ON THE QUASI-EXPONENT OF FINITE-DIMENSIONAL HOPF ALGEBRAS

PAVEL ETINGOF AND SHLOMO GELAKI

1. Introduction

In [EG] we introduced and studied a new invariant of Hopf algebras – the exponent. This invariant generalizes the exponent of a group (the least common multiple of orders of its elements). It has a number of interesting properties, and has found some applications [KSZ].

More specifically, consider Hopf algebras over $\mathbb{C}$. Then the exponent has a very nice behavior for *semisimple* Hopf algebras. Namely, it is shown in [EG] that in this case the exponent is finite and divides the cube of the dimension. On the contrary, for a non-semisimple Hopf algebra, the exponent is usually (and possibly always) infinite.

Thus, one is tempted to look for a more refined invariant of Hopf algebras, which would coincide with the exponent in the semisimple case, but would be finite if the Hopf algebra is finite-dimensional. This problem is solved in the present paper, by introduction of the *quasi-exponent*.

Recall from [EG] that the exponent of a finite-dimensional Hopf algebra H is the order of the Drinfeld element u of the Drinfeld double $D(H)$ of H. Recall also that while this order may be infinite, the eigenvalues of u are always roots of unity ([EG, Theorem 4.8]); i.e., some power of u is always unipotent. We are thus naturally led to define the *quasi-exponent* of a finite-dimensional Hopf algebra H to be the order of unipotency of u.

The goal of the paper is to create a theory of quasi-exponent, which would be parallel to the theory of the exponent developed in [EG]. In particular, similarly to [EG], we give two other equivalent definitions of the quasi-exponent, and prove that it is invariant under twisting. Furthermore, we prove that the quasi-exponent of a finite-dimensional pointed Hopf algebra H is equal to the exponent of the group $G(H)$ of grouplike elements of H. (In particular, the order of the squared antipode of H divides $\exp(G(H))$.) As an application, we find that if H is obtained by twisting the quantum group at root of unity $U_q(\mathfrak{g})$ then the order of any grouplike element in H divides the order p of q. Moreover, we prove that if $\mathfrak{g}$ is simple and p is sufficiently large and prime then the group $G(H)$ is an elementary abelian $p-$group, and if $\mathfrak{g}$ is one of the classical Lie algebras then the rank of $G(H)$ is at most the rank of $\mathfrak{g}$.

In this paper, all Hopf algebras will be over the field of complex numbers $\mathbb{C}$, unless otherwise is specified.

2. Definition and elementary properties of quasi-exponent

Let H be a complex finite-dimensional Hopf algebra with multiplication map m, comultiplication map Δ, unit map ι, counit map ε and antipode S. Recall that the Drinfeld double $D(H) = H^{*cop} \otimes H$ of H is a quasitriangular Hopf algebra with universal $R-$matrix $R = \sum_i h_i \otimes h_i^*$, where $\{h_i\}$, $\{h_i^*\}$ are dual bases for H and H^* respectively. Let

$$(1) \qquad\qquad u := m_{21}(Id \otimes S)(R) = \sum_i S(h_i^*)h_i,$$

be the Drinfeld element of $D(H)$, where Id denotes the identity map $H \to H$, S is the antipode of $D(H)$ and m_{21} is the composition of the usual permutation map with the multiplication map m of $D(H)$. Sometimes we shall use the notation u_H for emphasis. By [D],

$$(2) \qquad\qquad S^2(x) = uxu^{-1}, \ x \in D(H).$$

We start by making the following definition.

Definition 2.1. Let H be a finite-dimensional Hopf algebra over $\mathbb{C}$. The quasi-exponent of H, denoted by $\mathrm{qexp}\,(H)$, is the smallest positive integer n such that u^n is unipotent; that is, the smallest positive integer n such that $(1 - u^n)^N = 0$ for some positive integer N.

Remark 2.2. Let H be a finite-dimensional Hopf algebra over $\mathbb{C}$. Then the following hold:

1. If H is semisimple, $\mathrm{qexp}\,(H) = \exp(H)$ by [EG, Theorem 4.3].
2. By [EG, Theorem 4.8], $\mathrm{qexp}\,(H) < \infty$.
3. If $\exp(H) < \infty$ then $\mathrm{qexp}\,(H) = \exp(H)$.

