*Mathematical Research Letters* **9**, 129–137 (2002)

# FOCAL POINTS AND THE DECREASE OF CURVATURE: A SURPRISING EXAMPLE

Marlies Gerber

## 1. Introduction

If $\gamma(t)$ is a unit speed geodesic on a manifold $M$ and $J(t)$ is a non-trivial perpendicular Jacobi field along $\gamma$ such that $J(t_0) = 0$ and $(||J||^2)'(t_1) = 0$ for some $t_0 < t_1$, then the geodesic that is tangent to $J(t_1)$ at $\gamma(t_1)$ is said to have a *focal point* at $\gamma(t_0)$. If this does not happen for any times $t_0 < t_1$ and any non-trivial perpendicular Jacobi field $J$, then we say that there are no focal points along $\gamma$. Moreover, if there are no focal points along any geodesic in $M$, then we say $M$ has no focal points. Although manifolds with no focal points can have sectional curvatures of both signs [4], they have many of the properties of manifolds of nonpositive curvature that are of interest in the study of geodesic flows. (See, for example, [2], [5], and [6].)

In the case when $M$ is a surface $S$ with Gaussian curvature $K$, the existence of a focal point along $\gamma$ is equivalent to the following: There is a solution $u$ to the scalar Riccati equation

$$u^2(t) + u'(t) + K(\gamma(t)) = 0$$

defined on $(t_0, t_1]$ such that $\lim_{t\to t_0^+} u(t) = \infty$ and $u(t_1) = 0$.

The following comparison lemma (see [1]) is often used in estimating solutions to the Riccati equation.

**Lemma 1.1.** *Suppose $u_i$, $i = 1, 2$, satisfy*

$$u_i^2(t) + u_i'(t) + K_i(t) = 0$$

*on an interval $[t_0, t_1]$, where $K_1$ and $K_2$ are continuous functions on $[t_0, t_1]$ with $K_2 \le K_1$. If $u_1(t_0) \le u_2(t_0)$, then $u_1(t_1) \le u_2(t_1)$.*

According to this lemma, if a geodesic $\gamma$ on a surface $S$ does not have any focal points when restricted to $[t_0, t_1]$ and $\sigma$ is another geodesic on $S$ with $K(\sigma(t)) \le K(\gamma(t))$ for all $t \in [t_0, t_1]$, then $\sigma$ also does not have any focal points when restricted to $[t_0, t_1]$. Now suppose that $\gamma$ is a closed unit speed geodesic on a surface $S$. For simplicity, let us assume that small neighborhoods of $\gamma$ are orientable. Suppose that there are no focal points along $\gamma$ and that the Gaussian curvature $K$ is strictly decreasing as we move away from $\gamma$ along geodesics

Received December 9, 2001.

perpendicular to $\gamma$ in a small neighborhood of $\gamma$. Moreover, if $(s,x)$ are Fermi coordinates along $\gamma(s)$, we assume that

$$\frac{\partial K}{\partial x}(s,0) \equiv 0 \tag{1.1}$$

and

$$\frac{\partial^2 K}{\partial x^2}(s,0) < 0 \tag{1.2}$$

for all $s$. Consequently, if we think of the graph of $K(s,x)$ over a thin strip $(-\infty,\infty)\times[-\epsilon,\epsilon]$, then the part of this graph corresponding to $\gamma$ (where $x=0$), forms a ridge and the graph drops down on both sides of this ridge. Now suppose that $\sigma(t)$ is another unit speed geodesic such that $\lim_{t\to-\infty}\operatorname{dist}(\sigma(t),\gamma(t))=0$. Then for $t$ large and negative, $\sigma$ is very close to $\gamma$ and very nearly parallel to $\gamma$ in the $(s,x)$ coordinate system. From the comparison lemma, we might expect that for $T$ sufficiently large, there are no focal points along the restriction of $\sigma$ to $(-\infty,-T)$. Our intuitive reasoning proceeds as follows: As $\sigma$ approaches $\gamma$, the time parameter $t$ along $\sigma$ could be taken approximately equal to the Fermi coordinate $s$. For $x$ very close to 0, but not equal to 0, we have $K(s,x) < K(0,x)$ and we would then apply the comparison lemma. This argument appears even more plausible if we take into account the length of the $\partial/\partial s$ vector field. This vector field is a Jacobi field along geodesics perpendicular to $\gamma$, and by considering the Jacobi equation along such geodesics we see that

