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SCATTERING THEORY AND ADIABATIC DECOMPOSITION OF THE ζ-DETERMINANT OF THE DIRAC LAPLACIAN

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Abstract. In this note we announce the adiabatic decomposition formula for the ζ-determinant of the Dirac Laplacian. Theorem 1.1 of this paper extends the result of our earlier work (see [8] and [9]), which covered the case of the invertible tangential operator. The presence of the non-trivial kernel of the tangential operator requires careful analysis of the small eigenvalues of the Dirac Laplacian, which employs elements of scattering theory.

1. Statement of the Result

Let $\mathcal{D} : C^\infty(M; S) \to C^\infty(M; S)$ denote a compatible Dirac operator acting on sections of a bundle of Clifford modules S over a closed manifold M of dimension $2k+1$. Assume that we have a decomposition of M as $M_1 \cup M_2$, where M_1 and M_2 are compact manifolds with boundary so that

$$M = M_1 \cup M_2 \;\; , \;\; M_1 \cap M_2 = Y = \partial M_1 = \partial M_2 \;\; . \tag{1.1}$$

We assume that M and the operator $\mathcal{D}$ have product structures in a neighborhood of the boundary Y. More precisely, we assume that there is a bicollar neighborhood $N = [-1, 1] \times Y$ of Y in M such that both the Riemannian structure on M and the Hermitian structure on S are products when restricted to N. This implies that $\mathcal{D}$ has the following form when restricted to the submanifold N

$$\mathcal{D} = G(\partial_u + B) \;\; . \tag{1.2}$$

Here u denotes a normal variable, $G : S|Y \to S|Y$ is a bundle automorphism and B is the corresponding Dirac operator on Y. Moreover, G and B do not depend on u and they satisfy

$$G^* = -G \;\; , \;\; G^2 = -Id \;\; , \;\; B = B^* \;\; \textit{and} \;\; GB = -BG \;\; . \tag{1.3}$$

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The operator B has a discrete spectrum with infinitely many positive and infinitely many negative eigenvalues. Let $\Pi_>$ (resp. $\Pi_<$) denote the spectral projections onto the subspaces spanned by the eigensections of B corresponding to the positive (resp. negative) eigenvalues and
$\sigma_1, \sigma_2 : ker\ B \to ker\ B$ denote the involutions of kernel of B such that

$$G\sigma_i = -\sigma_i G \ . \tag{1.4}$$

Let $\pi_i = \frac{Id-\sigma_i}{2}$ denote the orthogonal projections of the kernel of B onto -1 eigenspace of σ_i . The orthogonal projections $P_1 = \Pi_< + \pi_1$ and $P_2 = \Pi_> + \pi_2$ provide elliptic self-adjoint boundary conditions for the operators $\mathcal{D}_1 = \mathcal{D}|M_1$ and $\mathcal{D}_2 = \mathcal{D}|M_2$ respectively. This means that the associated operators

$$(\mathcal{D}_i)_{P_i} : dom\ (\mathcal{D}_i)_{P_i} \to L^2(M_i; S|M_i)$$

with domains $\mathrm{dom}(\mathcal{D}_i)_{P_i} = \{s \in H^1(M_i; S|M_i); P_i(s|Y) = 0\}$ are self-adjoint Fredholm operators with $ker((\mathcal{D}_i)_{P_i}) \subset C^\infty(M_i; S|M_i)$ and they both have discrete spectrum (see [1], [11]).