Remark 2.3. Our introduction and study of the exponent of Hopf algebras in [EG] were motivated by Kashina's works [K1,K2] in which she studies the exponent of Hopf algebras whose antipode is involutive. In fact, for such Hopf algebras the notion of exponent has existed for over 30 years. For example, Kashina gives an elementary Hopf-algebraic proof of the fact that if H is either commutative or cocommutative of dimension n over any field, then its exponent is at most n. This statement was originally proved in the 60-s using the language of affine group schemes (see [Ka1]).

Let $m_0 = \varepsilon$, $m_1 = Id$, $\Delta_0 = \iota$ and $\Delta_1 = Id$, and for any integer $n \geq 2$ let $m_n : H^{\otimes n} \to H$ and $\Delta_n : H \to H^{\otimes n}$ be defined by $m_n = m \circ (m_{n-1} \otimes Id)$, and $\Delta_n = (\Delta_{n-1} \otimes Id) \circ \Delta$. Let $T_n : H \to H$ be the linear map given by

$$T_n(h) = m_n \circ (Id \otimes S^{-2} \otimes \cdots \otimes S^{-2n+2}) \circ \Delta_n(h)$$

and set

$$R_n := R(Id \otimes S^2)(R) \cdots (Id \otimes S^{2n-2})(R).$$

Proposition 2.4. *Let H be a finite-dimensional Hopf algebra, and $f(t) = \sum_{i=0}^{m} a_i t^i$ be a polynomial in $\mathbb{C}[t]$. The following conditions are equivalent:*

1. $f(u) = 0$.
2. $\sum_{i=0}^{m} a_i T_i = 0$.
3. $\sum_{i=0}^{m} a_i R_i = 0$.

Proof. $(3) \Rightarrow (1)$: By a simple induction on i, $m_{21}(Id \otimes S)(R_i) = u^i$ (here we use equation (2)). Hence (1) follows by applying $m_{21}(Id \otimes S)$ to the equation $\sum_{i=0}^{m} a_i R_i = 0$.

$(2) \Leftrightarrow (3)$: It is straightforward to show that the Hexagon axiom for R implies that $(\Delta_n \otimes Id)(R) = R_{1,n+1}R_{2,n+1}\cdots R_{n,n+1}$. Therefore, $(T_i \otimes Id)(R) = R_i$. Thus applying (2) to the first component of R we get (3). Conversely, the equation $(\sum_{i=0}^{m} a_i T_i \otimes Id)(R) = 0$ implies (2).

$(1) \Rightarrow (3)$: Since $m_{21}(Id \otimes S)(R_i) = u^i$, one has $m_{21}(Id \otimes S)(\sum_{i=0}^{m} a_i R_i) = 0$. But the multiplication map $H^* \otimes H \to D(H)$ is a linear isomorphism [D], so $\sum_{i=0}^{m} a_i R_i = 0$, as desired. $\square$

Proposition 2.4 allows us to give two equivalent definitions of the quasi-exponent.

Corollary 2.5. *Let H be a finite-dimensional Hopf algebra over $\mathbb{C}$. Then:*

1. $\sum_{k=0}^{N}(-1)^k \begin{pmatrix} N \\ k \end{pmatrix} R_{nk} = 0$ *for some positive integer N if and only if n is divisible by* $\operatorname{qexp}(H)$.
2. $\sum_{k=0}^{N}(-1)^k \begin{pmatrix} N \\ k \end{pmatrix} T_{nk} = 0$ *for some positive integer N if and only if n is divisible by* $\operatorname{qexp}(H)$.

Thus, $\operatorname{qexp}(H)$ *may be defined as the smallest positive integer satisfying either (1) or (2).*

Let us now list some of the elementary properties of $\operatorname{qexp}(H)$.

Proposition 2.6. *Let H be a finite-dimensional Hopf algebra over $\mathbb{C}$. Then:*

1. *The quasi-exponents of Hopf subalgebras and quotients of H divide* $\operatorname{qexp}(H)$.
2. *The order of any grouplike element g of H divides* $\operatorname{qexp}(H)$.
3. $\operatorname{qexp}(H^*) = \operatorname{qexp}(H)$.
4. $\operatorname{qexp}(H_1 \otimes H_2)$ *equals the least common multiple of* $\operatorname{qexp}(H_1)$ *and* $\operatorname{qexp}(H_2)$.
5. *The order of S^2 divides* $\operatorname{qexp}(H)$ *(in particular, $u^{\operatorname{qexp}(H)}$ is central in $D(H)$).*
6. *If* $\operatorname{qexp}(H) = 1$ *then $H = \mathbb{C}$.*
7. $\operatorname{qexp}(H^{*cop}) = \operatorname{qexp}(H)$.

