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HARMONIC MAPS AND THE TOPOLOGY OF CONFORMALLY COMPACT EINSTEIN MANIFOLDS

Naichung C. Leung* and Tom Y. H. Wan**

Abstract. We study the topology of a complete asymptotically hyperbolic Einstein manifold of which its conformal boundary has positive Yamabe invariant. We prove that all maps from such manifold into any nonpositively curved manifold are homotopically trivial. Our proof is based on a Bochner type argument on harmonic maps.

Introduction

While studying the AdS/CFT correspondence in physics, Witten and Yau [15] proved that for any conformally compact Einstein manifold (M^{n+1}, g) with $\mathrm{Ric} = -ng$ and a conformal infinity of positive scalar curvature, one has $H_n(M, \mathbb{Z}) = 0$. In particular, the conformal boundary of such manifold is connected. This result has opened up a new and exciting direction in geometric analysis. Since then, there have been several different proofs of the Witten-Yau Theorem by Anderson [1], Cai and Galloway [2] and X. Wang [13, 14].

In terms of homotopy theory, the above result can be expressed as

$$\left[(M, \partial M), (\mathbb{S}^1, *)\right] = 1,$$

i.e., any continuous map from M to $\mathbb{S}^1$ which maps the boundary into the marked point is homotopic to the constant map. So it is natural, from the point of view of harmonic maps, to ask whether the same is true if we replace $\mathbb{S}^1$ by any nonpositively curved manifold. We answered this question affirmatively. More precisely, we have:

Theorem 1. *Let (M^{n+1}, g), $n \geq 2$, be a conformally compact Einstein manifold of order $C^{3,\alpha}$ with $\mathrm{Ric}_g = -ng$ such that the conformal infinity of M has positive Yamabe invariant. Suppose that N is a compact nonpositively curved manifold. Then the homotopy classes $[(M, \partial M), (N, *)]$ are trivial.*

In case that $\mathrm{Ric} \geq 0$, the usual Bochner technique on harmonic maps is a standard tool in proving similar types of theorems. When $\mathrm{Ric} = -ng$, there is a Matsushima formula, a version of Bochner type formula, which can be used to obtain vanishing results under special situations. For example, Jost-Yau [5] and Mok-Siu-Yeung [11] used this idea to obtain superrigidity results. In [13, 14], X.

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Wang showed that the lower bound on the smallest L^2-eigenvalue can be used to balance the negativity of the Ricci curvature in the Bochner argument and hence, a new proof of the Witten-Yau Theorem via the result on the smallest eigenvalue by J. Lee [7]. Moreover, he found the sharp lower bound on the eigenvalue such that the method works. We want to remark that Lee's result is an estimate on the eigenvalue by the boundary geometry. Hence, this method is in fact using the boundary geometry to absorb the Ricci curvature term. This idea is also first introduced and used by Witten and Yau in [15]. Recently, P. Li and J. Wang [8] were able to show that without assuming the manifold is conformally compact, the conditions $\mathrm{Ric} \geq -n$ and $\lambda \geq n-1$ are sufficient to prove that the manifold has only one end of infinite volume or it is a wrap product.

We extend X. Wang's arguments to harmonic maps and generalize his result in this paper. Suppose that $\left(M^{n+1}, g\right)$ is a conformally compact manifold of order $C^{k,\alpha}$, i.e. there exists a smooth defining function t for ∂M on $\bar{M}$ such that $\bar{g}_{ij} = t^2 g_{ij}$ defines a metric on $\bar{M}$ which is $C^{k,\alpha}$ up to the boundary. By a defining function t, we mean $t > 0$ in M and t vanishes to first order on ∂M. If g satisfies the Einstein equation $\mathrm{Ric} = -ng$ and is regular enough, then $|\nabla t|_{\bar{g}} = 1$ on ∂M. Since the sectional curvature is asymptotic to $-|\nabla t|^2_{\bar{g}}$ on ∂M for conformally compact metric [10], the Einstein metric g is asymptotically hyperbolic according to Lee's terminology (called "weakly asymptotically hyperbolic" by X. Wang). By denoting the first L^2-eigenvalue of M as λ_g, we can now state our main results in which M is not necessarily Einstein.