We now introduce the manifold M_R equal to the manifold M with N replaced by $N_R = [-R, R] \times Y$ and $M_{1,R}$, $M_{1,\infty}$, $M_{2,R}$, $M_{2,\infty}$ which are manifolds M_1 or M_2 with the semicylinder $[0, R] \times Y$, $[0, \infty) \times Y$ or $[-R, 0] \times Y$, $(-\infty, 0] \times Y$ attached to them. Let $\mathcal{D}_R$, $\mathcal{D}_{i,R}$, $\mathcal{D}_{i,\infty}$ denote the natural extension of $\mathcal{D}$ to M_R , $M_{i,R}$, $M_{i,\infty}$ for $i = 1, 2$. We also use $\mathcal{D}_{1,\infty}$, $\mathcal{D}_{2,\infty}$ to denote the unique closed self-adjoint extension of those operators in the spaces $L^2(M_{1,\infty}; S)$ and $L^2(M_{2,\infty}; S)$. The operator $\mathcal{D}_R$ is a self-adjoint operator on $L^2(M_R; S)$ and as such it has a discrete spectrum only. Analysis of the eigenvalues shows that they fall into three different categories. We have large eigenvalues ($l-values$) bounded away from 0 . Then there is infinitely many small eigenvalues ($s-values$) , which are of the size $O(\frac{1}{R})$. Last, we have a finite amount of eigenvalues, which decay exponentially with R ($e - values$). There exists R_0 , such that for any $R > R_0$ number h_M of $e - values$ does not depend on R and we have the formula

$$h_M = \dim\left(ker_{L^2}(\mathcal{D}_{1,\infty})\right) + \dim\left(ker_{L^2}(\mathcal{D}_{2,\infty})\right) + \dim\left(L_1 \cap L_2\right)$$

(see [3], see also [6] and [12] for additional discussion). Here, $L_i \subset ker(B)$ denotes the space of the extended L^2-solutions of $\mathcal{D}_{i,\infty}$.

We define a modified zeta function of $\mathcal{D}_R^2$ by the formula

$$\zeta_{\mathcal{D}_R^2}(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} Tr'(e^{-t\mathcal{D}_R^2})\ dt$$

where $Tr'(\cdot)$ is taken over all the eigenvalues with the exception of $e - values$. The operators $(\mathcal{D}_{i,R})_{P_i}$ do not have $e - values$ (see [6]) and $h_i = \dim(ker(\mathcal{D}_{i,R})_{P_i})$ is equal to

$$\dim(ker_{L^2}(\mathcal{D}_{i,\infty})) + \dim(L_i \cap ker(\sigma_i - 1)) \ .$$

We define the zeta functions of $(\mathcal{D}_{i,R})^2_{P_i}$ by

$$\zeta_{(\mathcal{D}_{i,R})^2_{P_i}}(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} Tr'(e^{-t(\mathcal{D}_{i,R})^2_{P_i}})\, dt$$

where $Tr'(\cdot)$ is taken over the non-zero eigenvalues for $i = 1, 2$. The zeta functions $\zeta_{\mathcal{D}^2_R}(s)$, $\zeta_{(\mathcal{D}_{i,R})^2_{P_i}}(s)$ are regular at $s = 0$ and we can define the ζ-regularized determinants for these operators using the standard formula

$$ln \ \det{}_\zeta \mathfrak{D}^2_R = -\frac{d}{ds}\{\zeta_{\mathfrak{D}^2_R}(s)\}|_{s=0} \ , \tag{1.5}$$

where $\mathfrak{D}^2_R$ denotes one of the aforementioned operators. In this announcement we study the following adiabatic limit

$$\lim_{R\to 0} \frac{\det_\zeta \mathcal{D}^2_R}{\det_\zeta(\mathcal{D}_{1,R})^2_{P_1} \cdot \det_\zeta(\mathcal{D}_{2,R})^2_{P_2}} \ . \tag{1.6}$$

We have to introduce elements of *Scattering Theory* in order to present the formula for the limit (1.6). The operators $\mathcal{D}^2_{i,\infty}$ over $M_{i,\infty}$ have continuous spectrum equal to $[0, \infty)$. The number $\lambda \in [0, \infty)$ and $\phi \in ker(B)$ determine a generalized eigensection of $\mathcal{D}_{1,\infty}$, which has the following form on $[0, \infty) \times Y \subset M_{1,\infty}$ (see (4.24) in [6])

$$E(\phi, \lambda) = e^{-i\lambda u}(\phi - iG\phi) + e^{i\lambda u} C_1(\lambda)(\phi - iG\phi) + \theta(\phi, \lambda)$$

where $\theta(\phi, \lambda)$ is a square integrable section of S on $M_{1,\infty}$ which is orthogonal to $\ker(B)$, when restricted to $\{u\} \times Y$, and $C_1(\lambda)$ is the scattering matrix. There is also corresponding scattering matrix $C_2(\lambda)$ determined by the operator $\mathcal{D}_{2,\infty}$ over $M_{2,\infty}$. We refer to [6] and [7] (see also [5]) for the presentation of the necessary material from *Scattering Theory*.