Proof. (1) Follows from part (2) of Corollary 2.5.
(2) The order of g divides $\exp(G(H)) = \exp(\mathbb{C}[G(H)])$ which divides $\operatorname{qexp}(H)$ by part (1).
(3) Clear from part (2) of Corollary 2.5, as T_n for H^* and T_n for H are dual maps.

(4) Clearly, $u_{H_1 \otimes H_2} = u_{H_1} \otimes u_{H_2}$. Hence if $u_{H_1}^n$, $u_{H_2}^n$ are both unipotent then so is $u_{H_1 \otimes H_2}^n$. Conversely, if $u_{H_1 \otimes H_2}^n$ is unipotent, so is the operator $u_{H_1 \otimes H_2}^n$ restricted to $D(H_1) \otimes \mathbb{C}\lambda$, where λ is a non-zero left integral of $D(H_2)$. But this operator equals $u_{H_1}^n$ since $\varepsilon(u) = 1$. Similarly, $u_{H_2}^n$ is unipotent as well.

(5) Let $n := \mathrm{qexp}\,(H)$. Since $S^{2n} = \mathrm{Ad}u^n$, S^{2n} is unipotent. However, it is semisimple by [R], as it is of finite order. Thus, $S^{2n} = Id$.

(6) By part (5), $S^2 = Id$, so by [LR], H is semisimple. But then $1 = \mathrm{qexp}\,(H) = \exp(H)$ which implies that $Id = \varepsilon$, and hence that $H = \mathbb{C}$, as desired.

(7) Since $(D(H^{*cop}), R) \cong (D(H)^{op}, R_{21})$ as quasitriangular Hopf algebras, it follows that $u_H = u_{H^{*cop}}$. $\square$

Example 2.7. Let H be Sweedler's $4-$dimensional non-semisimple Hopf algebra generated by the grouplike element g and the skew-primitive element x with $\Delta(x) = x \otimes g + 1 \otimes x$ [Sw]. Using the basis $\{1, g, x, gx\}$ of H it is straightforward to verify that $\varepsilon - 2m_2(Id \otimes S^{-2})\Delta_2 + m_4(Id \otimes S^{-2} \otimes S^{-4} \otimes S^{-6})\Delta_4 = 0$. Therefore, $\mathrm{qexp}\,(H)$ divides 2. Since it is not equal to 1, we have $\mathrm{qexp}\,(H) = 2 = \exp(G(H))$. In Section 4 we will generalize this to any finite-dimensional pointed Hopf
algebra.

3. Invariance of quasi-exponent under twisting

In this section we study the invariance of $\mathrm{qexp}\,(H)$ under twisting. But first let us recall Drinfeld's notion of a twist for Hopf algebras for the convenience of the reader.

Definition 3.1. Let H be a Hopf algebra. A twist for H is an invertible element $J \in H \otimes H$ which satisfies

$$(\Delta \otimes Id)(J)(J \otimes 1) = (Id \otimes \Delta)(J)(1 \otimes J) \ \ and \ \ (\varepsilon \otimes Id)(J) = (Id \otimes \varepsilon)(J) = 1.$$

Given a twist J for H, one can construct a new Hopf algebra H^J, which is the same as H as an algebra, with coproduct Δ^J given by

$$\Delta^J(x) = J^{-1}\Delta(x)J, \ x \in H,$$

and antipode S^J given by

$$S^J(x) = Q^{-1}S(x)Q, \ x \in H,$$

where $Q = m \circ (S \otimes Id)(J)$ and $Q^{-1} = m \circ (Id \otimes S)(J^{-1})$.