$$\frac{\partial^2}{\partial x^2}\left(\|\partial/\partial s\|\right)(s,0) = -K(s,0)\ .$$

We also have

$$\frac{\partial}{\partial x}\left(\|\partial/\partial s\|\right)(s,0) = 0,$$

by the first variation formula. Therefore $\|\partial/\partial s\|(s,x)$ is increasing [decreasing] as we move away from $\gamma$ along $s=s_0$ curves for those $s_0$ values where $K(s_0,0)$ is negative [positive]. This effect would contribute to $\sigma$ spending slightly more time in regions of negative curvature and slightly less time in regions of positive curvature than $\gamma$.

In this paper we show that contrary to the above reasoning, there is an example of a surface containing a closed geodesic $\gamma$ along which there are no focal points and (1.1) and (1.2) are satisfied, but there are focal points along every geodesic $\sigma(t)$ that is asymptotic to $\gamma$ as $t\to-\infty$. Although our proof uses a different idea, an intuitive explanation of what is happening is that the angle $\phi$ that $\sigma'$ makes with the $\partial/\partial s$ vector field can have a greater effect than the decrease of the curvature. If we consider $\phi$ as a function of $s$, then by Lemma 3.3 in [3],

$$\phi'(s) = \frac{\partial}{\partial x}(||\partial/\partial s||)(s,x) \approx -K(s,0)x, \tag{1.3}$$

for small values of $x$. Along $\sigma$, we also have

$$\frac{dt}{ds} = \frac{||\partial/\partial s||}{\cos\phi} \approx \frac{1 - K(s,0)x^2}{1 - \phi^2/2} \approx 1 - K(s,0)x^2 + \phi^2/2, \tag{1.4}$$

for small $x$ and $\phi$. If we assume there are no focal points, then by [2], horocycles are convex, and we see that $\phi \geq 0$ along $\sigma$. By (1.3) it is possible, for suitably chosen $K(s,0)$, to have the $\phi^2/2$ term in (1.4) exceed the absolute value of the $K(s,0)x^2$ term, for some values of $s$. In regions where $K(s,0) > 0$, this would contribute to $\sigma$ spending more time in positive curvature than $\gamma$. This suggests that we should not assume $\phi \equiv 0$ in our estimates. More precisely, if we use a coordinate system $(t,y)$, to be described in §2, in which $y = 0$ corresponds to $\gamma$ and $y =$ constant curves are geodesics (with arc length parameter $t$) that are asymptotic to $\gamma$ as $t \to -\infty$, then $\partial K/\partial y \equiv 0$ along $\gamma$, but $\partial^2 K/\partial y^2$ is positive at some points of $\gamma$. This shows that Lemma 1.1 does not apply.

In our example, the unstable solution $v$ of the Riccati equation along $\gamma$ satisfies $v \geq 0$, but there is a point $p$ where $v(p) = 0$. If $K$ were to vanish identically along $\gamma$ and the curvature decreases on the average as we move away from $\gamma$ (as defined in [3]), then there would be no focal points near $\gamma$ (see Theorem 4.3 in [3]).