Theorem 2. *Suppose that (M^{n+1}, g), $n \geq 2$, is an asymptotically hyperbolic conformally compact manifold of order C^1 such that*

$$\mathrm{Ric}_g \geq -ng \text{ and } \lambda_g \geq n-1.$$

Suppose that $f : M \to N$ is a smooth harmonic map of finite total energy from M into a complete nonpositively curved manifold N. If $\lambda_g > n-1$, then f is a constant map. If $\lambda_g = n-1$, then either f is a constant map or $M = \mathbb{R} \times \Sigma$ with the warped product metric $g = dt^2 + \cosh^2(t)h$, where (Σ, h) is a compact manifold with $\mathrm{Ric}_h \geq -(n-1)$.

After finishing the first draft of this paper, the second author was told by P. Li that, using the method in [8], Theorem 2 remains true without assuming the conformally compactness of the manifold M if the harmonic map is assumed to be asymptotically constant at infinity.

Combining Theorem 2 with results of Li-Tam [9] and Liao-Tam [6] concerning harmonic map heat flow, we can prove the corresponding topological result for such a manifold easily. Finally, together with Lee's result [7], we can generalize the Witten-Yau Theorem as stated precisely in Theorem 1.

To prove Theorem 2, we proceed as in [13, 14] and consider the $(n-1)/2n$-power of the energy density. The technical part is to show that under our assumption, this power of energy density still belongs to $L^{1,2}(M)$. Our method is to show that the energy decay is sufficiently fast, as precisely state in Lemma 1.1. This decay Lemma may be interesting in itself.

This paper is arranged as follows: In section 1, we first study the decay rate of the energy for harmonic maps with finite total energy on each end and prove a couple of lemmas which are needed in the proof of our theorems. We will prove our main theorem in section 2, and apply it in section 3 to prove the desired and other topological theorems.

1. Energy decay of harmonic maps with finite total energy

Let (M^{n+1}, g) be an asymptotically hyperbolic conformally compact manifold of order C^1. Its boundary components and the corresponding ends will be denoted by $\Sigma_0^i = \partial M^i$ and E^i respectively. Under our assumption, there exists a *special* defining function t in the following sense [4]: each end E^i can be parametrized by (t, x) such that the metric can be written as

$$g = t^{-2}\left(dt^2 + h(t, x)\right),$$

(i.e. $\bar{g} = dt^2 + h$) where $x \in \Sigma_0^i$, $t > 0$ is small and $h(t, \cdot)$ is a family of metric defined on Σ_0^i.

For any $0 < t_1 < t_2$ sufficiently small, we define

$$E_{t_1}^i = \{p \in E^i : 0 < t(p) < t_1\},$$

$$A_{t_1,t_2}^i = \{p \in E^i : t_1 < t(p) < t_2\},$$

$$\Sigma_{t_1}^i = \{p \in E^i : t(p) = t_1\},$$

and

$$M_{t_1} = M \setminus \cup_i E_{t_1}^i.$$

In order to simplify notations, we will omit the superscript i and write E_{t_1}, A_{t_1,t_2}, or Σ_{t_1} if we are working on a fixed end.

Lemma 1.1. *Suppose that M^{n+1}, $n \geq 2$, is a conformally compact manifold of order C^1 with $\mathrm{Ric} \geq -n$ and N is a complete nonpositively curved manifold. Let $f : E_{t_0} \to N$ be a smooth harmonic map from an end $E_{t_0} \subset M$ into N of finite total energy. Then*

$$\int_{E_t} |\nabla f|^2 = O\left(t^n\right) \quad \text{as } t \to 0.$$

Proof. We may assume that $t_0 > 0$ is sufficiently small so that E_{t_0} is parametrized by (t, x), where x belongs to the corresponding boundary component Σ_0 and $t \in (0, t_0)$.