If (A, R) is quasitriangular then so is A^J with the universal R-matrix

$$R^J := J_{21}^{-1}RJ.$$

In particular, since H is a Hopf subalgebra of $D(H)$, we can twist $D(H)$ using the twist J, considered as an element of $D(H) \otimes D(H)$, and obtain $(D(H)^J, R^J)$. In [EG, Proposition 3.2] we proved that there exists a canonical isomorphism of quasitriangular Hopf algebras between $(D(H)^J, R^J)$ and $(D(H^J), R^{(J)})$ (where $R^{(J)}$ is the universal $R-$matrix of $D(H^J)$); namely, the span of the first components of R^J is equal to the Hopf subalgebra H^J of $D(H)^J$, the span of the second

components of R^J is a Hopf subalgebra of $D(H)^J$ which is naturally isomorphic to $(H^J)^*$ and the multiplication map $H^J \otimes (H^J)^* \to D(H)^J$ induces a Hopf algebra isomorphism between $D(H^J)$ and $D(H)^J$. We thus can identify $D(H)^J$ with $D(H^J)$, and using this identification it is straightforward to check that the Drinfeld element of $(D(H^J), R^J)$ is given by

$$(3) \qquad u^J = Q^{-1}S(Q)u.$$

We start with the following useful result.

Proposition 3.2. *Let H be a finite-dimensional Hopf algebra, and let $g \in G(H)$ be such that gu^n is unipotent. Then $g = 1$.*

Proof. First note that g and u commute by equation (2). Therefore, $(1-gu^n)^N = 0$ for some integer $N > 0$ is equivalent to $\sum_{k=0}^{N}(-1)^k \begin{pmatrix} N \\ k \end{pmatrix} u^{nk}g^k = 0$, which in turn is equivalent to $\sum_{k=0}^{N}(-1)^k \begin{pmatrix} N \\ k \end{pmatrix} R_{nk}(g^k \otimes 1) = 0$ by Proposition 2.4(3). Now, apply $1 \otimes \varepsilon$ to the last equation to get $\sum_{k=0}^{N}(-1)^k \begin{pmatrix} N \\ k \end{pmatrix} g^k = 0$, i.e., $(1 - g)^N = 0$. However, $1 - g$ is semisimple, hence $g = 1$, as desired. $\qquad \square$

As a corollary we can prove our first main result.

Theorem 3.3. *Let H be a finite-dimensional Hopf algebra and let J be a twist for H. Set $n := \mathrm{qexp}\,(H)$. Then $u^n = (u^J)^n$, and in particular $\mathrm{qexp}\,(H^J) = n$.*

Proof. Recall the formula

$$(4) \qquad \Delta(Q^{-1}S(Q)) = J(Q^{-1}S(Q) \otimes Q^{-1}S(Q))(S^2 \otimes S^2)(J^{-1})$$

(see e.g. [AEGN]). Set $g := Q^{-1}S(Q)S^2(Q^{-1})S^3(Q) \cdots S^{2n-2}(Q^{-1})S^{2n-1}(Q)$. Since $S^{2n} = Id$, we have by equation (4) that g is a grouplike element in H^J. By equation (3), $g^{-1}(u^J)^n = u^n$, so $g^{-1}(u^J)^n$ is unipotent, and the result follows from Proposition 3.2. $\qquad \square$

Remark 3.4. Theorem 3.3 shows that $\mathrm{qexp}\,(H)$ is an invariant of the tensor category of representations of H. It would thus be interesting to develop a theory of quasi-exponent for an arbitrary tensor category.

Let us point out a direct corollary of Theorem 3.3, which may be of use in future applications.

Corollary 3.5. *Let H be a finite-dimensional Hopf algebra and J be a twist for H. For any grouplike element g in H^J, $g^{\mathrm{qexp}\,(H)} = 1$.*

As another consequence we obtain the following.

Corollary 3.6. *Let H be a finite-dimensional Hopf algebra. Then $\mathrm{qexp}\,(D(H)) = \mathrm{qexp}\,(H)$.*

Proof. By Proposition 2.6(3), $\text{qexp}(D(H)) = \text{qexp}(D(H)^*)$. Now, it is known that $D(H)^* = (H^{op} \otimes H^*)^R$ is obtained by twisting the Hopf algebra $H^{op} \otimes H^*$ using the $R-$matrix R of $D(H)$. So by Theorem 3.3, $\text{qexp}(D(H)) = \text{qexp}(H^{op} \otimes H^*)$. However, by Proposition 2.6(4),(7), $\text{qexp}(H^{op} \otimes H^*) = \text{qexp}(H)$, and we are done. □

4. Quasi-exponent of finite-dimensional pointed Hopf algebras

Recall that a Hopf algebra is called pointed if all its irreducible corepresentations are $1-$dimensional (e.g. quantum universal enveloping algebras and quantum groups at roots of unity). In this section we calculate the quasi-exponent of finite-dimensional pointed Hopf algebras.