Let $C : W \to W$ denote a unitary operator acting on the finite dimensional vector space W. We introduce the operator $D(C)$ equal to the differential operator $-i\frac{1}{2}\frac{d}{du}$ acting on $L^2(S^1, E_C)$ where E_C is the flat vector bundle over $S^1 = \mathbb{R}/\mathbb{Z}$ with the complex conjugate of C as the holonomy group. We define operators

$$I_1 = (G + i) : \ker(B) \to \ker(G - i) \ ,$$

$$I_2 = (G - i) : \ker(B) \to \ker(G + i) \quad ,$$

$$P_{\sigma_i} = \frac{1}{2}(\sigma_i - 1) : \ker(B) \to \ker(\sigma_i + 1)$$

and

$$S_{\sigma_i}(\lambda) = -P_{\sigma_i} \circ C_i(\lambda) \circ I_i|_{ker(\sigma_i+1)} : ker(\sigma_i + 1) \to ker(\sigma_i + 1) \quad .$$

Then $C_{12} := C_1(0) \circ C_2(0)|_{\ker(G+i)}$, $S_{\sigma_1} := S_{\sigma_1}(0)$ and $S_{\sigma_2} := S_{\sigma_2}(0)$ are the unitary operators acting on the finite dimensional vector spaces and we have well-defined self-adjoint, elliptic operators $D(C_{12}), D(S_{\sigma_1}), D(S_{\sigma_2})$. Now, we are ready to state the main result

Theorem 1.1. *The following formula holds:*

$$\lim_{R\to\infty} \frac{R^{2h} \cdot \det_\zeta \mathcal{D}_R^2}{\det_\zeta(\mathcal{D}_{1,R})^2_{P_1} \cdot \det_\zeta(\mathcal{D}_{2,R})^2_{P_2}} = \frac{2^{-\zeta_{B^2}(0)} \cdot \det_\zeta \frac{1}{4} D(C_{12})^2}{\det_\zeta D(S_{\sigma_1})^2 \cdot \det_\zeta D(S_{\sigma_2})^2} \tag{1.7}$$

where $h = h_M - h_1 - h_2 = \dim(L_1 \cap L_2) - \dim(L_1 \cap ker(\sigma_1 - 1)) - \dim(L_2 \cap \ker(\sigma_2 - 1))$ *and* $P_1 = \Pi_< + \pi_{\sigma_1}, P_2 = \Pi_> + \pi_{\sigma_2}$.

Remark 1.2. (1) A special case of the Theorem 1.1 was proved in [8], [9], where it was assumed that

$$\ker_{L^2} \mathcal{D}_{1,\infty} = \ker_{L^2} \mathcal{D}_{2,\infty} = \ker\, B = \{0\} \quad . \tag{1.8}$$

Assumption (1.8) implies the vanishing of all $s-values$ and $e-values$, in other words, all eigenvalues of all operators involved are bounded away from 0 . This reduces formula (1.7) to the equality

$$\lim_{R\to\infty} \frac{\det_\zeta \mathcal{D}_R^2}{\det_\zeta \mathcal{D}^2_{1,R,\Pi_<} \cdot \det_\zeta \mathcal{D}^2_{2,R,\Pi_>}} = 2^{-\zeta_{B^2}(0)} \quad .$$

(2) The main issue in the present work is the analysis of $s - values$. The proof of Theorem 1.1 uses some ideas of the work in W. Müller (see [6]).

