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COMPLETELY INTEGRABLE TORUS ACTIONS ON SYMPLECTIC CONES

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Abstract. We study completely integrable torus actions on symplectic cones (equivalently, completely integrable torus actions on contact manifolds). We show that if the cone in question is the punctured cotangent bundle of a torus then the action has to be free. From this it follows easily, using hard results of Mañe and of Burago and Ivanov, that a metric on a torus whose geodesic flow admits global action-angle coordinates is necessarily flat thereby proving a conjecture of Toth and Zelditch.

1. Introduction

The main result of this paper is:

Theorem 1. *Suppose that an n-torus G acts effectively on the co-sphere bundle $M := S(T^*\mathbb{T}^n)$ of the standard n-torus $\mathbb{T}^n$ preserving the standard contact structure on M. Then the action of G is free.*

The motivation for proving the theorem comes from a recent work of Toth and Zelditch who studied the relation between the dynamics of the geodesic flow on a compact Riemannian manifold (Q, g) and the growth rate of L^∞ norms of L^2-normalized eigenfunctions of the Laplace operator Δ_g [TZ]. They showed that if the square root of the Laplace operator $\sqrt{\Delta_g}$ is "quantum completely integrable" and has uniformly bounded eigenfunctions then the metric g is flat, and hence by the Bieberbach theorems Q is finitely covered by a torus. The proof is particularly transparent when the geodesic flow is *toric integrable*. The latter means that there is an effective action of a torus $\mathbb{T}^n$, $n = \dim Q$, on the punctured cotangent bundle $T^*Q \smallsetminus Q$ of Q which

1. commutes with dilations $\rho_t : T^*Q \smallsetminus Q \to T^*Q \smallsetminus Q$, $\rho(q,p) = (q, e^t p)$,
2. preserves the standard symplectic form on T^*Q and
3. preserves the energy function $h(q,p) = g^*_q(p,p)$, where g^* denotes the metric on T^*Q dual to g. (The Hamiltonian flow of h is the geodesic flow.)

Note that any symplectic group action on the punctured cotangent bundle which commutes with dilations preserves the Liouville 1-form and is, therefore, Hamiltonian. Consequently if a metric on a manifold Q is toric integrable, the pull-back

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metric on a finite cover of Q is toric integrable as well. One is therefore lead to wonder if in the case of tori the boundedness of eigenfunctions is necessary for the flatness of the metric or if toric integrability by itself is enough. The main goal of this paper is to prove that, as conjectured by Toth and Zelditch in [TZ], toric integrable metrics on tori are flat:

Theorem 2. *Suppose that g is a toric integrable metric on a torus $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$. Then g is flat.*

The term "toric integrable" is apparently due to Toth and Zelditch (but the objects involved have been studied since the late 70's, e.g. by Colin de Verdière, Duistermaat and Guillemin). It describes a class of completely integrable systems slightly more general than the systems with global action-angle variables. Toric integrable systems are much more manageable than arbitrary completely integrable systems, and one can use the tools of symplectic and contact geometry to investigate them.

Recall that a symplectic cone is a symplectic manifold (M, ω) with a free proper action ρ_t of the real line which expands the symplectic form exponentially: ${\rho_t}^*\omega = e^t\omega$. For example the punctured cotangent bundle $T^*Q \smallsetminus Q$ with the standard symplectic form is a symplectic cone: the real line acts by dilations $\rho_t(q,p) = (q, e^t p)$ for all $q \in Q$, $p \in T_q^*Q$. Given a symplectic cone (M, ω, ρ_t), a function $h \in C^\infty(M)$ is *toric integrable* if there is an effective action of a torus G with $\dim G = \frac{1}{2}\dim M$ which preserves the symplectic form ω and the function h and commutes with dilations ρ_t. Any action of a torus G on a symplectic cone (M, ω, ρ_t) that commutes with dilations preserves a 1-form α with $d\alpha = \omega$ and hence is Hamiltonian. Thus toric integrability of a function h on a symplectic cone M amounts to the existence of $n = \frac{1}{2}\dim M$ functions $f_1, \dots, f_n$ which are homogeneous with respect to dilations, Poisson commute with each other and with h and whose Hamiltonian flows are all 2π-periodic.