Lemma 4.1. *Let X be an affine connected algebraic variety, and let H be a Hopf algebra over the coordinate ring $\mathbb{C}[X]$ of X, which is a finitely generated free module of rank r. For any $x \in X$ let $I_x \subseteq \mathbb{C}[X]$ be the corresponding maximal ideal, and set $H_x := H/I_x H$; it is a Hopf algebra of dimension r over $\mathbb{C}$. Then $\text{qexp}(H_x)$ is a constant which does not depend on the point x.*

Proof. Let u_x be the Drinfeld element of $D(H_x)$. Then we know the eigenvalues of u_x are roots of unity (see [EG]), hence the coefficients of the characteristic polynomial of u_x are algebraic numbers. However, they must also continuously depend on x. This means they are constant, and so the spectrum of u_x is independent on x, as desired. □

Lemma 4.2. *Let H be a $\mathbb{Z}_+-$filtered Hopf algebra and let grH be its associated graded Hopf algebra. Then $\text{qexp}(H) = \text{qexp}(grH)$.*

Proof. It is well known that there exists a Hopf algebra $\overline{H}$ over $\mathbb{C}[t]$ such that $\overline{H}_0 = grH$ and $\overline{H}_t = H$ for all $t \neq 0$. Thus the lemma follows from Lemma 4.1. □

Proposition 4.3. *Let H be a $\mathbb{Z}_+-$graded Hopf algebra with zero part H_0. Then $\text{qexp}(H) = l.c.m.(\text{qexp}(H_0), |S^2|)$, where $|S^2|$ denotes the order of S^2 in H.*

Proof. Set $n := l.c.m.(\text{qexp}(H_0), |S^2|)$, and let R_0 be the universal $R-$matrix of $D(H_0)$. Since $\text{qexp}(H_0)$ divides n, one has $\sum_{k=0}^{N}(-1)^k \begin{pmatrix} N \\ k \end{pmatrix} (R_0)_{nk} = 0$ for some positive integer N. But $S^{2n} = Id$ on H_0, so this equation is equivalent to $\sum_{k=0}^{N}(-1)^k \begin{pmatrix} N \\ k \end{pmatrix} (R_0)_n^k = 0$. That is, $(R_0)_n$ is unipotent. This implies that $(R_0)_n$ is unipotent in the quotient algebra $H_0/\text{Rad}(H_0) \otimes H_0^*/\text{Rad}(H_0^*)$. However,

$$H/\text{Rad}(H) \otimes H^*/\text{Rad}(H^*) = H_0/\text{Rad}(H_0) \otimes H_0^*/\text{Rad}(H_0^*)$$

(since $H_{\geq 1}$, $H_{\geq 1}^*$ are nilpotents ideals), and moreover $R = R_0$ in this quotient algebra. So R_n is unipotent in $H/\text{Rad}(H) \otimes H^*/\text{Rad}(H^*)$, hence in $H \otimes H^*$,

which is equivalent to saying that $\sum_{k=0}^{N}(-1)^k \begin{pmatrix} N \\ k \end{pmatrix} R_{nk} = 0$ for some integer $N > 0$ (since $S^{2n} = Id$ on H), hence n is divisible by qexp (H).

Since by Proposition 2.6, both qexp (H_0) and $|S^2|$ divide qexp (H), n divides qexp (H), and we are done. $\qquad\square$

Theorem 4.4. *Let H be a finite-dimensional pointed Hopf algebra over $\mathbb{C}$. Then $|S^2|$ divides* $\exp(G(H))$.

Proof. Let grH be the graded pointed Hopf algebra associated to H with respect to the coradical filtration of H (see e.g. [M]). Then $|S_H^2| = |S_{grH}^2|$ and $G(H) = G(grH)$. Therefore, it is sufficient to consider the graded case, so we will assume that $H = \bigoplus H_i$ is graded.

We have a $\mathbb{Z}_+-$filtration of H defined as follows: $F_i(H)$ is defined to be the subalgebra in H generated by $H_0, H_1, \ldots, H_i$. It is clearly a Hopf subalgebra in H.