2. Scattering Matrix and Slowly Decaying Eigenvalues

In this section we investigate the relation between the scattering matrices $C_1(\lambda)$, $C_2(\lambda)$ and the $s-values$ of the operators $\mathcal{D}_R$, $(\mathcal{D}_{1,R})_{P_1}$ and $(\mathcal{D}_{2,R})_{P_2}$.

Let φ be a eigensection of $\mathcal{D}_R$, which corresponds to $s-values$ $\lambda = \lambda(R)$, that is

$$\mathcal{D}_R\varphi = \lambda\varphi \quad \text{with} \quad |\lambda| < R^{-\kappa}$$

for some fixed κ with $0 < \kappa < 1$. Now we study the $s-values$ of the operator $\mathcal{D}_R$ over M_R . We introduce the manifold

$$\bar{M}_R = M_{1,R} \sqcup M_{2,R}$$

The boundary of $\bar{M}_R$ is equal to the sum of two copies of Y. We consider $\mathcal{D}_R$, the operator on a closed manifold M_R, as the Dirac operator on $\bar{M}_R$, which satisfies the transmission boundary condition. In particular, the corresponding eigensection φ to $s-values$ $\lambda = \lambda(R)$ satisfies the *"transmission boundary condition"*

$$\varphi|_{\partial M_{1,R}} = \varphi|_{\partial M_{2,R}}$$

We refer to [2] for more detailed discussion of the transmission problem (see also [8]). We want to warn the reader that, when we discuss transmission boundary condition, it would be natural to consider $M_{1,R}$ as $M_{1,R} = M_1 \cup [-R, 0] \times Y$, but in the following we parametrize cylindrical parts as

$$M_{1,R} = M_1 \cup [0, R] \times Y \quad , \quad M_{2,R} = M_2 \cup [-R, 0] \times Y.$$

The section φ can be represented in the following way on $[0, R] \times Y \subset M_{1,R}$

$$\varphi = e^{-i\lambda u}\psi_1 + e^{i\lambda u}\psi_2 + \varphi_1$$

where $\psi_1 \in \ker(G - i), \psi_2 \in \ker(G + i)$ and φ_1 is orthogonal to $\ker B$ when restricted to $\{u\} \times Y$. The eigenvalue $\lambda(R)$ is not bounded away from 0 , hence ψ_1, ψ_2 are non-trivial sections of $S|Y$ (see Theorem 2.2 in [12]). Choose $\phi \in L_1$ such that $\psi_1 = \phi - iG\phi$. Then the generalized eigensection $E(\phi, \lambda)$ associated to ϕ is given by

$$E(\phi, \lambda) = e^{-i\lambda u}(\phi - iG\phi) + e^{i\lambda u}C_1(\lambda)(\phi - iG\phi) + \theta(\phi, \lambda) \quad .$$

Following [6], we introduce $F = \varphi|_{M_{1,R}} - E(\phi, \lambda)|_{M_{1,R}}$. We know that there exist positive constants c_1, c_2 , such that

$$||\varphi_1|_{\{R\}\times Y}|| < c_1 e^{-c_2 R}$$

(see for instance Lemma 2.1 in [12]). Green's Theorem gives us

$$0 = \langle DF, F\rangle_{M_{1,R}} - \langle F, DF\rangle_{M_{1,R}} = \int_Y \langle GF, F\rangle dy \ .$$

This leads to

$$\int_Y \langle GF, F\rangle dy = -i||C_1(\lambda)\psi_1 - \psi_2||^2 + O(e^{-cR})$$

for some positive constant c , and we get the following inequality

$$\|C_1(\lambda)\psi_1 - \psi_2\|^2 < c_1 e^{-c_2 R} \ , \tag{2.1}$$

for some constants c_1, c_2 (compare [6]). Similarly, we have

$$\varphi = e^{i\lambda u}\psi_3 + e^{-i\lambda u}\psi_4 + \varphi_2$$

over $[-R, 0] \times Y \subset M_{2,R}$, where $\psi_3 \in ker(G + i), \psi_4 \in ker(G - i)$ and φ_2 is orthogonal to $\ker(B)$ when restricted to $\{u\} \times Y$. Again we have the expected estimate