Recall that if $\{f_1, \dots, f_n\}$ is a completely integrable system on a symplectic manifold (M, ω) and if the fibers of the map $f = (f_1, \dots f_n) : M \to \mathbb{R}^n$ are compact, then, by Arnold-Liouville theorem, in a neighborhood of every point of M the Hamiltonian vector fields of the functions $f_1, \dots f_n$ generate a Hamiltonian action of the n-torus $\mathbb{T}^n$ [A]. According to Duistermaat there are obstructions to these "local" $\mathbb{T}^n$ actions to patch up to an action of $\mathbb{T}^n$ on M — the monodromy of the period lattice [Du, GS]. Strictly speaking Duistermaat only considered patching together free torus actions, but essentially the same argument works in general [BoM]. If these "local" $\mathbb{T}^n$ actions patch to a global $\mathbb{T}^n$ action, there is a further obstruction to the existence of global action-angle variables: the "Chern class of the fibration $f : M \to \mathbb{R}^n$." (The expression is in quotation marks because if the torus action is not free then f is not a fibration. None the less one can still speak of the "Chern class" of f [BoM].) The second obstruction is easily seen to be nontrivial — there are completely integrable systems with global torus actions but no global action-angle variables. See, for example,

[Bt]. Thus toric integrability is weaker than the existence of global homogeneous action variables.

Toric integrable systems have not been studied systematicly. We will see in this paper that a good way to understand them is through the study contact toric manifolds. It appears that toric integrability is rare. It would be interesting to classify all compact manifolds admitting toric integrable geodesic flows. In particular it would be interesting to find out if there are manifolds other that S^2, S^3 and tori that admit such flows. (Toric integrable metrics on S^2 other than the round one were described by Colin de Verdière [CdV].) This will be addressed elsewhere. See [L1] for a first step in that direction.

A note on notation. Throughout the paper the Lie algebra of a Lie group denoted by a capital Roman letter will be denoted by the same small letter in the fraktur font: thus $\mathfrak{g}$ denotes the Lie algebra of a Lie group G etc. The identity element of a Lie group is denoted by 1. The natural pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$ will be denoted by $\langle\cdot,\cdot\rangle$.

When a Lie group G acts on a manifold M we denote the action by an element $g \in G$ on a point $x \in M$ by $g \cdot x$; $G \cdot x$ denotes the G-orbit of x and so on. The vector field induced on M by an element X of the Lie algebra $\mathfrak{g}$ of G is denoted by X_M. The isotropy group of a point $x \in M$ is denoted by G_x; the Lie algebra of G_x is denoted by $\mathfrak{g}_x$ and is referred to as the isotropy Lie algebra of x. We recall that $\mathfrak{g}_x = \{X \in \mathfrak{g} \mid X_M(x) = 0\}$.

If P is a principal G-bundle then $[p, m]$ denotes the point in the associated bundle $P \times_G M = (P \times M)/G$ which is the orbit of $(p, m) \in P \times M$.

2. From toric integrability to contact toric manifolds

In this section we recall an argument of Toth and Zelditch that Theorem 2 follows from Theorem 1. As the first step let us reduce the proof of Theorem 2 to a statement about actions of tori on their punctured cotangent bundles. To wit, suppose we know that any action of an n-torus G on $M = T^*\mathbb{T}^n \smallsetminus \mathbb{T}^n$ which is symplectic and commutes with the action of $\mathbb{R}$, is actually a free action. Then, as indicated in [TZ] we can apply a theorem of Mañe