Let $n := \exp(G(H))$, and let us show that $S^{2n} = Id$ on $F_i(H)$ by induction in i.

For $i = 0$, this is clear, since $F_0(H) = H_0 = \mathbb{C}[G(H)]$. Suppose the statement is known for $i = k - 1$, and let us prove it for $i = k$. Clearly, it is sufficient to show that $S^{2n} = Id$ on H_k.

Recall that H_k is a bicomodule over H_0. Indeed, $\Delta : H_k \to \bigoplus_{i+j=k} H_i \otimes H_j$, so we can write $\Delta = \bigoplus_{i+j=k} \Delta_{(i,j)}$, where $\Delta_{(i,j)} : H_k \to H_i \otimes H_j$. Now, the maps $\Delta_{(0,k)}$, $\Delta_{(k,0)}$ define the H_0-bicomodule structure on H_k. Therefore, since H_0 is cocommutative, we can restrict our attention to elements of H_k which belong to a $1-$dimensional bicomodule; i.e., elements $x \in H_k$ for which

$$\Delta(x) = g \otimes x + x \otimes h + \xi(x),$$

where $\xi(x) \in \bigoplus_{i=1}^{k-1} H_i \otimes H_{k-i}$, and g, h are grouplike elements. In fact, we can assume $h = 1$ (by multiplying x by a grouplike element, if necessary).

Now, apply $m(S \otimes Id)$ to this equation. Since $\varepsilon(x) = 0$, we get $S(x) = -g^{-1}x$ modulo $F_{k-1}(H)$. Thus, $S^2(x) = g^{-1}xg$ modulo $F_{k-1}(H)$, and hence $S^{2n}(x) = x$ modulo $F_{k-1}(H)$ (as $g^n = 1$). Also by induction assumption, $S^{2n} = Id$ on $F_{k-1}(H)$. But S is semisimple [R], hence $S^{2n}(x) = x$, and we are done. $\qquad\square$

As a corollary, we have the following weak version of a recent result by Radford and Schneider [RS].

Corollary 4.5. *Let H be a finite-dimensional pointed Hopf algebra over $\mathbb{C}$. Then $|S^2|$ divides* $\dim(H)$.

Radford and Schneider proved that $|S^2|$ divides $\dim(H)/|G(H)|$ [RS].

We can now prove our second main result.

Theorem 4.6. *Let H be a finite-dimensional pointed Hopf algebra over $\mathbb{C}$. Then* qexp $(H) = \exp(G(H))$.

Proof. One has qexp $(H) =$ qexp $(grH) = l.c.m.(\exp(G(H)), |S^2|) = \exp(G(H))$ by Lemma 4.2, Proposition 4.3 and Theorem 4.4. $\qquad\square$

As an application we can relate the exponents of the groups $G(H)$ and $G(H^J)$.

Corollary 4.7. *Let H be a finite-dimensional pointed Hopf algebra over $\mathbb{C}$, and let J be a twist for H. Then $\exp(G(H^J))$ divides $\exp(G(H))$.*

Corollary 4.8. *If two finite-dimensional pointed Hopf algebras H_1, H_2 are twist equivalent, then $\exp(G(H_1)) = \exp(G(H_2))$.*

Example 4.9. For example, Corollary 4.8 implies that two isocategorical groups (see [EG1]) have the same exponent.

Example 4.10. Theorem 4.6 can be applied to calculate the quasi-exponent of quantum groups at roots of unity (the finite-dimensional version). Indeed, if $\mathfrak{g}$ is a finite-dimensional semisimple Lie algebra, $\mathfrak{b}_+$ its Borel subalgebra, and q is a primitive root of unity of order l, then $\mathrm{qexp}(U_q(\mathfrak{b}_+)) = \mathrm{qexp}(U_q(\mathfrak{g})) = l$.

As a consequence we have the following result.

Corollary 4.11. *For any twist J of $U_q(\mathfrak{g})$ and any grouplike element g of $U_q(\mathfrak{g})^J$, the order of g divides l.*

Remark 4.12. We expect that moreover $G(U_q(\mathfrak{g})^J)$ is abelian (see Section 5).

Let us conclude the section by calculating the quasi-exponent of finite-dimensional triangular Hopf algebras H. By [EG4], H has the Chevalley property (see [AEG], [EG2]).