$$\|C_2(\lambda)\psi_3 - \psi_4\|^2 < c_1 e^{-c_2 R} \ . \tag{2.2}$$

The transmission boundary condition over $Y \sqcup Y = \partial M_{1,R} \sqcup \partial M_{2,R}$ implies the equalities

$$e^{-i\lambda R}\psi_1 + e^{i\lambda R}\psi_2 = e^{-i\lambda R}\psi_3 + e^{i\lambda R}\psi_4$$

so that

$$\psi_1 = e^{2i\lambda R}\psi_4, \quad \psi_3 = e^{2i\lambda R}\psi_2 \ . \tag{2.3}$$

By (2.1), (2.2), (2.3), we have

$$\|e^{2i\lambda R} C_1(\lambda)\psi_4 - \psi_2\|^2 < c_1 e^{-c_2 R} \ ,$$

$$\|e^{2i\lambda R} C_2(\lambda)\psi_2 - \psi_4\|^2 < c_1 e^{-c_2 R} \ .$$

We combine these inequalities and obtain

$$\|e^{4i\lambda R} C_1(\lambda) \circ C_2(\lambda)\psi_2 - \psi_2\|^2 < c_3 e^{-c_4 R} \ ,$$

$$\|e^{4i\lambda R} C_2(\lambda) \circ C_1(\lambda)\psi_4 - \psi_4\|^2 < c_3 e^{-c_4 R}$$

for some positive constants c_3, c_4. The unitary operator $C_1(\lambda) \circ C_2(\lambda)$ is an analytic function of λ for sufficiently small λ. This whole analysis follows the method presented in [6] and results in the following Proposition.

Proposition 2.1. *There exists R_0 such that for $R > R_0$ the $s - value$ $\lambda(R)$ of $\mathcal{D}_R$ satisfies*

$$4R\lambda(R) + \alpha_j(\lambda(R)) = 2\pi k + O(e^{-cR}) \tag{2.4}$$

for an integer k with $|k| < R^{1-\kappa}$, where $\exp(i\alpha_j(\lambda(R)))$ is an eigenvalue of the restriction of the unitary operator $C_1(\lambda(R)) \circ C_2(\lambda(R))$ to $\ker(G+i) \subset \ker(B)$.

Remark 2.2. The map $C_1(0) \circ C_2(0)$ on $\ker(G+i)$ is not the identity map. However, it is equal to the identity, when restricted to the subspace $I_2(L_1 \cap L_2)$, where

$$I_2 = (G - i) : \ker(B) \to \ker(G+i)$$

It follows that the number of j's such that $\alpha_j(0) = 0$ is equal to $\dim(L_1 \cap L_2)$. This is the dimension of the space of eigensections corresponding to $e - values$, which are not determined by $\ker_{L^2}(\mathcal{D}_{i,\infty})$.

Similarly, we have the corresponding analysis for the slowly decaying eigenvalues of the operators $(\mathcal{D}_{i,R})_{P_i}$ for $i = 1, 2$.

Proposition 2.3. *There exists R_0 such that for $R > R_0$ the $s - value$ $\lambda = \lambda(R)$ of $(\mathcal{D}_{i,R})_{P_i}$ satisfies*

$$2R\lambda(R) + \beta_j(\lambda(R)) = 2\pi k + O(e^{-cR}) \tag{2.5}$$

for an integer k with $|k| < R^{1-\kappa}$, and $\exp(i\beta_j(\lambda(R)))$ an eigenvalue of the unitary operator $S_{\sigma_i}(\lambda(R)) = -P_{\sigma_i} \circ C_i(\lambda(R)) \circ I_{\sigma_i} : \ker(\sigma_i + 1) \to \ker(\sigma_i + 1)$ and $i = 1, 2$.