Theorem 3 (Mañe, [M]). *Let (Q, g) be a Riemannian manifold with a geodesic flow $\phi_t : T^*Q \smallsetminus Q \to T^*Q \smallsetminus Q$. Suppose the flow ϕ_t preserves the leaves of a non-singular Lagrangian foliation of $T^*Q \smallsetminus Q$. Then (Q, g) has no conjugate points.*

to conclude that the toric integrable metric on $\mathbb{T}^n$ has no conjugate points. Finally, following [TZ] again, and applying

Theorem 4 (Burago-Ivanov, [BI]). *A metric on a torus $\mathbb{T}^n$ with no conjugate points is flat.*

we can conclude that a toric-integrable metric is flat. To summarize, in order to prove Theorem 2 it is enough to show

Proposition 2.1. *Suppose that an n torus G acts effectively and symplecticly on the punctured cotangent bundle $T^*\mathbb{T}^n \smallsetminus \mathbb{T}^n$, and suppose that the action commutes with the action of $\mathbb{R}$. Then the action of G is free.*

Clearly the lift of the left multiplication on $\mathbb{T}^n$ to the cotangent bundle $T^*\mathbb{T}^n$ satisfies both the hypotheses of the proposition and the conclusion. The crux is to show that an arbitrary action of $G \simeq \mathbb{T}^n$ satisfying the hypotheses has to be free.

Under the hypotheses of the proposition, the action of G descends to an action on the co-sphere bundle $M := (T^*\mathbb{T}^n \smallsetminus \mathbb{T}^n)/\mathbb{R}$. Moreover this induced action G preserves the natural contact structure ξ on M (we'll discuss contact structures in more detail in the next section). Consequently the proof of Theorem 2 reduces to Theorem 1.

Our strategy for proving Theorem 1 is to study completely integrable torus actions on arbitrary (compact connected co-oriented) contact manifolds and to show that if an action is not free then the underlying manifold cannot be the product of a torus and a sphere of the appropriate dimensions.

3. Group actions on contact manifolds

Recall that a co-oriented contact manifold is a pair (M, ξ) where $\xi \subset TM$ is a distribution globally given as the kernel of a 1-form α such that $d\alpha|_\xi$ is nondegenerate. Such a 1-form α is called a contact form and the distribution ξ is called a contact structure. A co-sphere bundle $S(T^*N)$ of a manifold N (defined with respect to some metric) is a natural example of a contact manifold: the contact form is the restriction of the Liouville 1-form to the co-sphere bundle.

The condition that a distribution $\xi \subset TM$ is contact is equivalent to: the punctured line bundle $\xi^\circ \smallsetminus M$ is a symplectic submanifold of the punctured cotangent bundle $T^*M \smallsetminus M$, where ξ° denotes the annihilator of ξ in T^*M. Note that if $\xi = \ker\alpha$ then the 1-form α is a nowhere zero section of the line bundle $\xi^\circ \to M$. (Conversely any nowhere vanishing section of ξ° is a contact form.) Thus if $\xi = \ker\alpha$ then $\xi^\circ \smallsetminus M$ has two components. If a compact connected Lie group G acts on M and preserves the contact distribution ξ, then the action of G on ξ° maps the components of $\xi^\circ \smallsetminus M$ into themselves. Hence given a contact 1-form α with $\xi = \ker\alpha$, we can average it over G and obtain a G-invariant contact form $\bar\alpha$ with $\xi = \ker\bar\alpha$. Note that each component of $\xi^\circ \smallsetminus M$ is the symplectization of (M, ξ).

Definition 3.1. An action of a torus G on a contact manifold (M, ξ) is *completely integrable* if it is effective, preserves the contact structure ξ and if $2\dim G = \dim M + 1$.

A *contact toric G-manifold* is a co-oriented contact manifold with a completely integrable action of a torus G.

Note that if an action of a torus G on (M, ξ) is completely integrable, then the action of G on a component ξ°_+ of $\xi^\circ \smallsetminus M$ is a completely integrable Hamiltonian action and thus ξ°_+ is a symplectic toric manifold (for more information on symplectic toric manifolds and orbifolds see [D] and [LT]).

