} \setminus \overline{\Sigma}) \setminus K$ is not connected. The preceding discussion implies that $\{\zeta \in \mathbb{C}\colon\ |\Phi(\zeta)| > r\}$ has a bounded component which contradicts the maximum principle. Thus, $(\mathbb{C}\setminus\overline{\Sigma})\setminus K$ is connected which implies that for each $q \in \Phi(\mathbb{C}) \setminus (\Phi(\overline{\Sigma}) \cup r\overline{B})$ there is a path $\eta\colon\ [0,1) \to \Phi(\mathbb{C}) \setminus (\Phi(\overline{\Sigma}) \cup r\,\overline{\mathbb{B}})$ such that $\eta(0) = q$ and $|\eta(t)| \to +\infty$ as $t \to 1$. The statement of the lemma now follows from [BF, Lemma 3.1]. This completes the proof. ☐

Remark. It is easy to see that the proof of Lemma 3.1 in [BF] works for $\mathbb{C}^N,\ N \geq 2,$ and so Lemma 5.1. holds for proper holomorphic embeddings $\Phi\colon\ \mathbb{C} \to \mathbb{C}^N,\ N \geq 2$.

Proof of the induction step, continued. We have already mentioned that for each m, $f_{m+1} = (\Psi_{m+1} \circ \Theta_{m+1} \circ A_m) \circ (G_{m+1} \circ g_m \circ p_{m+1}) = A_{m+1} \circ g_{m+1}$. Thus, $f_n = H_n \circ g_1 \circ (p_2 \circ \cdots \circ p_n)$ where $H_n = (\Psi_n \circ \Theta_n) \circ \cdots \circ (\Psi_2 \circ \Theta_2) \circ (G_n \circ \cdots \circ G_2)$ is a holomorphic automorphism of $\mathbb{C}^2$. It follows that $f_n(K_n)$ is a compact subset of $(H_n \circ g_1)(\mathbb{C})$, a closed submanifold of $\mathbb{C}^2$ biholomorphically equivalent to $\mathbb{C}$, whose boundary $f_n(bK_n)$ is a smooth Jordan curve which is, by (ii), contained in $\mathbb{C}^2 \setminus r_n\overline{\mathbb{B}}$. By Lemma 5.1 the set $f_n(K_n) \cup r_n\overline{\mathbb{B}}$ is polynomially convex. By (ii) $f_n(K_n) \cup r_n\mathbb{B}$ contains no point of $f_n((K_{n+1} \setminus K_n) \cap S)$. Since f_n is one to one it follows that $f_n(\xi) \neq f_n(\eta)$ if $\xi, \eta \in (K_{n+1} \setminus K_n) \cap S$. By (i), $\varphi((K_{n+1} \setminus K_n) \cap S)$ does not meet $r_n\overline{\mathbb{B}}$. However, some points of $\varphi((K_{n+1} \setminus K_n) \cap S)$ may lie in $f_n(K_n)$. Since $f_n(K_n)$ is contained in $(H_n \circ g_1)(\mathbb{C})$, a closed one dimensional complex submanifold of $\mathbb{C}^2$, one can change φ slightly on $K_{n+1} \cap S$ to $\tilde{\varphi}$ so that

$$|\tilde{\varphi} - \varphi| < \theta_n \quad \text{on } K_{n+1} \cap S,$$

so that $\tilde{\varphi}$ is one to one on $K_{n+1} \cap S$ and that $f_n(K_n) \cup r_n\overline{\mathbb{B}}$ contains no point of $\tilde{\varphi}((K_{n+1} \setminus K_n) \cap S)$. By [FGS] there is an automorphism Θ_{n+1} of $\mathbb{C}^2$ which fixes each point of $f_n(K_n \cap S)$, that moves each point $f_n(\zeta),\ \zeta \in (K_{n+1} \setminus K_n) \cap S$ to $\tilde{\varphi}(\zeta)$, and that satisfies

$$|\Theta_{n+1} - \mathrm{Id}| < \theta_n \quad \text{on } f_n(K_n) \cup r_n\overline{\mathbb{B}}. \tag{5.2}$$