## 2. Construction of the example in terms of the functions $v$ and $w$

We will use a "horocyclic coordinate system," which K. Burns had suggested in the context of an earlier, simpler example that is used in §6 of [3]. This coordinate system $(t,y)$ will be defined on $(-\infty,\infty) \times (-1,1)$ such that the curves $y =$ constant will be geodesic segments and the curves $t =$ constant will be segments of unstable horocycles. The $t$ coordinate will agree with the $s$ coordinate of the $(s,x)$ Fermi coordinate system along the geodesic $\gamma(s)$, where $y = 0$. We will construct $C^\infty$ periodic functions $v$ and $w$ of period $\ell$ such that $v > 0$ on $[0,\ell] \setminus \{p\}$ and $v(p) = 0$ for some $p \in (0,\ell)$; $w \equiv 0$ on $[-\delta,\delta]$, for some small $\delta > 0$; $w(p) < 0$; and $v$ and $w$ satisfy the inequality

$$(2w' + 4vw)(1/j)^2 + (v'' + 2vv')v > 0 \tag{2.1}$$

on $[0,\ell]$, where $j(t) = \exp\left[\int_0^t v(\xi)\, d\xi\right]$. The left hand side of (2.1) turns out to be $-\partial^2 K/\partial x^2$ along $\gamma$ (as will be shown below), and consequently (2.1) is equivalent to (1.2).

For $(t,y) \in (-\infty,\infty) \times (-1,1)$, let $\bar{v}(t,y) = v(t) + y^2 w(t)$ and let $\bar{\jmath}(t,y) = \exp[\int_0^t \bar{v}(\xi,y)\, d\xi]$. We define a Riemannian metric $\langle\ ,\ \rangle$ on $(-\infty,\infty) \times (-1,1)$ by $\langle \partial/\partial t, \partial/\partial t\rangle = 1$, $\langle \partial/\partial t, \partial/\partial y\rangle = 0$, and $\langle \partial/\partial y, \partial/\partial y\rangle = \bar{\jmath}^2(t,y)$. Then for any $y_0 \in (-1,1)$, the function $\bar{\jmath}(t,y_0)$, $-\infty < t < \infty$, is a solution to the Jacobi equation along the geodesic $y = y_0$ and $\bar{v}(t,y) = ((\partial\bar{\jmath}/\partial t)(t,y_0))/\bar{\jmath}(t,y_0)$ is the corresponding solution to the Riccati equation. Let $z(t,y) = \int_0^y \bar{\jmath}(t,\eta)\, d\eta$. Then $z$ is an arc length parameter along the curves with constant $t$ coordinate.

In order to prove the smoothness of the metric obtained with the identification of points along $t = 0$ and $t = \ell$ to be described below, we define the map

$$F(\ell + t, y) = (t, z(\ell, y)),$$

for $|t| < \delta$ and $y \in (-1, 1)$ sufficiently small so that $z(\ell, y) \in (-1, 1)$. Then

$$dF\,(\partial/\partial y)_{(\ell+t,y)} = \bar{\jmath}(\ell, y)\,(\partial/\partial y)_{(t,z(\ell,y))}\,.$$

Since $w \equiv 0$ on $[-\delta, \delta]$, $\bar{\jmath}(t, z(\ell, y)) = \bar{\jmath}(t, y)$, and we obtain

$$\begin{aligned}\|dF\,(\partial/\partial y)_{(\ell+t,y)}\,\| &= \bar{\jmath}(\ell, y)\bar{\jmath}(t, z(\ell, y)) = \bar{\jmath}(\ell, y)\bar{\jmath}(t, y)\\ &= \bar{\jmath}(\ell + t, y) = \|\,(\partial/\partial y)_{(\ell+t,y)}\,\|.\end{aligned}$$

It follows that $F$ preserves the Riemannian metric $\langle\ ,\ \rangle$. If $y, y' \in (-1, 1)$ are such that $z(\ell, y) = y'$, then we identify the points $(0, y')$ and $(\ell, y)$. After making this identification we can extend a small neighborhood of $y = 0$ in $[0, \ell] \times (-1, 1)$ to a compact surface $S$, and $\gamma$ becomes a closed geodesic of period $\ell$. Since $F$ preserves $\langle\ ,\ \rangle$, the metric obtained with this identification is consistently defined and smooth in a neighborhood of $y = 0$.