Completely integrable torus actions on co-oriented contact manifolds and contact toric manifolds have been studied by Banyaga and Molino [BM1, BM2, B] and by Boyer and Galicki [BG]. To state their results it would be convenient to first digress on the subject of moment maps for group actions on contact manifolds.

If a Lie group G acts on a manifold M preserving a contact form α, the corresponding *α-moment map* $\Psi_\alpha : M \to \mathfrak{g}^*$ is defined by

$$\langle \Psi_\alpha(x), X\rangle = \alpha_x(X_M(x)) \tag{3.1}$$

for all $x \in M$ and all $X \in \mathfrak{g}$, where X_M denotes the vector field corresponding to X induced by the infinitesimal action of the Lie algebra $\mathfrak{g}$ of the group G: $X_M(x) = \frac{d}{dt}|_{t=0}(\exp tX)\cdot x$.

Note that if f is a G-invariant nowhere zero function, then $\alpha' = f\alpha$ is also a G-invariant contact form defining the same contact structure. Clearly the corresponding moment map $\Psi_{\alpha'}$ satisfies $\Psi_{\alpha'} = f\Psi_\alpha$. Thus the definition of a contact moment map above is somewhat problematic: it depends on a choice of an invariant contact form rather then solely on the contact structure and the action. Fortunately there is also a notion of a contact moment map that doesn't have this problem. Namely, suppose again that a Lie group G acts on a manifold M preserving a contact structure ξ (and its co-orientation). The lift of the action of G to the cotangent bundle then preserves a component ξ°_+ of $\xi^\circ \smallsetminus M$. The restriction $\Psi = \Phi|_{\xi^\circ_+}$ of the moment map Φ for the action of G on T^*M to depends only on the action of the group and on the contact structure. Moreover, since $\Phi : T^*M \to \mathfrak{g}^*$ is given by the formula

$$\langle \Phi(q,p), X\rangle = \langle p, X_M(q)\rangle$$

for all $q \in M$, $p \in T^*_qM$ and $X \in \mathfrak{g}$, we see that if α is any invariant contact form with $\ker\alpha = \xi$ then $\langle \alpha^*\Psi(q,p), X\rangle = \langle \alpha^*\Phi(q,p), X\rangle = \langle \alpha_q, X_M(q)\rangle = \langle \Psi_\alpha(q), X\rangle$. Here we think of α as a section of $\xi^\circ_+ \to M$. Thus $\Psi \circ \alpha = \Psi_\alpha$, that is, $\Psi = \Phi|_{\xi^\circ}$ is a "universal" moment map.

There is another reason why the universal moment map $\Psi : \xi^\circ_+ \to \mathfrak{g}^*$ is a more natural notion of the moment map than the one given by (3.1). The vector fields induced by the action of G preserving a contact distribution ξ are contact. The space of contact vector fields is isomorphic to the space of sections of the bundle $TM/\xi \to M$. Thus a contact group action gives rise to a linear map

$$\mathfrak{g} \to \Gamma(TM/\xi), \quad X \mapsto X_M \mod \xi. \tag{3.2}$$

The moment map should be the transpose of the map (3.2). The total space of the bundle $(TM/\xi)^*$ naturally maps into the space dual to the space of sections $\Gamma(TM/\xi)$:

$$(TM/\xi)^* \ni \eta \mapsto (s \mapsto \langle \eta, s(\pi(\eta))\rangle),$$

where $\pi : (TM/\xi)^* \to M$ is the projection and $\langle\cdot,\cdot\rangle$ is the paring between the corresponding fibers of $(TM/\xi)^*$ and TM/ξ. In other words, the transpose

$\Psi : (TM/\xi)^* \to \mathfrak{g}^*$ of (3.2) should be given by

$$\langle \Psi(\eta), X\rangle = \langle \eta, X_M(\pi(\eta)) \mod \xi\rangle \tag{3.3}$$

Under the identification $\xi^\circ \simeq (TM/\xi)^*$, the equation above becomes

$$\langle \Psi(q,p), X\rangle = \langle p, X_M(q)\rangle$$

for all $q \in M$, $p \in \xi^\circ_q$ and $X \in \mathfrak{g}$, which is the definition of Ψ given earlier as the restriction to ξ°_+ of the moment map for the lifted action of G on the cotangent bundle T^*M.