By (iv) we have $f_n|K_n \cap S = \varphi|K_n \cap S$. Almost the same equality holds for $\Theta_{n+1} \circ f_n$ in place of f_n since $\Theta_{n+1} \circ f_n|K_{n+1} \cap S = \tilde{\varphi}|K_{n+1} \cap S$. Applying on both sides on the left an automorphism Ψ provided by Lemma 3.1 which satisfies $\Psi \circ \tilde{\varphi} = \varphi$ on $K_{n+1} \cap S$, would produce a map from Δ to $\mathbb{C}^2$ that would satisfy (iv) with n replaced by $n+1$. However, such a map does not necessarily satisfy (ii) with $n+1$ in place of n or (iii) since we have no control over what Θ_{n+1} does with $f_n(\Delta \setminus K_n)$.

6. The induction step, Part 3

We perform our induction process in such a way that

$$(6.1') \qquad A_n(\{(z,w)\colon\ |w| > T_n/2\}) \text{ misses } 2r_{n-1}\mathbb{B} \ \text{ if } n \text{ is odd.}$$

and

$$(6.1'') \qquad A_n(\{(z,w)\colon\ |z| > S_n/2\}) \text{ misses } 2r_{n-1}\mathbb{B} \ \text{ if } n \text{ is even.}$$

Recall that (6.1′) holds for $n = 1$. We are describing the induction step for odd n so assume that (6.1′) holds. To handle the problem described at the end of the previous section we replace g_n in $\Theta_{n+1} \circ A_n \circ g_n = \Theta_{n+1} \circ f_n$ by $G_{n+1} \circ g_n$ where G_{n+1} is an automorphism of $\mathbb{C}^2$ of the form (2.1′). Passing to a slightly smaller U_n if necessary we may assume that $g_n(U_n)$ is bounded. We want that G_{n+1} changes g_n only slightly on K_n and on $K_{n+1} \cap S$ while it maps $g_n(U_n \setminus \mathrm{Int}Q_{n+1})$ so far from the origin that

$$(6.2) \qquad (\Theta_{n+1} \circ A_{n+1}) \circ (G_{n+1} \circ g_n)(U_n \setminus \mathrm{Int}Q_{n+1}) \subset \mathbb{C}^2 \setminus 2r_{n+1}\overline{\mathbb{B}}$$

which, since $g_n(U_n)$ is bounded, and since $\Theta_{n+1} \circ A_{n+1}$ is an automorphism of $\mathbb{C}^2$, holds if

$$(6.3) \qquad \left| S_{n+1} \left(\frac{w}{T_n} \right)^{M_{n+1}} \right| \geq \rho_n \qquad (|w| \geq T_{n2})$$

provided that ρ_n is sufficiently large. Choose $\tau_n > 0$ so small that

$$(6.4) \quad |(\Theta_{n+1} \circ A_n)(p) - (\Theta_{n+1} \circ A_n)(q)| < \theta_n \quad (q \in g_n(P_{n+1}),\ |p - q| < 2\tau_n).$$

We want that

$$(6.5) \qquad \left| S_{n+1} \left(\frac{w}{T_n} \right)^{M_{n+1}} \right| \leq \tau_n \qquad (|w| \leq T_{n1})$$

which will imply that G_{n+1} changes g_n on P_{n+1} for at most τ_n. Let

$$S_{n+1} = \rho_n \left(\frac{T_n}{T_{n2}} \right)^{M_{n+1}}.$$

Notice that S_{n+1} is arbitrarily large provided that M_{n+1} is large enough. The choice of S_{n+1} implies (6.3) while (6.5) becomes equivalent to

$$(6.7) \qquad \rho_n \left(\frac{T_{n1}}{T_{n2}} \right)^{M_{n+1}} < \tau_n$$

which will hold provided that M_{n+1} is large enough. Choose M_{n+1} so large that S_{n+1} becomes so large that

$$(6.8) \qquad (\Theta_{n+1} \circ A_n)(\{|z| > S_{n+1}/2\}) \ \text{ misses } \ (2r_n + \varepsilon_n)\overline{B}.$$