Proposition 4.13. *Let H be a finite-dimensional triangular Hopf algebra over $\mathbb{C}$, and let $H_s := H/\mathrm{Rad}(H)$ be its semisimple part. Then $\mathrm{qexp}(H) = \exp(H_s)$.*

Proof. It is enough to consider the non-semisimple case. We have that H_s^* is the zero part in the coradical filtration of H^*, therefore by Proposition 4.3, $\mathrm{qexp}(H) = \mathrm{qexp}(H^*) = l.c.m.(\mathrm{qexp}(H_s^*), |S_H^2|)$. Now by [AEG], $|S_H^2| = 2$. However, the Drinfeld element u of H is a grouplike element satisfying $u^2 = 1$, $u \neq 1$. Hence, $2 = |S_H^2|$ divides $\exp(H_s)$, and we are done. $\square$

Remark 4.14. Proposition 4.13 gives an efficient way of calculating $\mathrm{qexp}(H)$ since by [EG3], $H_s = \mathbb{C}[G]^J$ for some finite group G, hence $\exp(H_s) = \exp(G)$ by [EG, Theorem 3.3].

Remark 4.15. The results of this paper, so far, extend without changes to the case when the ground field $\mathbb{C}$ is replaced by an algebraically closed field k of odd characteristic relatively prime to the dimensions of the Hopf algebras involved. More precisely, all results remain true except Remark 2.2(3). (The assumption about odd characteristic is used when we claim that S^2 is semisimple.)

5. Applications to quantum groups at roots of unity

Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra and $p > 2$ an odd number (not necessarily a prime). Assume also that p is not divisible by 3 if $\mathfrak{g}$ is of type G_2. Let q be a primitive $p-$th root of unity, and $U_q(\mathfrak{g})$ denote the finite-dimensional quantum group associated to $\mathfrak{g}$. Recall that $U_q(\mathfrak{g})$ is generated

by e_i, f_i, K_i with the usual relations, and the additional relations $K_i^p = 1$, $e_i^p = f_i^p = 0$, and the similar relations for non-simple root elements.

The following theorem is essentially contained in [T] (see p. 404).

Theorem 5.1. *If p is relatively prime to the determinant of the Cartan matrix A of $\mathfrak{g}$, then the Hopf algebra $U_q(\mathfrak{g})$ is simple, i.e. does not have non-trivial Hopf ideals.*

Remark 5.2. We are grateful to N. Andruskiewitsch for explaining the proof of this theorem to us.

Proof. Let $\phi : U_q(\mathfrak{g}) \rightarrow L$ be a non-trivial Hopf algebra map of $U_q(\mathfrak{g})$ into another finite-dimensional Hopf algebra L. Our job is to show that this map is injective.

First of all, let us show that $\phi(e_i)$ is non-zero for all i. Indeed, consider the set I of all i for which $\phi(e_i) = 0$. We claim that this set is empty. Indeed, if $i \in I$ then $\phi([e_i, f_i]) = 0$, so $\phi(K_i) = 1$ (as p is odd), hence for any j connected to i, $\phi(e_j) = \phi(K_i e_j K_i^{-1}) = q^{a_{ij}} \phi(e_j)$, so $\phi(e_j) = 0$. So I is either empty of everything. But if I is everything, we get $\phi(e_i) = 0$ for all i, hence $\phi(K_i) = 1$ for all i, hence $\phi(f_i) = 0$ for all i, so ϕ is trivial. Contradiction.

Similarly, $\phi(f_i)$ is non-zero for all i.

Now, by Lemma 6.1.2 in [T], it is sufficient to check that the map ϕ is injective in degree 1 of the coradical filtration of $U_q(\mathfrak{g})$. First let us check that it is so in degree 0 of this filtration, i.e. that $\phi : \mathbb{C}[G] \rightarrow L$ is injective, where G is the group of grouplike elements generated by K_i. For this, it suffices to show that if $g \in G$ is non-trivial, then so is $\phi(g)$. Assume the contrary, i.e. that $\phi(g) = 1$. Since $(p, \det(A)) = 1$, there exists i such that $g e_i g^{-1} = b e_i$, where $b \neq 1 \in \mathbb{C}$. Thus, $\phi(e_i) = 0$. Contradiction.