Thus part of the above discussion can be summarized as

Proposition 3.2. *Let (M, ξ) be a co-oriented contact manifold with an action of a Lie group G preserving the contact distribution and its co-orientation. Suppose there exists an invariant 1-form α with $\ker \alpha = \xi$. Then the moment map Ψ_α for the action of G on (M, α) and the moment map Ψ for the action of G on the symplectization ξ°_+ are related by*

$$\Psi \circ \alpha = \Psi_\alpha.$$

Here ξ°_+ is the component of $\xi^\circ \smallsetminus 0$ containing the image of $\alpha : M \to \xi^\circ$.

Remark 3.3. We will refer to $\Psi : \xi^\circ_+ \to \mathfrak{g}^*$ as the moment map for the action of a Lie group G on a co-oriented contact manifold $(M, \xi = \ker \alpha)$. It is easy to show that Ψ is G-equivariant with respect to the given action of G on M and the coadjoint action of G on $\mathfrak{g}^*$. Hence for any invariant contact form α the corresponding moment map $\Psi_\alpha : M \to \mathfrak{g}^*$ is also G-equivariant.

Definition 3.4. Let $(M, \xi = \ker \alpha)$ be a co-oriented contact manifold with an action of a Lie group G preserving the contact distribution and its co-orientation. Let $\Psi : \xi^\circ_+ \to \mathfrak{g}^*$ denote the corresponding moment map. We define the *moment cone* $C(\Psi)$ to be the image of a connected component ξ°_+ of $\xi^\circ \smallsetminus M$ plus the origin:

$$C(\Psi) := \Psi(\xi^\circ_+) \cup \{0\}.$$

Remark that

$$C(\Psi) = \mathbb{R}^+ \Psi_\alpha(M) \cup \{0\}$$

where $\Psi_\alpha : M \to \mathfrak{g}^*$ is the α-moment map.

Note that the moment cone does not depend on the choice of a contact form; it is a true invariant of the co-oriented contact structure and the group action.

Remark 3.5. An action of a Lie group G on a manifold M preserving a contact form α is completely encoded in the moment map $\Psi_\alpha : M \to \mathfrak{g}^*$. Therefore it will be convenient for us to think of a *contact toric G-manifold* as an equivalence class of triples $(M, \alpha, \Psi_\alpha : M \to \mathfrak{g}^*)$ where the Ψ_α is the moment map for a completely integrable action of a torus G on a contact manifold (M, α), or, somewhat more sloppily, as a triple $(M, \alpha, \Psi_\alpha : M \to \mathfrak{g}^*)$.

4. Proof of Theorem 1

Banyaga and Molino made the first step towards classifying compact connected contact toric manifolds in [BM1]. A revised version of this paper circulated as the preprint [BM2]. The main classification result of [BM2] is cited in [B] roughly as follows:

Theorem 4.1. *Let $(M, \alpha, \Psi_\alpha : M \to \mathfrak{g}^*)$ be a compact connected contact toric G-manifold.*

Suppose the action of G on M is free. Then the orbit space M/G is diffeomorphic to a sphere. If additionally $\dim G > 2$ then the map $\bar{\Psi}_\alpha : M/G \to \mathfrak{g}^*$ induced by the moment map Ψ_α is an embedding. If furthermore $\dim G > 3$, then M is the co-sphere bundle of G, i.e., $M = S(T^*G)$.

Suppose the action of G on M is not free and suppose $\dim G > 2$. Then the moment cone $C(\Psi)$ is a convex polyhedral cone and the map $\bar{\Psi}_\alpha : M/G \to \mathfrak{g}^*$ induced by the moment map is an embedding. Moreover the cone $C(\Psi)$ determines the contact toric manifold.