Notice that if an automorphism $G\colon\ \mathbb{C}^2 \to \mathbb{C}^2$ satisfies $|G(z) - z| < \tau$ ($z \in R\mathbb{B}$) where $0 < \tau < R$ then $(R - \tau)\overline{\mathbb{B}} \subset G(R\overline{\mathbb{B}})$. Choose a compact set

$K'_n \subset \text{Int}K_n$ such that $f_n(\Delta \setminus K'_n) \subset f_n(\Delta \setminus K_n) + \theta_n\mathbb{B}$. Now, (ii) implies that $A_n(g_n(\Delta \setminus K'_n)) = f_n(\Delta \setminus K'_n)$ misses $(r_n - \theta_n)\overline{\mathbb{B}}$ and (5.2) implies that

$$(\Theta_{n+1} \circ A_n \circ g_n)(\Delta \setminus K'_n) \subset \mathbb{C}^2 \setminus (r_n - 2\theta_n)\overline{\mathbb{B}}. \tag{6.9}$$

By (6.5), $|G_{n+1} \circ g_n - g_n| \leq \tau_n$ on P_{n+1} so by (6.4)

$$|(\Theta_{n+1} \circ A_n \circ G_{n+1} \circ g_n)(\zeta) - (\Theta_{n+1} \circ A_n \circ g_n)(\zeta)| \leq \theta_n \quad (\zeta \in P_{n+1})$$

which, by (6.9) gives

$$(\Theta_{n+1} \circ A_n \circ G_{n+1} \circ g_n)(P_{n+1} \setminus K'_n) \subset \mathbb{C}^2 \setminus (r_n - 3\theta_n)\overline{\mathbb{B}}. \tag{6.10}$$

Let $\zeta \in Q_{n+1} \setminus P_{n+1}$. Since $g_n(Q_{n+1} \setminus P_{n+1}) \subset \{|w| > T_n/2\}$ and since G_{n+1} does not change the w coordinate we have $(G_{n+1} \circ g_n)(\zeta) \in \{|w| > T_n/2\}$ and so $(A_n \circ G_{n+1} \circ g_n)(\zeta) \in A_n(\{|w| > T_n/2\})$. By (6.1′) $A_n(\{|w| > T_n/2\})$ misses $2r_{n-1}\mathbb{B}$ which implies that $(A_n \circ G_{n+1} \circ g_n)(\zeta) \in \mathbb{C}^2 \setminus 2r_{n-1}\mathbb{B}$. By (5.2) it follows that $(\Theta_{n+1} \circ A_n \circ G_{n+1} \circ g_n)(\zeta) \in \mathbb{C}^2 \setminus s\overline{\mathbb{B}}$ where $s = \min\{r_n - \theta_n, 2r_{n-1} - \theta_n\}$, by (3.4), satisfies $s > r_{n-1} + \theta_n + \varepsilon_n$. By (6.10), (6.2) and (3.4) it follows that

$$(\Theta_{n+1} \circ A_n \circ G_{n+1} \circ g_n)(U_n \setminus K'_n) \subset \mathbb{C}^2 \setminus (r_{n-1} + \theta_n + \varepsilon_n)\overline{\mathbb{B}}. \tag{6.11}$$

7. The induction step, Part 4

Note first that $\Theta_{n+1} \circ A_n \circ g_n | K_{n+1} \cap S = \tilde{\varphi} | K_{n+1} \cap S$. This does not necessarily hold if we replace g_n by $G_{n+1} \circ g_n$. However, since all points of $K_{n+1} \cap S$ lie in P_{n+1}, since $|G_{n+1} \circ g_n - g_n| < \tau_n$ on P_{n+1} and since $|\varphi - \tilde{\varphi}| < \theta_n$ on $K_{n+1} \cap S$ it follows by (6.4) that

$$|\Theta_{n+1} \circ A_n \circ G_{n+1} \circ g_n - \varphi| < 2\theta_n \quad \text{on } K_{n+1} \cap S. \tag{7.1}$$

