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ON THE MINIMAL NUMBER OF CRITICAL POINTS OF FUNCTIONS ON h-COBORDISMS

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Abstract. Let (W, M_0, M_1) be a non-trivial h-cobordism (i.e., the Whitehead torsion of (W, M_0) is non-zero) with W compact, connected and $\dim W \geq 6$. We prove that every smooth function $f : W \to [0,1]$, $f(M_0) = 0, f(M_1) = 1$ has at least 2 critical points. This estimate is sharp: W possesses a function as above with precisely two critical points.

Introduction

Let (W, M_0, M_1) be an h-cobordism, [3]. Here W is always assumed to be smooth, connected and compact and $M_i, i = 0, 1$ is always assumed to be closed. Recall that an h-cobordism (W, M_0, M_1) is called trivial if there is a diffeomorphism $(W, M_0, M_1) \cong (M \times [0,1], M_0, M_0)$. We say that a function (not necessarily Morse) $f : W \to [0,1]$ is *regular* if $f^{-1}(M_0) = 0$, $f^{-1}(M_1) = 1$ and both values 0 and 1 are regular values of f. It is well known that an h-cobordism (W, M_0, M_1) is trivial if and only if W possesses a regular function without critical points. In this note we prove the following theorem.

Theorem. *Let (W, M_0, M_1) be a non-trivial h-cobordism with* $\dim W \geq 6$. *Then every regular function on W has at least two critical points. Moreover, this estimate is sharp: W possesses a regular function with precisely two critical points.*

We denote by I the closed interval $[0,1]$.

1. Preliminaries

Let $f : W \to I$ be a regular Morse function on an h-cobordism (W, M_0, M_1). Choose a Riemannian metric on W and consider integral trajectories for the vector field $-\operatorname{grad} f$, the so-called anti-gradient trajectories. We say that an anti-gradient trajectory $a = a(t)$ is *a special trajectory from p to q* if $\lim_{t\to-\infty} a(t) = p$ and $\lim_{t\to+\infty} a(t) = q$ where p and q are critical points of f such that the index of p is one more than the index of q. We can and shall assume that the number of special trajectories is finite (this is true for generic function and metric).

For every critical point of f we fix orientations of unstable disks (left-hand disks in terminology of [3]). Then every unstable sphere (in a certain level)

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gets an orientation. Moreover, every stable sphere gets a coorientation, i.e., an orientation of its normal bundle in the corresponding level set. Now, for every special trajectory a from p to q we define the number $\varepsilon(a) = \pm 1$ as follows. Take $c \in]f(q), f(p)[$. Then our trajectory a meets the level $f^{-1}(c)$ in a certain point x, which is a point of transversal intersection of the corresponding stable and unstable spheres. We define $\varepsilon(a)$ to be the intersection index at x.

2. Whitehead torsion

Given a ring R, we define a based R-module to be a free finite generated left R-module M with a fixed R-free basis.

Recall the definition of the Whitehead torsion of an h-cobordism (W, M_0, M_1). Given a group π, let $A = A(\pi)$ denote the set of long exact sequences

$$\cdots \longrightarrow C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots \longrightarrow 0$$

such that each C_i is a based $\mathbb{Z}[\pi]$-module and all but finite number of modules C_i are zero modules. Furthermore, each ∂_i is a $\mathbb{Z}[\pi]$-module homomorphism. Let us call the exact sequence of based $\mathbb{Z}[\pi]$-modules trivial if it has only two non-zero terms and the corresponding isomorphism is given by the identity matrix. The term-wise direct sum operation converts A into an abelian semigroup. Let R be the equivalence relation on A generated by the following operations:

- interchanging of the elements;
- replacement of a basis element by the sum of this element with the multiple of another basis element;
- addition of the trivial exact sequence;
- multiplication of any basis element by the element $\pm g, g \in \pi$.

The above mentioned operation in A induces a group structure in A/R. This groups is called the *Whitehead group* of π and is denoted by $\mathrm{Wh}(\pi)$, [4]. It turns out to be that $\mathrm{Wh}(\pi)$ is a functor of π. In particular, every homomorphism $\varphi : \pi \to G$ induces a homomorphism $\mathrm{Wh}(\varphi) : \mathrm{Wh}(\pi) \to \mathrm{Wh}(G)$. Namely, the homomorphism φ yields the homomorphism $\mathbb{Z}[\varphi] : \mathbb{Z}[\pi] \to \mathbb{Z}[G]$ of group rings, which turns $\mathbb{Z}[G]$ into the right $\mathbb{Z}[\pi]$-module $\mathbb{Z}[G]_{\mathbb{Z}[\varphi]}$. Now, for every based $\mathbb{Z}[\pi]$-module C we can form the based $\mathbb{Z}[G]$-module $\mathbb{Z}[G]_{\mathbb{Z}[\varphi]} \otimes C$. The sequence $\{\mathbb{Z}[G]_{\mathbb{Z}[\varphi]} \otimes C_n\}$ turns out to be exact because all the C_n's are free, etc.