For $y_0$ close to 0, but not equal to 0, there is a geodesic ray $\sigma_{y_0}(\tilde{t})$, $\tilde{t} \le 0$, such that $\sigma_{y_0}$ has constant $y$ coordinate, $y_k$, in each time interval $-(k+1)\ell < \tilde{t} < -k\ell$, for $k = 0, 1, 2, \ldots.$ The time coordinate $\tilde{t}$ and the $t$ coordinate of $\sigma_{y_0}(\tilde{t})$ are related by $\tilde{t} \equiv t \bmod \ell$. Since $y_k = \int_0^{y_{k+1}} \bar{\jmath}(\ell, \eta)\, d\eta$, for $k = 0, 1, 2, \ldots,$ and $\bar{\jmath}(\ell, \eta) > \exp\left[(1/2)\int_0^{\ell} v(\xi)\, d\xi\right] > 1$ for $\eta$ close to 0, it follows that $(y_k)$ is a decreasing sequence with $\lim_{k\to\infty} y_k = 0$. Consequently each such geodesic $\sigma_{y_0}$ is asymptotic to $\gamma$ as $\tilde{t} \to -\infty$. Hence the curves $t =$ constant are unstable horocycles. The geodesic curvature of the curve $t = t_0$ at $(t_0, y)$ is $\bar{v}(t_0, y)$, which is negative if $t_0 = p$ and $y \ne 0$. Thus there are focal points along $\sigma_{y_0}$, but not along $\gamma$.

Let $K$ be the curvature for the metric $\langle\ ,\ \rangle$ and its extension to $S$. Near $\gamma$, $K$ may be regarded as a function of the coordinates $(t, y)$, $0 \le t < \ell$, or the Fermi coordinates $(s, x)$ along $\gamma(s)$. The symmetry of the metric in the $(t, y)$ coordinate system with respect to $y = 0$ implies that

$$\partial K/\partial y \equiv 0 \tag{2.2}$$

along $\gamma$. Since $j(\partial/\partial x) = \partial/\partial y$ along $\gamma$, (1.1) is satisfied. It follows from the Riccati equation that

$$K(t, y) = -\frac{\partial \bar{v}}{\partial t}(t, y) - \bar{v}^2(t, y),$$

and by differentiating, we obtain

$$\frac{\partial^2 K}{\partial y^2} = -2w' - 4vw$$

and

$$\frac{\partial K}{\partial t} = -v'' - 2vv'$$

along $\gamma$.

Let $U$ be the gradient vector field for the function $t$. This vector field consists of unit normal vectors to the unstable horocycles. We have

$$\frac{\partial t}{\partial x} = \left\langle U, \frac{\partial}{\partial x} \right\rangle$$

and

$$\frac{\partial^2 t}{\partial x^2} = \frac{\partial}{\partial x} \left\langle U, \frac{\partial}{\partial x} \right\rangle = \left\langle \frac{DU}{dx}, \frac{\partial}{\partial x} \right\rangle + \left\langle U, \frac{D}{dx} \left( \frac{\partial}{\partial x} \right) \right\rangle .$$

The $(D/dx)(\partial/\partial x)$ term is 0, because $\partial/\partial x$ consists of vectors tangent to geodesics perpendicular to $\gamma$. Along $\gamma$ we have $\partial/\partial x = \partial/\partial z$ and

$$\frac{\partial^2 t}{\partial x^2} = \left\langle \frac{DU}{dz}, \frac{\partial}{\partial z} \right\rangle = v,$$

because $\langle DU/dz, \partial/\partial z \rangle$ is the curvature of the unstable horocycle.