The problem now is that $(G_{n+1} \circ g_n)^{-1}(\{(z, w) \colon \ |z| = S_{n+1}\})$ is not necessarily equal to $b\Delta$ so we cannot use $\Theta_{n+1} \circ A_n \circ G_{n+1} \circ g_n$ as f_{n+1} even after composing with a correction automorphism provided by Lemma 3.1. However, $(G_{n+1} \circ g_n)^{-1}(\{|z| = S_{n+1}\})$ is a real analytic curve that is arbitrarily small $\mathcal{C}^1$ perturbation of $b\Delta$ independently of M_{n+1} if only S_{n+1} is large enough [G, Sec. 5]; in our case this means if only M_{n+1} is large enough.

Thus, provided that M_{n+1} is large enough the conformal map p_{n+1} mapping Δ to the domain $(G_{n+1} \circ g_n)^{-1}(\{|z| < S_{n+1}\})$ and satisfying $p_{n+1}(0) = 0$, $p'_{n+1}(0) > 0$, is arbitrarily close to the identity on Δ provided that M_{n+1} is sufficiently large [P, p. 286]. Once we have chosen M_{n+1} the map p_{n+1} extends holomorphically to a neighbourhood $U_{n+1} \subset U_n$ of $\overline{\Delta}$ so that the extended map p_{n+1} maps U_{n+1} biholomorphically onto $p_{n+1}(U_{n+1})$ and so that the map $g_{n+1} = G_{n+1} \circ g_n \circ p_{n+1} \colon \ U_{n+1} \to \mathbb{C}^2$ is transverse to $\{(z, w) \colon \ |z| = S_{n+1}\}$ and satisfies $g_{n+1}^{-1}(\{|z| = S_{n+1}\}) = b\Delta$ [G].

Passing to a larger M_{n+1} if necessary we may assume that p_{n+1} is so close to the identity on $\overline{\Delta}$ that

$$|g_n \circ p_{n+1} - g_n| < \tau_n \quad \text{on } \overline{\Delta} \tag{7.2}$$

and that

$$\begin{cases} K_n \subset p_{n+1}^{-1}(P_{n+1}), & K_{n+1} \cap S \subset p_{n+1}^{-1}(P_{n+1}) \\ p_{n+1}^{-1}(Q_{n+1}) \subset \mathrm{Int}K_{n+1}, & p_{n+1}^{-1}(K_n') \subset K_n. \end{cases} \tag{7.3}$$

Since $|G_{n+1}\circ g_n - g_n| \le \tau_n$ on P_{n+1} it follows that $|G_{n+1}\circ g_n \circ p_{n+1} - g_n \circ p_{n+1}| \le \tau_n$ on $p_{n+1}^{-1}(P_{n+1})$ which, by (7.2) and (7.3) implies that

$$|G_{n+1}\circ g_n \circ p_{n+1} - g_n| < 2\tau_n \quad \text{on} \quad K_n \cup (K_{n+1}\cap S).$$

Since $K_n \cup (K_{n+1}\cap S) \subset P_{n+1}$, (6.4) implies that

$$|(\Theta_{n+1}\circ A_n \circ G_{n+1}\circ g_n \circ p_{n+1})(\zeta) - (\Theta_{n+1}\circ A_n \circ g_n)(\zeta)| < \theta_n \ \ (\zeta \in K_n \cup (K_{n+1}\cap S)).$$

By (5.2), $|\Theta_{n+1} f_n(\zeta) - f_n(\zeta)| < \theta_n$ $(\zeta \in K_n)$ so it follows that

$$|(\Theta_{n+1}\circ A_n \circ G_{n+1}\circ g_n \circ p_{n+1})(\zeta) - f_n(\zeta)| < 2\theta_n \quad (\zeta \in K_n). \tag{7.4}$$

Further, since $(\Theta_{n+1}\circ A_n \circ g_n)|K_{n+1}\cap S = \tilde\varphi$ and since $|\tilde\varphi - \varphi| < \theta_n$ on $K_{n+1}\cap S$ it follows also that

$$|(\Theta_{n+1}\circ A_n \circ G_{n+1}\circ g_n \circ p_{n+1})(\zeta) - \varphi(\zeta)| < 3\theta_n \quad (\zeta \in K_{n+1}\cap S). \tag{7.5}$$

