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THE GROTHENDIECK RING OF VARIETIES IS NOT A DOMAIN

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Abstract. If k is a field, the ring $K_0(\mathcal{V}_k)$ is defined as the free abelian group generated by the isomorphism classes of geometrically reduced k-varieties modulo the set of relations of the form $[X - Y] = [X] - [Y]$ whenever Y is a closed subvariety of X. The multiplication is defined using the product operation on varieties. We prove that if the characteristic of k is zero, then $K_0(\mathcal{V}_k)$ is not a domain.

1. The Grothendieck ring of varieties

Let k be a field. By a k-variety we mean a geometrically reduced, separated scheme of finite type over k. Let $\mathcal{V}_k$ denote the category of k-varieties. Let $K_0(\mathcal{V}_k)$ denote the free abelian group generated by the isomorphism classes of k-varieties, modulo all relations of the form $[X - Y] = [X] - [Y]$ where Y is a closed k-subvariety of a k-variety X. Here, and from now on, $[X]$ denotes the class of X in $K_0(\mathcal{V}_k)$. The operation $[X] \cdot [Y] := [X \times_k Y]$ is well-defined, and makes $K_0(\mathcal{V}_k)$ a commutative ring with 1. It is known as the *Grothendieck ring of k-varieties*. A completed localization of $K_0(\mathcal{V}_k)$ is needed for the theory of *motivic integration*, which has many applications: see [Loo00] for a survey.

Our main result is the following.

Theorem 1. *Suppose that k is a field of characteristic zero. Then $K_0(\mathcal{V}_k)$ is not a domain.*

Remark. We conjecture that the result holds also for fields k of characteristic p. But we use a result whose proof relies on resolution of singularities and weak factorization of birational maps, which are known only in characteristic zero.

2. Abelian varieties of GL_2-type

If A is an abelian variety over a field k_0, and k is a field extension of k_0, then $\mathrm{End}_k(A)$ denotes the endomorphism ring of the base extension $A_k := A \times_{k_0} k$, that is, the ring of endomorphisms defined over k.

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Lemma 2. *Let k be a field of characteristic zero, and let $\overline{k}$ denote an algebraic closure. There exists an abelian variety A over k such that* $\mathrm{End}_k(A) = \mathrm{End}_{\overline{k}}(A) \simeq \mathcal{O}$, *where $\mathcal{O}$ is the ring of integers of a number field of class number* 2.

Let us precede the proof of Lemma 2 with a few paragraphs of motivation. Our strategy will be to find a single abelian variety A over $\mathbb{Q}$ such that the base extension A_k works over k.

Let A be a simple abelian variety over $\mathbb{Q}$. Let $E = \mathrm{End}_{\mathbb{Q}}(A) \otimes \mathbb{Q}$. Since A is simple, E is a division algebra. The Lie algebra $\mathrm{Lie}\, A$ is a nonzero left E-vector space, so $[E : \mathbb{Q}] \le \dim_{\mathbb{Q}} \mathrm{Lie}\, A = \dim A$. If equality holds and E is commutative (hence a number field), then A is said to be of GL_2-*type*. (The terminology is due to the following: If A is of GL_2-type, then the action of the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on a Tate module $V_\ell A$ can be viewed as a representation $\rho_\ell : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(E \otimes \mathbb{Q}_\ell)$.)

Because $\mathbb{Q}$ has class number 1, we must take $[E : \mathbb{Q}] \ge 2$ to find an A over $\mathbb{Q}$ as in Lemma 2. The inequality $\dim A \ge [E : \mathbb{Q}]$ then forces $\dim A \ge 2$. Moreover, if we want $\dim A = 2$, then A must be of GL_2-type.