Remark 4.2. It is easy to construct examples of a completely integrable action of a 2-torus on a contact 3-torus for which the fibers of the corresponding moment map are not connected: let $M = \mathbb{T}^3$ with coordinates θ_1, θ_2 and t, let $\alpha = \cos 2t\, d\theta_1 + \sin 2t\, d\theta_2$ be the contact form, and let $\mathbb{T}^2$ act by $(\mu, \nu) \cdot (\theta_1, \theta_2, t) = (\theta_1 + \mu, \theta_2 + \nu, t)$. Also there are examples of completely integrable 2-torus actions on overtwisted lens spaces for which the corresponding moment cones are not convex. See [L2].

Contact toric manifolds have also been studied by Boyer and Galicki [BG]. The following result is implicit in their paper:

Theorem 4.3. *Let $(M, \alpha, \Psi_\alpha : M \to \mathfrak{g}^*)$ be a compact connected contact toric G-manifold. Suppose there exits a vector $X \in \mathfrak{g}$ such that the component of the moment map $\langle \Psi_\alpha, X \rangle$ is strictly positive on M. Then M is a Seifert bundle over a (compact) symplectic toric orbifold.*

We remind the reader that a symplectic toric orbifold is a symplectic orbifold with a completely integrable Hamiltonian torus action. Compact connected symplectic toric orbifolds were classified in [LT].

Proof of Theorem 4.3. Since M is compact, the image $\Psi_\alpha(M)$ is compact. Therefore the set of vectors $X' \in \mathfrak{g}$, such that the function $\langle \Psi_\alpha, X' \rangle$ is strictly positive on M, is open. Hence we may assume that X lies in the integral lattice $\mathbb{Z}_G := \ker(\exp : \mathfrak{g} \to G)$ of the torus G. Let $H = \{\exp tX \mid t \in \mathbb{R}\}$ be the corresponding circle subgroup of G.

Let $f(x) = 1/(\langle \Psi_\alpha(x), X \rangle)$ and let $\alpha' = f\alpha$. The form α' is another G-invariant contact form with $\ker \alpha' = \xi$. The moment map $\Psi_{\alpha'}$ defined by α' satisfies $\Psi_{\alpha'} = f \Psi_\alpha$. Therefore $\langle \Psi_{\alpha'}(x), X \rangle = 1$ for all $x \in M$.

Since the function $\langle \Psi_\alpha, X \rangle$ is nowhere zero, the action of H on M is locally free. Consequently the induced action of H on the symplectization $(N, \omega) =$

$(M \times \mathbb{R}, d(e^t\alpha'))$ is locally free as well. Hence any $a \in \mathbb{R}$ is a regular value of the X-component $\langle \Phi, X\rangle$ of the moment map Φ for the action of G on the symplectization (N, ω). Note that $\Phi(x,t) = -e^t\Psi_{\alpha'}(x)$. Now $M \times \{0\}$ is the -1 level set of $\langle \Phi, X\rangle$. Therefore $B := (\langle \Phi, X\rangle)^{-1}(-1)/H \simeq M/H$ is a (compact connected) symplectic orbifold with an effective Hamiltonian action of G/H. The orbit map $\pi : M \simeq (\langle \Phi, X\rangle)^{-1}(-1) \to B$ makes M into a Seifert bundle over B. A dimension count shows that the action of G/H on B is completely integrable. □

Remark 4.4. It is easy to see that the moment cone for the action of G on M is the cone on the moment polytope of B. In particular it is a proper polyhedral cone, that is, it contains no linear subspaces.

Corollary 4.5. *Let $(M, \alpha, \Psi_\alpha : M \to \mathfrak{g}^*)$ be a (compact connected) contact toric manifold. Suppose there exits a vector $X \in \mathfrak{g}$ such that the component of the moment map $\langle \Psi_\alpha, X\rangle$ is strictly positive on M. Then* $\dim_{\mathbb{Q}} H^1(M, \mathbb{Q}) \leq 1$.