Using the fact that $\partial t/\partial x \equiv 0$ along $\gamma$, together with (2.2), and applying the chain rule twice, we find that

$$\begin{aligned} \frac{\partial^2 K}{\partial x^2} &= \frac{\partial^2 K}{\partial y^2} \left( \frac{\partial y}{\partial x} \right)^2 + \frac{\partial K}{\partial t} \frac{\partial^2 t}{\partial x^2} \\ &= (-2w' - 4vw)(1/j)^2 + (-v'' - 2vv')v. \end{aligned}$$

Thus (1.2) will follow from (2.1), once that is established in §3.

## 3. Construction of the functions $v$ and $w$

If we multiply (2.1) by the integrating factor $j^4(t)$, then we see that (2.1) is equivalent to

$$\text{(3.1)} \qquad \Big( w(t) j^2(t) \Big)' > \frac{-j^4(t)}{2} \Big( v''(t) + 2v(t)v'(t) \Big) v(t)$$

We will choose $v$ so that $v$ is $C^\infty$, has period $\ell$, $v \geq 0$,

$$\text{(3.2)} \qquad \Big( v''(0) + 2v(0)v'(0) \Big) v(0) > 0,$$

$$\text{(3.3)} \qquad v(p) = 0,$$

and

$$\text{(3.4)} \qquad \int_I j^4(t) \Big( v''(t) + 2v(t)v'(t) \Big) v(t) \; dt > 0$$

for $I = [0, p]$ and for $I = [p, \ell]$. We can define $w$ in terms of $v$, as follows. Choose $\delta > 0$ sufficiently small so that $\big( v''(t) + 2v(t)v'(t) \big) v(t) > 0$ for $t \in [0, \delta]$ and $t \in [\ell - \delta, \ell]$ and (3.4) holds for $I = [\delta, p]$ and for $I = [p, \ell - \delta]$. Let $f(t) = -(j^4(t)/2) \big( v''(t) + 2v(t)v'(t) \big) v(t)$.