For every h-cobordism (W, M_0, M_1) with $\pi_1(W) = \pi$ the Whitehead torsion $\tau(W, M_0, M_1) \in \mathrm{Wh}(\pi)$ is defined as follows. Consider a regular Morse function $f : W \to I$, Riemannian metric, etc. as in §1 . Fix a point $x_0 \in W$ and, for every critical point p of f, choose a path $u(p)$ from x_0 to p. Every special trajectory a from p to q gives us a map $a : \mathbb{R} \to W$ which is well defined up to shift of $t \in \mathbb{R}$. We define a path $v = v_a : I \to W$ as follows. Let $\lambda(t) :]0, 1[\to \mathbb{R}$ be a function such that

$$\lim \lambda(t)_{t\to 0} = -\infty, \quad \lim \lambda(t)_{t\to 1} = +\infty.$$

We set $v(0) = p$, $v(1) = q$, $v(t) = a(\lambda(t))$. Now, consider the loop $u(p) \circ v \circ (u(q))^{-1}$ (where $\circ$ denotes the product of paths) and define $g(a) \in \pi = \pi_1(W)$ as the based homotopy class of the loop constructed.

Let $p_1, \dots, p_k$ be all the critical points of the index n. Define C_n to be the free $\mathbb{Z}[\pi]$-module generated by symbols $[p_1], \dots, [p_k]$. In other words, C_n consists of formal linear combinations

$$\sum_{i=1}^{k} \alpha_i [p_i], \quad \alpha_i \in \mathbb{Z}[\pi].$$

We define the differential $\partial_n : C_n \to C_{n-1}$ to be a $\mathbb{Z}[\pi]$-module homomorphism such that

$$\partial_n [p] = \sum_q \sum_{a \in T(p,q)} \varepsilon(a) g(a) [q]$$

where q runs over all critical points of the index $n-1$ and $T(p,q)$ is the set of special trajectories from p to q.

It follows from the Morse theory that $H_*(\{C_n, \partial_n\}) = H_*(\widetilde{W}, \widetilde{M_0})$ where $(\widetilde{W}, \widetilde{M_0})$ is the universal covering of the pair (W, M_0). Since M_0 is a deformation retract of W, we conclude that $\widetilde{M_0}$ is a deformation retract of $\widetilde{W}$, and therefore the complex $\{C_n, \partial_n\}$ is acyclic, i.e. the sequence

$$\cdots \longrightarrow C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots \longrightarrow 0$$

is exact. Thus, the above sequence determines a certain element $\tau = \tau(W, M_0) \in \mathrm{Wh}(\pi)$, the so-called *Whitehead torsion* of the h-cobordism (W, M_0, M_1).

According to well-known Barden–Mazur–Stallings Theorem, [1, 2, 5], an h-cobordsim (W, M_0, M_1) with $\dim W \geq 6$ is trivial if and only if $\tau(W, M_0) = 0$.

2.1. Lemma. *Suppose that an h-cobordism (W, M_0, M_1) possesses a regular Morse function f such that all the critical points and special trajectories of f are contained in a simply connected domain U of W. Then $\tau(W, M_0) = 0$*

Proof. Since $\tau(W, M_0)$ does not depend on the choice of the based point x_0 and the paths $u(p)$, we can assume that $x_0 \in U$ and every path $u(p)$ belongs to U. Then, for every special trajectory a, $g(a)$ is the neutral element of $\pi = \pi_1(W)$. Thus,

$$\tau(W, M_0) \in \mathrm{Im}\{\mathrm{Wh}(j) : \mathrm{Wh}\{e\} \to \mathrm{Wh}(\pi)\}$$

where $j : \{e\} \to \pi$ is the inclusion of the trivial subgroup. But it follows from the elementary linear algebra that $\mathrm{Wh}\{e\} = 0$, see e.g. [4]. Thus, $\tau(W, M_0) = 0$.

3. Proof of the theorem

Let $f : M \to \mathbb{R}$ be a smooth function (not necessarily Morse) on a Riemannian manifold M. Let U be an open ball in M and suppose that U contains precisely one critical point o.