Proof. By Theorem 4.3 the manifold M is a Seifert bundle over a compact connected symplectic toric orbifold B. A generic component of the moment map on B is a Morse function with all indices even. The Morse inequalities hold rationally for Morse functions on orbifolds (see [LT]). Therefore the first cohomology $H^1(B, \mathbb{Q})$ is zero.

Next we apply the Gysin sequence to the map $\pi : M \to B$. Since the Gysin sequence comes from the collapse of the Leray-Serre spectral sequence for π and since rationally the "fibration" π is a circle bundle, the Gysin sequence does exist. We have $0 \to H^1(B, \mathbb{Q}) \to H^1(M, \mathbb{Q}) \stackrel{\pi_*}{\to} H^0(B, \mathbb{Q}) \to H^2(B, \mathbb{Q}) \to \cdots$. Since $H^0(B, \mathbb{Q}) = \mathbb{Q}$ and since $H^1(B, \mathbb{Q}) = 0$, the result follows. □

We conclude immediately

Corollary 4.6. *If M is a contact toric manifold satisfying the hypotheses of the Theorem 4.3, then M is not the co-sphere bundle of a torus.*

Combining Corollary 4.6 with Theorem 4.1 we see that if an n-dimensional torus G ($n > 2$) acts on the co-sphere bundle $M = S(T^*\mathbb{T}^n)$ preserving a contact form α and if the action is not free, then the corresponding moment cone $C(\Psi)$ contains a linear subspace P of positive dimension.

Proposition 4.7. *Suppose $(M, \alpha, \Psi_\alpha : M \to \mathfrak{g}^*)$ is a compact connected contact toric G-manifold of dimension $2n-1 > 3$, the action of G on M is not free and the moment cone $C(\Psi)$ contains a linear subspace P of dimension k, $0 < k < n$. Then* $\dim H^1(M, \mathbb{Q}) = k \neq n = \dim H^1(S(T^*\mathbb{T}^n), \mathbb{Q})$.

Proof. By Theorem 4.1 the fibers of the contact moment map Ψ_α are connected. Let $\Phi : M \times \mathbb{R} \to \mathfrak{g}^*$ denote the symplectic moment map for the Hamiltonian action of G on the symplectization $(M \times \mathbb{R}, d(e^t\alpha))$ of (M, α). It is given by $\Phi(m,t) = -e^t\Psi_\alpha(m)$. Thus $\Phi(M \times \mathbb{R}) \cup \{0\} = -C(\Psi)$ and the fibers of Φ are connected. The triple $(M \times \mathbb{R}, d(e^t\alpha), \Phi)$ is a symplectic toric manifold.

Since the image of Φ is contractible and since the fibers of Φ are connected, it follows from a result of Lerman, Tolman and Woodward (Lemma 7.2 and Proposition 7.3 in [LT]), that the image of Φ determines the symplectic toric manifold $(M \times \mathbb{R}, d(e^t\alpha), \Phi)$ uniquely.[1] In particular the image determines the first cohomology group of M.

A standard argument for Hamiltonian G-spaces implies that the subspace P is the annihilator of the Lie algebra $\mathfrak{h}$ of a subtorus H of G. Since H is a subtorus, the exact sequence