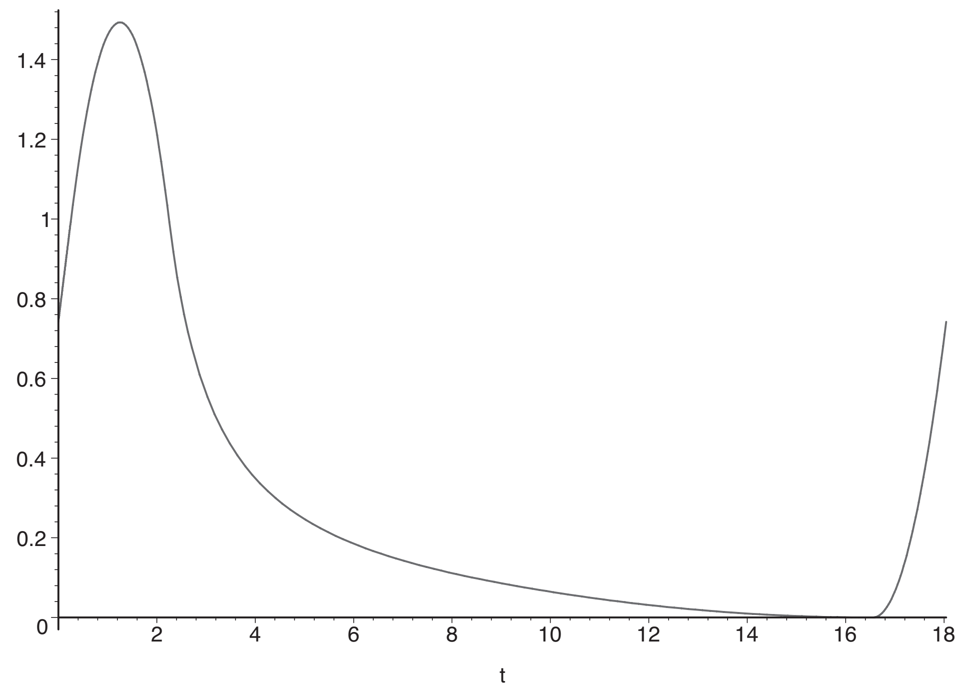


Figure 1

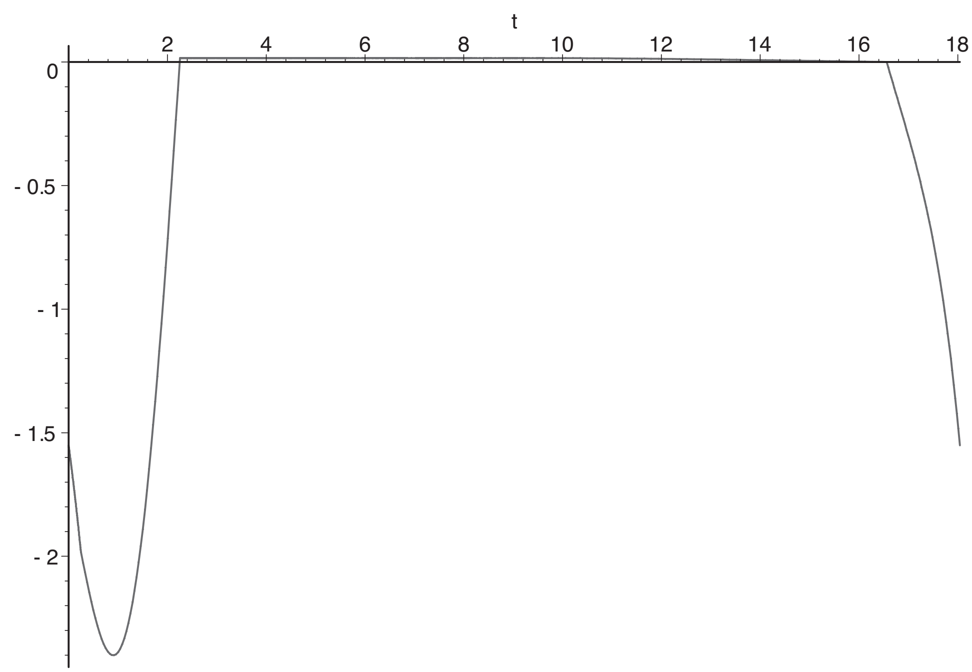


Figure 2

If $\delta > 0$ is sufficiently small, then there exists a $C^\infty$ function $g$ on $[0, \ell]$ such that $g \equiv 0$ in $[0, \delta] \cup [\ell - \delta, \ell]$, $g(t) > f(t)$ on $[0, \ell]$, $\int_0^p g(t)\, dt < 0$, and $\int_0^\ell g(t)\, dt = 0$. Let

$$w(t) = j^{-2}(t) \int_0^t g(\xi)\, d\xi. \tag{3.5}$$

Then $w \equiv 0$ on $[0, \delta] \cup [\ell - \delta, \ell]$, $w(p) < 0$, and $\big(w(t)j^2(t)\big)' = g(t) > f(t)$, which is (3.1). The function $w$ defined by (3.5) is not periodic due to the $j^{-2}(t)$ term, but we will just use (3.5) to define $w$ on $[0, \ell]$ and then extend $w$ periodically. Next we will define $v$ in terms of a parameter $\alpha$. This parameter will eventually be taken very small, but to define $v$, we only need $0 < \alpha < 1/2$. Figures 1 and 2 show the graphs of $v(t)$ and $K(t)$, respectively, for $\alpha = 1/8$. We calculate $K(t)$ from the Riccati equation $K(t) = -v'(t) - v^2(t)$. Our initial choice of $v$ will be $C^1$, but not $C^2$. At the end of our calculations we will explain how to modify $v$ to make it $C^\infty$, while maintaining the other properties that we need.