Abelian varieties of GL_2-type are closely connected to modular forms. For each $N \ge 1$, let $\Gamma_1(N)$ denote the classical modular group, let $X_1(N)$ denote the corresponding modular curve over $\mathbb{Q}$, and let $J_1(N)$ be the Jacobian of $X_1(N)$. G. Shimura, in Theorem 1 of [Shi73], attached to each weight-2 newform f on $\Gamma_1(N)$ an abelian variety quotient A_f of $J_1(N)$. (Previously, in Theorem 7.14 of [Shi71], he had attached to f an abelian *subvariety* of $J_1(N)$.) Let E_f be the number field generated over $\mathbb{Q}$ by the Fourier coefficients of f. Theorem 1 of [Shi73] shows also that $\dim A_f = [E_f : \mathbb{Q}]$, and that there is an injective $\mathbb{Q}$-algebra homomorphism $\theta : E_f \hookrightarrow E := \mathrm{End}_{\mathbb{Q}}(A_f) \otimes \mathbb{Q}$ mapping each Fourier coefficient to the endomorphism of A_f induced by the associated Hecke correspondence on $X_1(N)$. Corollary 4.2 of [Rib80] proves that θ is an isomorphism. It follows that A_f is of GL_2-type.

Conversely, it is conjectured that each simple abelian variety over $\mathbb{Q}$ of GL_2-type is $\mathbb{Q}$-isogenous to some A_f. See [Rib92] for more details. The $\dim A = 1$ case of this conjecture is the statement that elliptic curves over $\mathbb{Q}$ are modular, which is known [BCDT01].

Therefore we are led to consider A_f of dimension 2, where f is a newform as above.

Proof of Lemma 2. Tables [Ste] show that there exists a weight-2 newform $f = \sum_{n=1}^{\infty} a_n q^n$ on $\Gamma_0(590)$ (hence also on $\Gamma_1(590)$) such that $E_f = \mathbb{Q}(\sqrt{10})$ and $a_3 = \sqrt{10}$. Let $A = A_f$ be the corresponding abelian variety over $\mathbb{Q}$. Then $\dim A = [E_f : \mathbb{Q}] = 2$. But $\mathrm{End}_{\mathbb{Q}}(A)$ is an order of $E = E_f$ containing $a_3 = \sqrt{10}$, so $\mathrm{End}_{\mathbb{Q}}(A)$ is the maximal order $\mathbb{Z}[\sqrt{10}]$ of E. Since 590 is squarefree, A is semistable over $\mathbb{Q}$ by Theorem 6.9 of [DR73], and then Corollary 1.4(a) of [Rib75] shows that all endomorphisms of A over any field extension k of $\mathbb{Q}$ are defined over $\mathbb{Q}$. Finally, the class number of $\mathbb{Z}[\sqrt{10}]$ is 2. □

Remarks.

1. After one knows that $\mathrm{End}_{\mathbb{Q}}(A) = \mathbb{Z}[\sqrt{10}]$, another way to prove $\mathrm{End}_{\overline{\mathbb{Q}}}(A) = \mathbb{Z}[\sqrt{10}]$ is to use the fact that $\mathrm{End}_{\overline{\mathbb{Q}}}(A)$ injects into the endomorphism ring of the reduction A_p over $\overline{\mathbb{F}}_p$ for any prime p not dividing 590. The latter endomorphism rings can be computed using Eichler-Shimura theory and Honda-Tate theory. Combining the information from a few primes p yields the result.
2. The smallest N for which there exists a newform f on $\Gamma_0(N)$ with E_f of class number 2 is 276. The advantage of 590 is that it is squarefree. (In fact, our original proof applied the technique in the previous remark at level 276.)
3. The case $k = \mathbb{C}$ of Lemma 2 has an easy proof: let A be an elliptic curve over $\mathbb{C}$ with complex multiplication by $\mathbb{Z}[\sqrt{-5}]$.

3. Abelian varieties and projective modules

Let A be an abelian variety over a field k, and let $\mathcal{O} = \mathrm{End}_k(A)$. Given a finite-rank projective right $\mathcal{O}$-module M, we define an abelian variety $M \otimes_{\mathcal{O}} A$ as follows: choose a finite presentation $\mathcal{O}^m \to \mathcal{O}^n \to M \to 0$, and let $M \otimes_{\mathcal{O}} A$ be the cokernel of the homomorphism $A^m \to A^n$ defined by the matrix that gives $\mathcal{O}^m \to \mathcal{O}^n$. It is straightforward to check that this is independent of the presentation, and that $M \mapsto (M \otimes_{\mathcal{O}} A)$ defines a fully faithful functor T from the category of finite-rank projective right $\mathcal{O}$-modules to the category of abelian varieties over k. (Essentially the same construction is discussed in the appendix by J.-P. Serre in [Lau01].)