$$1 \to H \to G \to G/H \to 1$$

splits. Let $K \simeq G/H$ be a subtorus in G complementary to H and let $\mathfrak{g}^* = \mathfrak{h}^* \times \mathfrak{k}^*$ be the corresponding splitting of the duals of the Lie algebras. Then $P \simeq \mathfrak{k}^*$ and $C(\Psi_\alpha) = P \times W$ where $W \subset \mathfrak{h}^*$ is a proper cone. It follows from a theorem of Delzant [D] that the exists a basis $w_1, w_2, \dots, w_l$ of weight lattice of the torus H so that the edges of the cone W are of the form $\mathbb{R}^+ w_i$. In particular the representation of H on $\mathbb{C}^l$ defined by the infinitesimal characters $w_1, \dots, w_l$ has the property that the image of the corresponding moment map is W. Consequently we can realize $\Phi(M \times \mathbb{R})$ as the image of the moment map for the product action of $K \times H$ on $K \times \big(\mathfrak{k}^* \times \mathbb{C}^l \smallsetminus (0,0)\big) \subset T^*K \times \mathbb{C}^l$. Therefore $M \times \mathbb{R}$ is G-equivariantly symplectomorphic to $K \times \big(\mathfrak{k}^* \times \mathbb{C}^l \smallsetminus (0,0)\big)$, which is homotopy equivalent to $K \times S^{k+2l-1} = (S^1)^k \times S^{k+2l-1}$. Since $0 < k < n$ and since $l = n - k > 0$, $k = \dim H^1((S^1)^k \times S^{k+2l-1}) = \dim H^1(M)$. □

It remains to consider the case where $(M, \alpha, \Psi_\alpha : M \to \mathfrak{g}^*)$ is a contact toric manifold of dimension 3 and the action of the 2-torus G is not free.

Proposition 4.8. *Suppose $(M, \alpha, \Psi_\alpha : M \to \mathfrak{g}^*)$ is a compact connected contact toric G-manifold of dimension 3 and suppose the action of the 2-torus G is not free. Then M is a lens space and hence cannot be the co-sphere bundle of a 2-torus.*

Remark 4.9. We consider the 3-manifold $S^1 \times S^2$ a lens space.

Proof of Proposition 4.8. As we remarked earlier the symplectization $(M \times \mathbb{R}, d(e^t\alpha), \Phi(m,t) = -e^t\Psi_\alpha(m))$ is a symplectic toric manifold. Delzant [D] showed that for symplectic toric manifolds all the isotropy groups are connected and all fixed points are isolated. If a point $x \in M$ is fixed by the action of G then the line $\{x\} \times \mathbb{R}$ is fixed by the action of G on $M \times \mathbb{R}$. Therefore the action of G on a contact toric G-manifold has no fixed points (one can also give a direct proof of this fact).

Next we use the fact that $\dim M = 3$ and $\dim G = 2$. By the above observations the isotropy groups for the action of G on M are either trivial or circles. If the isotropy group of $x \in M$ is trivial, then a neighborhood of the orbit $G \cdot x$ in M is G-equivariantly diffeomorphic to $G \times (-\epsilon, \epsilon)$ for some small epsilon. If the isotropy group of a point $x \in M$ is a circle $H < G$, then a neighborhood of

[1] In proving the result Lerman, Tolman and Woodward rediscovered the ideas of Boucetta and Molino [BoM].

the orbit $G \cdot x$ in M is G-equivariantly diffeomorphic to $G \times_H D^2$ for a small disk $D^2 = \{z \in \mathbb{C} \mid |z| < \epsilon\}$. Moreover the action of H on D^2 must be effective; hence we may identify H with S^1 in such a way that the action of H on D^2 is given by $\lambda \cdot z = \lambda z$.

We conclude that locally M/G is homeomorphic to either $(-\epsilon, \epsilon)$ or to $D^2/S^1 \simeq [0, \epsilon)$. Thus if the action of G on M is not free, then M/G is a one-dimensional manifold with boundary. Since M is compact and connected, M/G has to be an interval. Therefore, by a theorem of Haefliger and Salem (Proposition 4.2 in [HS]), M as a G-space is uniquely determined by the isotropy representations at the points in M above the endpoints of the interval M/G. It is easy to see that in this case M is diffeomorphic to two solid tori glued along their boundaries. We conclude that if M is a three dimensional compact connected contact toric manifold and if the action of a 2-torus G is not free, then M is a lens space. In particular M is not diffeomorphic to $\mathbb{T}^3 = S(T^*\mathbb{T}^2)$. □

This finishes the proof of the main result, Theorem 1.

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