The choice of R and (3.3) imply that there is a holomorphic automorphism Ψ_{n+1} of $\mathbb{C}^2$ such that

$$|\Psi_{n+1} - \mathrm{Id}| < \varepsilon_n/2^{n+1} \quad \text{on} \quad R\mathbb{B} \tag{7.6}$$

and such that

$$(\Psi_{n+1}\circ\Theta_{n+1}\circ A_n \circ G_{n+1}\circ g_n \circ p_{n+1})(\zeta) = \varphi(\zeta) \quad (\zeta \in K_{n+1}\cap S). \tag{7.7}$$

Put $f_{n+1} = A_{n+1}\circ g_{n+1}$, where $A_{n+1} = \Psi_{n+1}\circ\Theta_{n+1}\circ A_n$, and $g_{n+1} = G_{n+1}\circ g_n \circ p_{n+1}$. By (7.7), (iv) is satisfied with $n+1$ in place of n. Since $\theta_n < \varepsilon_n/2^{n+2}$ and since $f_n(K_n) + \mathbb{B} \subset R\mathbb{B}$, (7.4) and (7.6) imply that $|f_{n+1}(\zeta) - f_n(\zeta)| < 2\theta_n + \varepsilon_n/2^{n+1} < \varepsilon_n/2^n$ $(\zeta \in K_n)$ so that (v) is satisfied.

By (7.3), $\zeta \in \Delta \setminus \mathrm{Int}K_{n+1}$ implies that $p_{n+1}(\zeta) \in U_n \setminus Q_{n+1}$ which, by (6.2) implies that $(\Theta_{n+1}\circ A_n \circ g_{n+1})(\zeta) \in \mathbb{C}^2 \setminus 2r_{n+1}\overline{\mathbb{B}}$. By (7.6), by the fact that $R > 2r_{n+1}$ and by (3.2) it follows that $f_{n+1}(\zeta) \in \mathbb{C}^2 \setminus (2r_{n+1} - \varepsilon_n/2^{n+1})\mathbb{B} \subset \mathbb{C}^2 \setminus (2r_{n+1} - r_1)\mathbb{B} \subset \mathbb{C}^2 \setminus (r_{n+1}\overline{\mathbb{B}})$. Thus (ii) holds with n replaced by $n+1$.

By (6.11)

$$(\Theta_{n+1}\circ A_n \circ G_{n+1}\circ g_n)(U_n \setminus K_n') \subset \mathbb{C}^2 \setminus (r_{n-1} + \theta_n + \varepsilon_n)\overline{\mathbb{B}}.$$

If $\zeta \in \Delta \setminus K_n$ then, by (7.3), $p_{n+1}(\zeta) \in p_{n+1}(\Delta) \setminus K_n' \subset U_n \setminus K_n'$ so

$$(\Theta_{n+1}\circ A_n \circ g_{n+1})(\Delta \setminus K_n) \subset \mathbb{C}^2 \setminus (r_{n-1} + \theta_n + \varepsilon_n)\overline{\mathbb{B}},$$

and since $R_{n-1} + \theta_n + \varepsilon_n < R$ it follows by (7.6) that $f_{n+1}(\Delta \setminus K_n) \subset \mathbb{C}^2 \setminus r_{n-1}\overline{\mathbb{B}}$, that is, (iii) is satisfied.

Finally, (6.8) implies that

$$(\Psi_{n+1}\circ\Theta_{n+1}\circ A_n)(\{|z| > S_{n+1}/2\}) \text{ misses } \Psi_{n+1}((2r_n + \varepsilon_n)\overline{\mathbb{B}}).$$

Since $2r_n + \varepsilon_n < R$, (7.6) implies that $2r_n\mathbb{B} \subset \Psi_{n+1}((2r_n + \varepsilon_n)\mathbb{B})$ so $A_{n+1}(\{|z| > S_{n+1}/2\})$ misses $2r_n\mathbb{B}$, that is, (6.1″) holds with n replaced by $n+1$.