Lemma 3. *Let k be a field of characteristic zero. There exist abelian varieties A and B over k such that $A \times A \simeq B \times B$ but $A_{\overline{k}} \not\simeq B_{\overline{k}}$.*

Proof. Let A and $\mathcal{O}$ be as in Lemma 2. Let I be a nonprincipal ideal of $\mathcal{O}$. Since $\mathcal{O}$ is a Dedekind domain, the isomorphism type of a direct sum of fractional ideals $I_1 \oplus \ldots \oplus I_n$ is determined exactly by the nonnegative integer n and the product of the classes of the I_i in the class group $\mathrm{Pic}(\mathcal{O})$. Since $\mathrm{Pic}(\mathcal{O}) \simeq \mathbb{Z}/2$, we have $\mathcal{O} \oplus \mathcal{O} \simeq I \oplus I$ as $\mathcal{O}$-modules. Applying the functor T yields $A \times A \simeq B \times B$, where $B := I \otimes_{\mathcal{O}} A$. Since $\mathrm{End}_{\overline{k}}(A)$ also equals $\mathcal{O}$, we have $B_{\overline{k}} = I \otimes_{\mathcal{O}} A_{\overline{k}}$. Since T for $\overline{k}$ is fully faithful, $A_{\overline{k}} \not\simeq B_{\overline{k}}$. □

4. Stable birational classes and Albanese varieties

For any extension of fields $k \subseteq k'$, there is a ring homomorphism $K_0(\mathcal{V}_k) \to K_0(\mathcal{V}_{k'})$ mapping $[X]$ to $[X_{k'}]$.

Let k be a field of characteristic zero. Smooth, projective, geometrically integral k-varieties X and Y are called *stably birational* if $X \times \mathbb{P}^m$ is birational to $Y \times \mathbb{P}^n$ for some integers $m, n \geq 0$. The set SB_k of equivalence classes of this relation is a monoid under product of varieties over k. Let $\mathbb{Z}[\mathrm{SB}_k]$ denote the corresponding monoid ring.

When $k = \mathbb{C}$, there is a unique ring homomorphism $K_0(\mathcal{V}_k) \to \mathbb{Z}[\mathrm{SB}_k]$ mapping the class of any smooth projective integral variety to its stable birational class [LL01]. (In fact, this homomorphism is surjective, and its kernel is the ideal generated by $\mathbb{L} := [\mathbb{A}^1]$.) The proof in [LL01] requires resolution of singularities and weak factorization of birational maps [AKMW00, Theorem 0.1.1], [Wło01, Conjecture 0.0.1]. The same proof works over any algebraically closed field of characteristic zero.

The set AV_k of isomorphism classes of abelian varieties over k is a monoid. The Albanese functor mapping a smooth, projective, geometrically integral variety to its Albanese variety induces a homomorphism of monoids $\mathrm{SB}_k \to \mathrm{AV}_k$, since the Albanese variety is a birational invariant, since formation of the Albanese variety commutes with products, and since the Albanese variety of $\mathbb{P}^n$ is trivial. Therefore we obtain a ring homomorphism $\mathbb{Z}[\mathrm{SB}_k] \to \mathbb{Z}[\mathrm{AV}_k]$.

5. Zerodivisors

Proof of Theorem 1. Let A and B be as in Lemma 3. Then $([A]+[B])([A]-[B]) = 0$ in $K_0(\mathcal{V}_k)$. On the other hand, $[A]+[B]$ and $[A]-[B]$ are nonzero, because their images under the composition

$$K_0(\mathcal{V}_k) \to K_0(\mathcal{V}_{\overline{k}}) \to \mathbb{Z}[\mathrm{SB}_{\overline{k}}] \to \mathbb{Z}[\mathrm{AV}_{\overline{k}}]$$

are nonzero. (The Albanese variety of an abelian variety is itself.) □

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