Remark 2.4. The map $S_{\sigma_i}(0)$ restricted to the subspace $\ker(\sigma_i + 1) \cap \ker(C_i(0) + 1) \subset \ker(\sigma_i + 1)$ is equal to I, and the number of j's such that $\beta_j(0) = 0$ is equal to $\dim(\ker(\sigma_{\mathrm{i}} + 1) \cap \ker(\mathrm{C_i}(0) + 1))$. This is the number of zero eigenvalues of $(\mathcal{D}_{i,R})_{P_i}$ which are not in $\ker_{L^2}(\mathcal{D}_{i,\infty})$ for $i = 1, 2$.

3. Sketch of the Proof of Theorem 1.1

In this section we briefly sketch the proof of Theorem 1.1. We refer to [10] for the detailed exposition.

We define

$$\zeta_{rel,R}(s) := \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} [Tr(e^{-t\mathcal{D}_R^2} - e^{-t(\mathcal{D}_{1,R})^2_{P_1}} - e^{-t(\mathcal{D}_{2,R})^2_{P_2}}) - h] \; dt \tag{3.1}$$

where $h = \dim(L_1 \cap L_2) - \dim(L_1 \cap \ker(\sigma_1 - 1)) - \dim(L_2 \cap \ker(\sigma_2 - 1))$. We decompose $\zeta_{rel,R}(s)$ into two parts

$$\zeta_s^R(s) = \frac{1}{\Gamma(s)} \int_0^{R^{2-\varepsilon}} (\cdot) \; dt \quad , \quad \zeta_l^R(s) = \frac{1}{\Gamma(s)} \int_{R^{2-\varepsilon}}^\infty (\cdot) \; dt$$

where $0 < \epsilon < 1$. The derivatives of $\zeta_s^R(s), \zeta_l^R(s)$ at $s = 0$ give the small and large time contribution. The standard computation shows that

$$(\zeta_s^R)'(0) = \int_0^{R^{2-\epsilon}} t^{-1} Tr(e^{-t\mathcal{D}_R^2} - e^{-t(\mathcal{D}_{1,R})^2_{P_1}} - e^{-t(\mathcal{D}_{2,R})^2_{P_2}}) dt + h\gamma - h(2-\epsilon)\log R,$$

where γ denotes the Euler constant. We analyze the *"small time"* contribution

$$\int_0^{R^{2-\epsilon}} t^{-1} [Tr(e^{-t\mathcal{D}_R^2} - e^{-t(\mathcal{D}_{1,R})^2_{P_1}} - e^{-t(\mathcal{D}_{2,R})^2_{P_2}}) - h] \; dt$$

using a method developed in [4] and explicit computations made in [10]. It follows that

$$\lim_{R\to\infty} \Big(\frac{d}{ds} \zeta_s^R(s)|_{s=0} + h(2-\varepsilon)\cdot ln \; R \Big) = -ln \; 2 \cdot \zeta_{B^2}(0) + h\gamma \quad . \tag{3.2}$$

We have the equality

$$(\zeta_l^R)'(0) = \int_{R^{-\epsilon}}^\infty t^{-1} [Tr(e^{-tR^2\mathcal{D}_R^2} - e^{-tR^2(\mathcal{D}_{1,R})^2_{P_1}} - e^{-tR^2(\mathcal{D}_{2,R})^2_{P_2}}) - h] dt$$

and the analysis of $s-values$ from Section 2 (see Proposition 2.1 and Proposition 2.3) provides the proof of the following result. The details will appear in [10].

Theorem 3.1.

$$\lim_{R\to\infty} \int_{R^{-\epsilon}}^\infty t^{-1} [Tr(e^{-tR^2\mathcal{D}_R^2} - e^{-tR^2(\mathcal{D}_{1,R})^2_{P_1}} - e^{-tR^2(\mathcal{D}_{2,R})^2_{P_2}}) - h] \; dt + h\gamma + h\epsilon \log R$$

$$= \frac{d}{ds}|_{s=0} \{ \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} [Tr(e^{-t\frac{1}{4}D(C_{12})^2} - e^{-tD(S_{\sigma_1})^2} - e^{-tD(S_{\sigma_2})^2}) - h] \; dt \} \quad .$$

This ends the proof of Theorem 1.1.

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