3.1. Lemma. *There exists a regular Morse function g which is C^∞-close to f in the Whitney topology and such that every special g-trajectory is contained in U whenever its ends are contained in U.*

Proof. Let $D(r) = \{m \in M \mid d(m,o) < r)$ where d is the distance function on M. We can and shall assume that the injectivity radius at o is at least one and that $U = D(1)$. Then there are positive constants C and E such that, for every function g which is C^∞-close to f, the following estimates hold in $D(1) \setminus D(1/2)$:

$$|\operatorname{grad} g| \geq E, \quad |L_{\operatorname{grad} g}\, d(m,o)| \leq C.$$

Choose a function g close to f. Let p and q be two critical points of g which belong to U. Suppose that there is a special trajectory $a(t)$ from p to q which meets the boundary of $D(3/4)$. We claim that in this case

$$g(p) - g(q) \geq \frac{E^2}{4C}.$$

Indeed, since $L_{\operatorname{grad} g}\, d(m,o)| \leq C$, we conclude that

$$a\left[t - \frac{1}{4C}, t + \frac{1}{4C}\right]$$

does not meet $D(1/2)$ whenever $a(t) \notin D(3/4)$. So, if $a(t_0) \notin D(3/4)$ then

$$g(p) - g(q) \geq \int_{t_0 - \frac{1}{4C}}^{t_0 + \frac{1}{4C}} dg(a(t)) = \int_{t_0 - \frac{1}{4C}}^{t_0 + \frac{1}{4C}} |\operatorname{grad} g|^2 dt \geq \frac{E^2}{4C}.$$

Now we can finish the proof as follows. Since f has only one critical point, there exists g close to f and such that $g(p) - g(q)$ is small enough for all critical points p and q of g. This is a contradiction. □

3.2. Corollary. *If an h-cobordism (W, M_0, M_1) possesses a regular function f with one critical point p, then $\tau(W, M_0) = 0$. In particular, if $\dim W \geq 6$ then the h-cobordism is trivial.*

Proof. Because of Lemma 3.1, we can perturb the function f in a small neighborhood of the critical point and get a function f_1 such that all its critical points and special trajectories belong to a disk neighborhood of p. Now the result follows from Lemma 2.1. □

3.3. Proposition. *Every h-cobordism (W, M_0, M_1) , $\dim W \geq 6$ possesses a regular function with at most 2 critical points.*

Proof. Consider a regular Morse function $f : W \to I$. Asserting as in [1, Lemme 1] and [3, §4], we can modify f and to get a regular Morse function which has at most two critical levels a, b, $a < b$ and index of each of critical points is equal to 2 or 3. Because of this, every critical level is path connected. Now, following [6, Th. 2.7 and Prop.2.9], we can contract the critical points in each of the levels and get a regular function with at most 2 critical points. □

Clearly, Corollary 3.2 and Proposition 3.3 together imply the Theorem.

3.4. Remarks. 1. Asserting as in 3.2, one can show that, for every regular function f on a non-trivial h-cobordism, the number of critical levels of f is at least 2 provided that all the critical points of f are isolated.

2. Every h-cobordism (W, M_0, M_1) possesses a regular function with 1 critical level. Namely, choose collars of the boundary components and define f to be constant on complements of collars and depending on the "vertical" coordinate only for collars. In greater detail, consider a smooth function

$$\varphi : I \to I, \quad \varphi(t) = \begin{cases} t & \text{if } \ 0 \le t \le \varepsilon/4 \text{ or } \ 1 - \varepsilon/4 \le t \le 1, \\ 1/2 & \text{if } \ \varepsilon/2 \le t \le 1 - \varepsilon/2 \end{cases}$$

for $\varepsilon > 0$ small enough. Choose collars $M_0 \times [0, \varepsilon]$ and $M_1 \times [1 - \varepsilon, 1]$ and define $f : W \to I$ by setting

$$f(x) = \begin{cases} \varphi(t) & \text{if } x = (m, t) \in M_0 \times [0, \varepsilon], \\ \varphi(t) & \text{if } x = (m, t) \in M_1 \times [1, 1 - \varepsilon], \\ 1/2 & \text{else.} \end{cases}$$

3. Every trivial h-cobordism $(M \times I, M, M)$ possesses a regular function with 1 critical point. Indeed, consider a function $\varphi : M \to I$ such that $\varphi^{-1}(1)$ is a point m_0 (and therefore m_0 is a critical point of φ) and define

$$f : M \times I \to I, \quad f(m, t) = (t - 1/2)(1 - \varphi(m)) + \varphi(m)(t - 1/2)^3.$$

It is easy to see that f has just one critical point $(m_0, 1/2)$.

4. Notice that, for every h-cobordism (W, M_0, M_1), the relative Lusternik–Schnirelmann category $\operatorname{cat}(W, M_0) = 0$, while every regular function on any non-trivial h-cobordism (W, M_0, M_1) has at least two critical points.

5. It is easy to see that, because of the collar theorem, the regularity condition for f in the Theorem can be weaken as follows: $f(M_0) = 0$ and $f(M_1) = 1$.

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