By straightforward calculations and the harmonicity of f, one has the following conservation law: for any smooth vector field X on E_{t_0},

$$\int_{\partial A_{\tau,t}} \frac{1}{2}|\nabla f|^2 \langle X, n\rangle d\sigma = \int_{\partial A_{\tau,t}} \langle df(X), df(n)\rangle d\sigma + \int_{A_{\tau,t}} \langle S_f, \nabla X\rangle,$$

where $S_f = \frac{1}{2}|\nabla f|^2 g - f^* ds_N^2$ is the stress-energy tensor of f. There is a distinguished vector field $X = -t\frac{\partial}{\partial t}$ on E_{t_0} and the conservation law applied to this X gives

$$\begin{aligned}
&\int_{\Sigma_\tau} \frac{1}{2}|\nabla f|^2 \left\langle -\tau\frac{\partial}{\partial t}, -\tau\frac{\partial}{\partial t}\right\rangle d\sigma_\tau + \int_{\Sigma_t} \frac{1}{2}|\nabla f|^2 \left\langle -t\frac{\partial}{\partial t}, t\frac{\partial}{\partial t}\right\rangle d\sigma_t \\
&\quad = \int_{A_{\tau,t}} \langle S_f, \nabla X\rangle + \int_{\Sigma_\tau} \left\langle df\left(-\tau\frac{\partial}{\partial t}\right), df\left(-\tau\frac{\partial}{\partial t}\right)\right\rangle d\sigma_\tau \\
&\qquad + \int_{\Sigma_t} \left\langle df\left(-t\frac{\partial}{\partial t}\right), df\left(t\frac{\partial}{\partial t}\right)\right\rangle d\sigma_t.
\end{aligned}$$

That is,

$$\begin{aligned}
&\frac{1}{2}\int_{\Sigma_\tau} |\nabla f|^2 d\sigma_\tau - \frac{1}{2}\int_{\Sigma_t} |\nabla f|^2 d\sigma_t \\
&\quad = \int_{\Sigma_\tau} \tau^2 \left|\frac{\partial f}{\partial t}\right|^2 d\sigma_\tau - \int_{\Sigma_t} t^2 \left|\frac{\partial f}{\partial t}\right|^2 d\sigma_t + \int_{A_{\tau,t}} \langle S_f, \nabla X\rangle. \qquad (1.1)
\end{aligned}$$

For a fixed point $p = (t, x) \in E_{t_0}$, one can choose a normal coordinates x^i, $i = 1, \ldots, n$, of the metric $h(t, \cdot)$ centered at p. Then $e_0 = t\frac{\partial}{\partial t}$, $e_i = t\frac{\partial}{\partial x^i}$, $i = 1, \ldots, n$, forms an orthonormal basis at p. By straightforward calculations,

$$\nabla_{e_0} X \equiv 0 \quad \text{and} \quad \nabla_{e_i} X = e_i - \frac{t}{2} h^{jk} \frac{\partial h_{ki}}{\partial t} e_j.$$

Therefore, at p,

$$\mathrm{div} X = n - \frac{t}{2}\mathrm{Tr}_h\left(h^{-1}\frac{\partial h}{\partial t}\right)$$

and

$$\langle \nabla_{e_i} X, e_j\rangle = \delta_{ij} - \frac{t}{2}\frac{\partial h_{ij}}{\partial t}$$

for $i, j = 1, \ldots, n$. Hence at p,

$$\begin{aligned}
\langle S_f, \nabla X\rangle &= \sum_{A,B=0}^{n} S_f(e_A, e_B)\, \langle \nabla_{e_A} X, e_B\rangle \\
&= \frac{1}{2}|\nabla f|^2 \mathrm{div} X - \sum_{A,B=0}^{n} \langle df(e_A), df(e_B)\rangle\, \langle \nabla_{e_A} X, e_B\rangle \\
&= \frac{1}{2}|\nabla f|^2 \left[n - \frac{t}{2}\mathrm{Tr}_h\left(h^{-1}\frac{\partial h}{\partial t}\right)\right] \\
&\quad - \sum_{i,j=0}^{n} \langle df(e_i), df(e_j)\rangle \left(\delta_{ij} - \frac{t}{2}\frac{\partial h_{ij}}{\partial t}\right) \\
&= \frac{1}{2}|\nabla f|^2 \left[n - \frac{t}{2}\mathrm{Tr}_h\left(h^{-1}\frac{\partial h}{\partial t}\right)\right]
\end{aligned}$$

$$-\sum_{i=1}^{n}|df(e_i)|^2+\frac{t}{2}\langle df(e_i),df(e_j)\rangle\frac{\partial h_{ij}}{\partial t}. \tag{1.2}$$

