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THE MAP $V \longrightarrow V//G$ NEED NOT BE SEPARABLE

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Abstract. We construct a vector space V with a linear action of a reductive group G such that the quotient map $V \longrightarrow V//G$ (in the sense of geometric invariant theory) fails to be separable. This gives a counterexample to an assertion of Bardsley and Richardson.

0. Introduction

Let G be a reductive algebraic group, possibly nonconnected, and let X be an irreducible affine G–variety. Suppose that the ground field k has characteristic $p > 0$. In their paper on étale slices in characteristic p, Bardsley and Richardson claim ([1], Section 2, 2.1.9(b)) that the canonical projection from X to the quotient $X//G$ is separable. We give a counterexample to show that this map need not be separable, not even when X is a vector space V and G acts linearly. Bardsley and Richardson extend Luna's important Étale Slice Theorem [2] from characteristic zero to characteristic p. At only one point (see [1], Section 4, 4.3) do they use the separability of the quotient map. There the group G is finite and their assertion is justified: for the function field $k(X//G)$ is the whole of the field of invariants $k(X)^G$ of the function field $k(X)$, by [1], Section 4, 4.3.1, and whenever a group Γ acts on a field K, the extension K/K^Γ is separable [3], IV.1, Lemma 1.5. The main results of [1], therefore, remain valid.

Separability questions come up when one tries to generalise or strengthen the results of Bardsley and Richardson's [1]. A major interest of the counterexample presented here is that it indicates limits on any possible Luna slice theorem in characteristic $p > 0$. The first author will explore this point further in a forthcoming paper.

Throughout this article, k will be an algebraically closed field. We denote by $X//G$ the quotient of X by G in the sense of geometric invariant theory; see [1] for details.

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1. The counterexample

Notation 1.1. Let W be a finite dimensional vector space over k, of dimension $d = \dim(W) \geq 2$. Let $G = GL(W)$ be the group of all linear automorphisms of W. Let W^* be the dual of W, with the usual G–action. For any integer $n > 0$, we let W^n stand for the direct sum $W \oplus W \oplus \cdots \oplus W$ of n copies of W. We first wish to consider the G–module $W^* \oplus W^n$.

This G–module is naturally an affine variety. Choose a basis $(x_1, x_2, \ldots, x_d)$ for W, and take the natural dual basis $(y_1, y_2, \ldots, y_d)$ for W^*. The polynomial functions on W^* form the ring $k[x_1, \ldots, x_d]$, while the polynomial functions on W form the ring $k[y_1, \ldots, y_d]$. The difference is that the G–actions on these rings are dual. The polynomial functions on $W^* \oplus W^n$ form a ring $R = k[x_j, y_j^i]$, with $1 \leq j \leq d$ and $1 \leq i \leq n$. Note that in y_j^i the i is a superscript; y_j^i stands for the j^{th} component of the i^{th} vector. We are not raising anything to the i^{th} power. Our notation for raising to the p^{th} power, in this article, will be $\{y_j^i\}^p$.

Lemma 1.2. *With the notation as in Notation 1.1, let $I \subset R = k[x_j, y_j^i]$ be the ideal generated by all $\{x_j, y_j^i \mid j \geq 2\}$. Then any G–invariant element of R that lies in the ideal I must vanish. In symbols, $I \cap R^G = 0$.*

Proof. In the G–orbit of any point of W^* there is a point $(\lambda, 0, \cdots, 0)$. Therefore in the G–orbit of any point of $W^* \oplus W^n$ there is a point whose coordinates are

$$\begin{pmatrix} \lambda \\ 0 \\ \vdots \\ 0 \end{pmatrix} ; \begin{pmatrix} \mu_1^1 \\ \mu_2^1 \\ \vdots \\ \mu_d^1 \end{pmatrix}, \begin{pmatrix} \mu_1^2 \\ \mu_2^2 \\ \vdots \\ \mu_d^2 \end{pmatrix}, \cdots, \begin{pmatrix} \mu_1^n \\ \mu_2^n \\ \vdots \\ \mu_d^n \end{pmatrix}$$

Now the element of $GL(W)$ given by the diagonal matrix

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & t & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & t \end{pmatrix}$$

takes the above to the point

$$\begin{pmatrix} \lambda \\ 0 \\ \vdots \\ 0 \end{pmatrix} ; \begin{pmatrix} \mu_1^1 \\ t\mu_2^1 \\ \vdots \\ t\mu_d^1 \end{pmatrix}, \begin{pmatrix} \mu_1^2 \\ t\mu_2^2 \\ \vdots \\ t\mu_d^2 \end{pmatrix}, \cdots, \begin{pmatrix} \mu_1^n \\ t\mu_2^n \\ \vdots \\ t\mu_d^n \end{pmatrix}$$

Taking the limit as $t \longrightarrow 0$, we have that the closure of any G–orbit must contain a point of the form

$$\begin{pmatrix} \lambda \\ 0 \\ \vdots \\ 0 \end{pmatrix}; \begin{pmatrix} \mu_1^1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} \mu_1^2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} \mu_1^n \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Now any polynomial in the ideal $I \subset R$ vanishes on the points above. Since G–invariant polynomials are constant on closures of orbits, any G–invariant polynomial in I must vanish on all closures of all orbits; in other words, it must be identically zero. □

Definition 1.3. *Let the notation be as in Lemma 1.2. For every $1 \leq i \leq n$, we may form the polynomial*

$$Y_i = x_1 y_1^i + x_2 y_2^i + \cdots + x_d y_d^i.$$

The Y_i's are obviously G–invariant.