Now, the degree 1 term of the coradical filtration has the form

$$\mathbb{C}[G] + \sum_i \mathbb{C}[G] e_i + \sum_i \mathbb{C}[G] f_i.$$

Assume that ϕ is not injective in degree 1. Since $\text{Ker}(\phi)$ is obviously stable under conjugation by G, there exists an element of $\text{Ker}(\phi)$ which is in $\mathbb{C}[G] e_i$ for some i or $\mathbb{C}[G] f_i$ for some i. Let us assume that the former is the case (the other case is analogous). Then we have $\phi(\sum_{g \in G} a_g g e_i) = 0$. Calculating the coproduct of this, one easily concludes that $\phi(e_i) = 0$. Contradiction. The theorem is proved. $\square$

From now on we suppose that p is *prime*.

Let Y be a non-trivial irreducible representation of $\mathfrak{g}$, and assume that $p > \dim(Y)$.

Theorem 5.3. *For any twist J of $U_q(\mathfrak{g})$, the group $G := G(U_q(\mathfrak{g})^J)$ is an elementary abelian $p - group$.*

Proof. By Corollary 4.11, $\exp(G) = p$. Thus, it remains to show that G is abelian. Consider the smallest abelian tensor subcategory of $\text{Rep}(U_q(\mathfrak{g})) =$

$\mathrm{Rep}(U_q(\mathfrak{g})^J)$, which contains Y. Since by Theorem 5.1, $U_q(\mathfrak{g})$ does not have non-trivial Hopf algebra quotients, this category must coincide with the whole $\mathrm{Rep}(U_q(\mathfrak{g}))$. This means that any irreducible representation occurs as a subquotient in the tensor product of several copies of Y.

Now, since G is a $p-$group, the dimensions of its irreducible representations are powers of p. This means that the restriction of Y to G is a direct sum of $1-$dimensional representations of G (as $p > \dim(Y)$). Hence, the same holds for tensor powers of Y, and therefore for their subquotients; i.e., for all irreducible representations of $U_q(\mathfrak{g})$. But then the regular representation of $U_q(\mathfrak{g})$ is a sum of $1-$dimensional representations for G, which implies that G is abelian. $\qquad\square$

Example 5.4. For $\mathfrak{g} := sl(n)$, there is an $n-$dimensional vector representation, so the condition on p is $p > n$. For $\mathfrak{g} := sp(2n)$ the condition is $p > 2n$. For $\mathfrak{g} := o(n)$, the condition is $p > n$ $(n \geq 7)$.

Conjecture 5.5. Under the conditions of Theorem 5.3, the rank of G is at most the rank of the Lie algebra $\mathfrak{g}$.

Remark 5.6. There are examples (twist corresponding to Belavin-Drinfeld triples, see [EN]) where the rank of G is actually less than the rank of g.

Theorem 5.7. *Conjecture 5.5 holds for the classical Lie algebras (series $A-D$).*

Proof. Type $A_{n-1} = sl(n)$: Consider the restriction of the $n-$dimensional irreducible representation Y to G. Then $Y = a_1 + \cdots + a_n$, where a_i are characters of G. Since $Y^{\otimes n}$ contains the trivial representation as a subquotient (quantum exterior power), there is at least one relation between $a_1, \ldots, a_n$. This implies that the subgroup K in G^* generated by a_i is of rank $n-1$. By the proof of Theorem 5.3, the characters of G which occur in any irreducible representation of $U_q(\mathfrak{g})$ must lie in K, so $K = G^*$, and the rank of G^* is at most $n-1$, so we are done.

Types $C_n = sp(2n)$, $D_n = o(2n)$: Take the $2n-$dimensional irreducible representation. Since it is self-dual, the characters of G that occur in it are $a_i^{\pm 1}$, $i = 1, \ldots, n$. But they generate G^*, so the rank of G is at most n.

Type $B_n = o(2n+1)$: Take the $(2n+1)-$dimensional irreducible representation. It is self-dual, so the characters of G that occur are $1, a_i^{\pm 1}$, $i = 1, \ldots, n$, so the rank of G^* is at most n.

The theorem is proved. $\qquad\square$

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DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MA 02139, U.S.A.
    E-mail address: `etingof@math.mit.edu`

DEPARTMENT OF MATHEMATICS, TECHNION-ISRAEL INSTITUTE OF TECHNOLOGY, HAIFA 32000, ISRAEL.
    E-mail address: `gelaki@math.technion.ac.il`