By the assumption on g, we first observe that

$$|\langle S_f,\nabla X\rangle|\le C|\nabla f|^2$$

and hence $|\langle S_f,\nabla X\rangle|$ is integrable since f has finite total energy. So, we can let τ in equation (1.1) tend to 0. As in [13, 14], we can also choose a sequence of τ's tending to 0, such that

$$\int_{\Sigma_\tau}|\nabla f|^2\to 0\quad\text{as } \tau\to 0.$$

So, we conclude that for all $t\in(0,t_0)$,

$$\int_{E_t}\langle S_f,\nabla X\rangle=\int_{\Sigma_t}t^2\left|\frac{\partial f}{\partial t}\right|^2 d\sigma_t-\frac{1}{2}\int_{\Sigma_t}|\nabla f|^2 d\sigma_t. \tag{1.3}$$

Secondly, the assumption on g and (1.2) together imply that there exist $t_1\in(0,t_0)$ and $C_1>0$, such that for any $t\in(0,t_1)$,

$$\begin{aligned}\langle S_f,\nabla X\rangle &\ge \frac{n-C_1t}{2}|\nabla f|^2-(1+C_1t)\sum_{i=1}^{n}|df(e_i)|^2\\ &= \frac{n-C_1t}{2}t^2\left|\frac{\partial f}{\partial t}\right|^2+\frac{n-2-3C_1t}{2}\sum_{i=1}^{n}|df(e_i)|^2.\end{aligned}$$

To simplify notation, we write

$$|\nabla_t f|^2=t^2\left|\frac{\partial f}{\partial t}\right|^2\quad\text{and}\quad|\nabla_x f|^2=\sum_{i=1}^{n}|df(e_i)|^2.$$

Putting these into (1.3), we have

$$\begin{aligned}\frac{n-C_1t}{2}\int_{E_t}|\nabla_t f|^2 &+ \frac{n-2-3C_1t}{2}\int_{E_t}|\nabla_x f|^2\\ &\le \frac{1}{2}\int_{\Sigma_t}t^2|\nabla_t f|^2 d\sigma_t-\frac{1}{2}\int_{\Sigma_t}|\nabla_x f|^2 d\sigma_t.\end{aligned} \tag{1.4}$$

If we further write

$$F(t)=\int_{E_t}|\nabla_t f|^2\quad\text{and}\quad G(t)=\int_{E_t}|\nabla_x f|^2$$

and note that

$$tF'(t)=\int_{\Sigma_t}|\nabla_t f|^2 d\sigma_t\quad\text{and}\quad tG'(t)=\int_{\Sigma_t}|\nabla_x f|^2 d\sigma_t,$$

then the above inequality becomes

$$(n-C_1t)F+(n-2-3C_1t)G\le tF'-tG'.$$

This immediately implies

$$\frac{d}{dt}\left[t^{n-2}e^{-3C_1t}\left(F-G\right)\right] \geq (2n-2-4C_1t)t^{n-3}e^{-3C_1t}F.$$

So if we set $t_2 = \min\{t_1, 1/2C_1\}$, then for all $t \in (0, t_2)$,

$$\frac{d}{dt}\left[t^{n-2}e^{-3C_1t}\left(F-G\right)\right] \geq 0.$$

As f has finite total energy, $\lim_{t\to 0} F(t) = \lim_{t\to 0} G(t) = 0$, and hence

$$F(t) \leq G(t) \quad \forall\, t \in (0, t_2).$$

Put this back into (1.4), we first have

$$tF' \geq (n - 4C_1t)F \quad \text{for } n = 2.$$

And then for $n \geq 3$, one simply drops the term that involve G and conclude that

$$tF' \geq (n - C_1t)F \geq (n - 4C_1t)F.$$

Hence, in all cases,

$$\frac{d}{dt}\left(t^{-n}e^{4C_1t}F(t)\right) \geq 0.$$

Integrating this inequality from t to t_2, we see that there is a constant $C > 0$ such that

$$F(t) \leq Ct^n \quad \forall\, t \in (0, t_2).$$

Together with $G \leq F$, we have

$$\int_{E_t} |\nabla f|^2 \leq 2Ct^n$$

and the proof of the lemma is completed. □

Lemma 1.2. *Let f be a smooth harmonic map from an $(n+1)$-dimensional manifold, then*

$$|\nabla^2 f|^2 \geq \left(1 + \frac{1}{n}\right)|\nabla|\nabla f||^2.$$

Proof. With respect to normal coordinates, we have $\sum_i f^\alpha_{ii} = 0$ at the center. Then for each α, one can show, as in the gradient estimate in [12], that

$$|\nabla^2 f^\alpha|^2 \geq \left(1 + \frac{1}{n}\right)|\nabla|\nabla f^\alpha||^2.$$