This completes the proof of the induction step.

Since the map φ is proper, (vii) and the fact that (3.1) holds for every n imply that $r_n \to +\infty$ as $n \to \infty$. The proof of Theorem 1.1 is complete. □

8. Remarks

Theorem 1.1 holds with $\mathbb{C}^2$ replaced by $\mathbb{C}^N$, $N \geq 2$.

Theorem 8.1 *Let $N \geq 2$. Given a discrete set $S \subset \Delta$ and a proper injection $\varphi\colon\ S \to \mathbb{C}^N$ there is a proper holomorphic embedding $f\colon\ \Delta \to \mathbb{C}^N$ that extends φ.*

If $N \geq 3$ then one proves Theorem 8.1. as in the case $N = 2$ with a slight modification: Let $\iota\colon\ \mathbb{C}^2 \to \mathbb{C}^N$ be the standard embedding $\iota(\zeta_1, \zeta_2) = (\zeta_1, \zeta_2, 0, \cdots, 0)$. In the proof we replace $f_n = A_n \circ g_n$ by $f_n = A_n \circ \iota \circ g_n$ where A_n is a holomorphic automorphism of $\mathbb{C}^N$ and g_n, as in the proof in the case $N = 2$, is a one to one and regular holomorphic map from an open neighbourhood U_n of $\overline{\Delta}$ to $\mathbb{C}^2$ which, for even n is transverse to $\{(z,w)\colon\ |z| = S_n\}$ and satisfies $g_n^{-1}(\{|z| = S_n\}) = b\Delta$, and for odd n, is transverse to $\{(z,w)\colon\ |w| = T_n\}$ and satisfies $g_n^{-1}(\{|w| = T_n\}) = b\Delta$. Also, in the induction step, the maps Θ_{n+1} and Ψ_{n+1} are automorphisms of $\mathbb{C}^N$ and G_{n+1} is an automorphism of $\mathbb{C}^2$. In (6.1′) and (6.1″) we replace A_n by $A_n \circ \iota$.

We say that two proper holomorphic embeddedings $f_1, f_2\colon\ \Delta \to \mathbb{C}^N$ are $\mathrm{Aut}(\mathbb{C}^N)$-*equivalent* if there is an automorphism $\Psi\colon\ \mathbb{C}^N \to \mathbb{C}^N$ such that $f_2 = \Psi \circ f_1$.

Corollary 8.2 *Let $N \geq 2$. The set of* $\mathrm{Aut}(\mathbb{C}^N)$*-equivalence classes of proper holomorphic embeddedings of Δ into $\mathbb{C}^N$ is uncountable.*

Proof. [BFo] It is known [RR, Remark 5.2] that there is an uncountable family E of discrete injective sequences in $\mathbb{C}^N$ such that if $\{z_n,\ n \in \mathbb{N}\}$, $\{w_n,\ n \in \mathbb{N}\}$ are distinct elements of E then there is no automorphism Ψ of $\mathbb{C}^N$ such that $\Psi(z_n) = w_n$ $(n \in \mathbb{N})$. Let $\{\zeta_n\} \subset \Delta$ be an injective sequence, $\lim_{n\to\infty} |\zeta_n| = 1$, and let $\{z_n,\ n \in \mathbb{N}\}$, $\{w_n,\ n \in \mathbb{N}\}$ be distinct elements of E. By Theorem 8.1 there are proper holomorphic embeddings $f_1, f_2\colon\ \Delta \to \mathbb{C}^N$ such that $f_1(\zeta_j) = z_j$, $f_2(\zeta_j) = w_j$ $(j \in \mathbb{N})$. Every automorphism Ψ of $\mathbb{C}^N$ such that $f_2 = \Psi \circ f_1$ would have to satisfy $\Psi(z_n) = w_n$ $(n \in \mathbb{N})$ and there is no such Ψ. Thus, in this way, each element of E produces a proper holomorphic embedding of Δ into $\mathbb{C}^N$ and the embeddings associated with distinct elements of E are not $\mathrm{Aut}(\mathbb{C}^N)$-equivalent. This completes the proof. □

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