Let $\beta = (\sin^{-1}\alpha)/\alpha$, $a = 1/(2\beta)$, $b = \sqrt{1-\alpha^2} + \beta/2$, $c = 8\alpha^3/3^{3/2}$, $p = 1/4 + \beta + \pi/(3\alpha) + c^{-1/3}$, and $\ell = p + 2\sqrt{1-\alpha^2} - 1/2$. Define closed intervals $I_i = [a_i, b_i]$, $i = 1, \ldots, 5$, by $a_1 = 0$, $a_2 = 1/4$, $a_3 = 1/4 + 2\beta$, $a_4 = a_3 - \beta + \pi/(3\alpha) = p - c^{-1/3}$, $a_5 = p$, $b_5 = \ell$, and $a_{i+1} = b_i$, for $i = 1, \ldots, 4$. Let $c_1 = -1/4 + \sqrt{1-\alpha^2}$, $c_2 = -c^{-1/3} + \pi/(6\alpha)$, $c_3 = c/2$, $c_4 = (1/4)c_1^{-1}$, and $q = \beta + 1/4$. We define

$$v(t) = \begin{cases} t + c_1, & \text{for } t \in I_1, \\ -a(t-q)^2 + b, & \text{for } t \in I_2, \\ -\alpha \tan\Big(\alpha(t - p - c_2)\Big), & \text{for } t \in I_3, \\ c_3(t-p)^2, & \text{for } t \in I_4, \\ c_4(t-p)^2, & \text{for } t \in I_5, \end{cases}$$

and extend $v$ periodically to $(-\infty, \infty)$ with period $\ell$. It is clear that (3.3) is satisfied, and it is easy to verify that $v$ is $C^1$ and $v > 0$ on $[0, \ell]\backslash\{p\}$. Moreover, (3.2) is satisfied if $v''(0)$ is replaced by the second derivative of $v$ from the right or the left at 0.

We now prove that (3.4) holds for $I = [0, p]$. On $I_2$, $v''(t) = -2a$, $v'(t) = -2a(t - 1/4 - \beta)$, and $\sqrt{1-\alpha^2} \le v(t) \le b$. Also $1 \le \beta \le \pi/2$, $1/\pi \le a \le 1/2$, $b < 1 + \pi/4$, and $v'(t) \ge -1$. Thus $(v'' + 2vv')v \ge (-2a - 2v)v \ge (-2a - 2b)b \ge -(3 + \pi/2)(1 + \pi/4) > -9$ on $I_2$.

We obtain the following table:

| INTERVAL | $\big(v''(t) + 2v(t)v'(t)\big)v(t)$ |
|---|---|
| $I_1$ | positive |
| $I_2$ | greater than $-9$ |
| $I_3$ | 0 |

| | |
|---|---|
| $I_4$ | $\big(c+c^2(t-p)^3\big)(c/2)(t-p)^2$ |
| $I_5$ | positive except at $p$ |

Note that the entry in this table for $\big(v''(t)+2v(t)v'(t)\big)v(t)$ on $I_4$ is positive except at the endpoints of $I_4$ and that $\int_{I_4}\big(v''(t)+2v(t)v'(t)\big)v(t)\ dt = c/12 = (2/3^{5/2})\alpha^3$. Then

$$\begin{aligned}\int_0^p j^4(t)\Big(v''(t)+2v(t)v'(t)\Big)v(t)\ dt \geq& \\ & j^4(b_2)(b_2-a_2)(-9)+j^4(a_4)\int_{I_4}\Big(v''(t)+2v(t)v'(t)\Big)v(t)\ dt \\ & \geq -18\beta j^4(1/4+2\beta)+(2/3^{5/2})\alpha^3 j^4(p-c^{-1/3}). \qquad (3.6)\end{aligned}$$