Therefore,

$$\begin{aligned}
|\nabla|\nabla f||^2 &= \left|\nabla\sqrt{\sum_\alpha |\nabla f^\alpha|^2}\right|^2 \\
&= \left|\frac{\sum_\alpha |\nabla f^\alpha||\nabla|\nabla f^\alpha|}{\sqrt{\sum_\alpha |\nabla f^\alpha|^2}}\right|^2
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_\alpha |\nabla|\nabla f^\alpha||^2 \\
&\leq \left(1+\frac{1}{n}\right)^{-1} \sum_\alpha \left|\nabla^2 f^\alpha\right|^2 \\
&= \left(1+\frac{1}{n}\right)^{-1} \left|\nabla^2 f\right|^2.
\end{aligned}$$

□

Lemma 1.3. *Let M, N be as in Lemma 1.1 and $f : M \to N$ be a smooth harmonic map with finite total energy, then*

$$\int_M |\nabla|\nabla f||^2 < +\infty.$$

Proof. We only need to show this on one of the end E_{t_1}. Let η be a smooth cutoff function such that $0 \leq \eta \leq 1$, $\eta(\beta) \equiv 1$ for $\beta \in (0,1]$, $\mathrm{supp}\eta \subset [0,2)$, and $|\eta'| \leq C$ for some absolute constant $C > 0$. Then for any $r > 0$, consider the function on E_{t_1} defined by

$$\phi(x,t) = \eta\left(\frac{1}{r}\left|\log\frac{t_1}{t}\right|\right).$$

Then $0 \leq \phi \leq 1$, $\phi(\cdot,t) \equiv 1$ for all $t \in [e^{-r}t_1, t_1]$, $\phi(\cdot,t) \equiv 0$ for all $t \in (0, e^{-2r}t_1]$, and $|\nabla\phi|^2 = t^2\left(\frac{\partial\phi}{\partial t}\right)^2 \leq \frac{C^2}{r^2}$. Then, multiplying ϕ^2 by the Bochner formula of f and integrating, one obtains

$$\frac{1}{2}\int_{E_{t_1}} \phi^2\Delta|\nabla f|^2 \geq \int_{E_{t_1}} \phi^2|\nabla^2 f|^2 - n\int_{E_{t_1}} \phi^2|\nabla f|^2.$$

By Lemma 1.2, we have

$$\begin{aligned}
\left(1+\frac{1}{n}\right)\int_{E_{t_1}} \phi^2\left|\nabla|\nabla f|\right|^2 &\leq n\int_{E_{t_1}} \phi^2|\nabla f|^2 - 2\int_{E_{t_1}} \phi|\nabla f|\nabla\phi\cdot\nabla|\nabla f| \\
&\quad + \frac{t_1}{2}\int_{\Sigma_{t_1}} \left|\frac{\partial}{\partial t}|\nabla f|^2\right| \\
&\leq n\int_{E_{t_1}} \phi^2|\nabla f|^2 + \int_{E_{t_1}} \phi^2\left|\nabla|\nabla f|\right|^2 \\
&\quad + \int_{E_{t_1}} |\nabla\phi|^2|\nabla f|^2 + \frac{t_1}{2}\int_{\Sigma_{t_1}} \left|\frac{\partial}{\partial t}|\nabla f|^2\right|.
\end{aligned}$$

This implies

$$\frac{1}{n}\int_{A_{e^{-r}t_1,t_1}} |\nabla|\nabla f||^2 \leq \left(n+\frac{C^2}{r^2}\right)\int_{E_{t_1}} |\nabla f|^2 + \frac{t_1}{2}\int_{\Sigma_{t_1}} \left|\frac{\partial}{\partial t}|\nabla f|^2\right|.$$

Letting $r \to \infty$, we conclude that $\int_{E_{t_1}} |\nabla|\nabla f||^2 < +\infty$ which completes the proof of the lemma. □

2. Vanishing Theorem of Harmonic Maps

We will prove Theorem 2 in this section. As we mentioned in the introduction, we proceed as in [13, 14] and consider the $(n-1)/2n$-power of the energy density. The technical part is to show that, under our assumption, this power of the energy density still belongs to $L^{1,2}(M)$.