Proposition 1.4. *Let the notation be as in Definition 1.3. The subring $R^G \subset R$, of all G–invariant polynomials in R, is generated by the Y_i's.*

Proof. It is easy to prove Proposition 1.4 as a consequence of the First Main Theorem of classical invariant theory. Instead, we will give an equally easy, self-contained proof.

The center $Z(G)$ of G, that is the set of non-zero scalar matrices, stabilises I. Therefore $Z(G)$ acts on R/I. It is very obvious that the $Z(G)$–invariant polynomials in R/I are generated by the monomials $x_1 y_1^i$, and Y_i is congruent mod I to $x_1 y_1^i$.

Take any G–invariant polynomial $f \in R^G$. Then f gives a $Z(G)$–invariant polynomial modulo I, and by the above paragraph, there exists a polynomial P in n variables so that f is congruent to $P(Y_1, \dots, Y_n)$ mod I. But then

$$f - P(Y_1, \dots, Y_n)$$

is a G–invariant element of I, and by Lemma 1.2 it must vanish. □

Lemma 1.5. *With the notation as above, the $Y_i \in R^G$ are algebraically independent. Even better: any monomial in $\{x_j, y_j^i\}$ can occur in the expansion of at most one monomial $Y_1^{M_1} Y_2^{M_2} \cdots Y_n^{M_n}$.*

Proof. By checking the degrees of the monomials in the vectors $(y_1^i, y_2^i, \dots, y_d^i)$ for different i. □

Definition 1.6. *Suppose now that k is of characteristic $p > 0$. Put $X_j = \{x_j\}^p$. The ring $R = k[x_j, y_j^i]$ contains a subring $S = k[X_j, y_j^i]$. The ring S is not just* *a subring of R, it is also a G–submodule. In fact, S can be thought of as the ring of polynomial functions on the G–module $\pi_* W^* \oplus W^n$. Here, $\pi_* W^*$ is the Frobenius twist of W^*. The vector spaces W^* and $\pi_* W^*$ are identical. A matrix in GL(d) acts on a vector in $\pi_* W^*$ by raising the entries of the matrix to the p^{th} power, followed by the usual action on W^*.*

The polynomials

$$\begin{aligned} \overline{Y}_i &= X_1\{y_1^i\}^p + X_2\{y_2^i\}^p + \cdots + X_d\{y_d^i\}^p \\ &= x_1^p\{y_1^i\}^p + x_2^p\{y_2^i\}^p + \cdots + x_d^p\{y_d^i\}^p \\ &= Y_i^p \end{aligned}$$

are clearly G–invariant elements of the ring S.

Proposition 1.7. *Let the notation be as in Definition 1.6. The subring $S^G \subset S$, of all G–invariant elements of S, is generated by the $\overline{Y}_i$'s.*

Proof. The ring $S = k[X_j, y_j^i]$ is a subring and G–submodule of $R = k[x_j, y_j^i]$. By Proposition 1.4 we know that R^G is generated by

$$Y_i = x_1 y_1^i + x_2 y_2^i + \cdots + x_d y_d^i.$$

The ring S^G is nothing more than the intersection of R^G with S.

By Lemma 1.5, the elements $Y_1^{M_1} Y_2^{M_2} \cdots Y_n^{M_n} \in R^G$ have disjoint monomial expansions. A linear combination of $Y_1^{M_1} Y_2^{M_2} \cdots Y_n^{M_n}$'s will lie in S if and only if every term does. Suppose therefore that some $Y_1^{M_1} Y_2^{M_2} \cdots Y_n^{M_n}$ belongs to S.

In the expansion of the product, there is a term

$$x_2^{M_1}\{y_2^1\}^{M_1} \prod_{i=2}^n x_1^{M_i}\{y_1^i\}^{M_i}$$

and since this lies in S, it follows that p must divide M_1. By symmetry, p must divide M_i for every i. That is, our monomial is really a monomial in $Y_i^p = \overline{Y}_i$. □

Theorem 1.8. *There exists a vector space V, and a reductive group G acting on V, so that the geometric invariant theory map $V \longrightarrow V//G$ is not separable.*

Proof. Put $V = \pi_* W^* \oplus W^n$ as above, with $n > d = \dim(W)$. We assert that the map $V \longrightarrow V//G$ is not separable. The map corresponds to the inclusion $S^G \subset S$. We know that S^G is the polynomial algebra $k[\overline{Y}_1, \ldots, \overline{Y}_n]$. The derivative of the inclusion $S^G \subset S$ takes $d\overline{Y}_i$ to

$$\{y_1^i\}^p dX_1 + \{y_2^i\}^p dX_2 + \cdots + \{y_d^i\}^p dX_d$$

which is in the linear span of $\{dX_1, dX_2, \ldots, dX_d\}$. The image is therefore contained in a d–dimensional vector subspace of the 1–forms on V. Since the dimension of $V//G$ is $n > d$, the map $\Omega^1_{V//G} \longrightarrow \Omega^1_V$ cannot be generically injective. □

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