We will show that the first term in (3.6) is bounded from below, independently of $\alpha$, while the second term goes to $\infty$ as $\alpha\to 0$. Since $v(t)\leq b$ on $I_1\cup I_2$,

$$\begin{aligned}-18\beta j^4(1/4+2\beta) &= -18\beta\exp\Big[4\textstyle\int_0^{1/4+2\beta} v(\xi)\ d\xi\Big] \\ &\geq -18\beta\exp\Big[4b(1/4+2\beta)\Big] \\ &\geq -9\pi\exp\Big[(4+\pi)(1/4+\pi)\Big].\end{aligned}$$

Also,

$$\begin{aligned}\int_{I_3} v(\xi)\ d\xi = \int_{-\frac{\pi}{2\alpha}+\beta}^{-\frac{\pi}{6\alpha}} -\alpha\tan(\alpha\tau)\ d\tau &= \log(\cos(\alpha\tau))\Big|_{-\pi/(2\alpha)+\beta}^{-\pi/(6\alpha)} \\ &= \log(\sqrt{3}/2)-\log\alpha.\end{aligned}$$

Therefore

$$\alpha^3 j^4(p-c^{-1/3})\geq \alpha^3\exp\Big[4\int_{I_3} v(\xi)\ d\xi\Big]\geq(\sqrt{3}/2)^4(1/\alpha).$$

Consequently, the expression in (3.6) goes to infinity as $\alpha$ goes to 0. We now fix a choice of $\alpha$ so that the expression in (3.6) is positive. Then (3.4) holds for $I=[0,p]$. It is clear that (3.4) holds for $I=[p,\ell]$, since the integrand is positive except at $p$. Thus $K$ satisfies condition (1.2).

We now replace $v$ by a $C^\infty$ function $\tilde{v}$ such that $\tilde{v}$ has period $\ell$, $\tilde{v}>0$ on $[0,\ell]\setminus\{p\}$, and $\tilde{v}$ satisfies (3.2), (3.3), and (3.4). (In fact, $\tilde{v}$ will be real analytic, but $w$ is only $C^\infty$.) Let $\tilde{v}_1(t)=\int_{-\infty}^{+\infty} v(\xi)P_\mu(t-\xi)\ d\xi$, where $P_\mu$ is the Poisson kernel, $P_\mu(t)=(\mu/\pi)/(t^2+\mu^2)$, and $\mu>0$. Then $\tilde{v}_1$ is $C^\infty$, has period $\ell$, and satisfies $\tilde{v}_1>0$. If $\mu$ is small, then $\tilde{v}_1$ has a minimum at $t=\tilde{p}$, where $\tilde{p}$ is close to $p$. Let $\tilde{v}(t)=\tilde{v}_1(t+\tilde{p}-p)-\tilde{v}_1(\tilde{p})$. For $\mu$ sufficiently small, $\tilde{v}$ satisfies (3.2), (3.3), and (3.4).

## References

[1] Ballmann, W., Brin, M., and Burns, K., *On surfaces with no conjugate points,* J. Differential Geom. **25** (1987), no. 2, 249-273.

[2] Eschenburg, J., *Horospheres and the stable part of the geodesic flow,* Math. Z. **153** (1977), no. 3, 237-251.

[3] Gerber, M., *On the existence of focal points near closed geodesics,* preprint (2001).

[4] Gulliver, R., *On the variety of manifolds without conjugate points,* Trans. Amer. Math. Soc. **210** (1975), 185-201.

[5] Hurley, D., *Ergodicity of the geodesic flow on rank one manifolds without focal points,* Proc. Roy. Irish Acad. Sect. A **86** (1986), no. 1, 19–30.

[6] O'Sullivan, John J., *Manifolds without conjugate points.* Math. Ann. **210** (1974), 295–311.

Department of Mathematics, Indiana University, Bloomington, IN 47405, U.S.A.
*E-mail address*: `gerber@indiana.edu`