Proof of Theorem 2. Let us consider the function

$$\zeta = |\nabla f|^{\beta}.$$

We first prove that for $\beta > 1/2$, $\zeta \in L^2(M)$.

For each fixed end E_{t_1}, we set $t_k = 2^{-k+1}t_1$ and consider the integral of ζ^2 on the annulus A_{t_{k+1},t_k}. Then the Hölder inequality implies that

$$\begin{aligned}
\int_{A_{t_{k+1},t_k}} \zeta^2 &= \int_{A_{t_{k+1},t_k}} |\nabla f|^{2\beta} dv_g \\
&\le \left(\int_{A_{t_{k+1},t_k}} |\nabla f|^2 dv_g\right)^{\beta} \left(\int_{A_{t_{k+1},t_k}} dv_g\right)^{1-\beta} \\
&\le \left(\int_{A_{t_{k+1},t_k}} |\nabla f|^2 dv_g\right)^{\beta} \left(\sup_{0<t<t_1} \mathrm{vol}_h(\Sigma_t) \int_{t_{k+1}}^{t_k} \frac{dt}{t^{n+1}}\right)^{1-\beta} \\
&\le \left(\frac{\sup_{0<t<t_1} \mathrm{vol}_h(\Sigma_t)}{n}\right)^{1-\beta} \left(\int_{A_{t_{k+1},t_k}} |\nabla f|^2\right)^{\beta} \cdot t_{k+1}^{-n(1-\beta)}.
\end{aligned}$$

By Lemma 1.1,

$$\begin{aligned}
\int_{A_{t_{k+1},t_k}} \zeta^2 &\le C \left(\frac{\sup_{0<t<t_1} \mathrm{vol}_h(\Sigma_t)}{n}\right)^{1-\beta} t_k^{n\beta} \cdot t_{k+1}^{-n(1-\beta)} \\
&\le C_1 \left(\frac{\sup_{0<t<t_1} \mathrm{vol}_h(\Sigma_t)}{n}\right)^{1-\beta} \left(2^{-\gamma}\right)^k,
\end{aligned}$$

for some constants C and $C_1 > 0$ independent of k and β, and

$$\gamma = n\beta - n(1-\beta) = n(2\beta - 1) > 0.$$

Hence for all $K > 0$,

$$\int_{A_{t_{K+1},t_1}} \zeta^2 \le C_1 \left(\frac{\sup_{0<t<t_1} \mathrm{vol}_h(\Sigma_t)}{n}\right)^{1-\beta} \frac{1}{1-2^{-\gamma}}.$$

Since the upper bound is independent of K, we have proved that

$$\int_{E_{t_1}} \zeta^2 \le C_1 \left(\frac{\sup_{0<t<t_1} \mathrm{vol}_h(\Sigma_t)}{n}\right)^{1-\beta} \frac{1}{1-2^{-\gamma}}. \tag{2.1}$$

As the end E_{t_1} is arbitrary, this proves that $\zeta \in L^2(M)$.

Now we consider a differential inequality of ζ which is a direct consequence of the Bochner formula of f and Lemma 1.2. To simplify notation, we use $\mathbb{K}_N \le 0$

to denote the curvature term of N in the Bochner formula, which is in the order of $|\nabla f|^4$. In this notation, the inequality becomes

$$\Delta\zeta^2 \geq \frac{2}{\beta}\left(\frac{1}{n}+2\beta-1\right)|\nabla\zeta|^2 - 2n\beta\zeta^2 - \zeta^{-\frac{2(1-\beta)}{\beta}}\mathbb{K}_N.$$

With this inequality and $\zeta \in L^2(M)$, we conclude as in the proof of Lemma 1.3 that $|\nabla\zeta|^2$ is also integrable. Hence $\zeta \in L^{1,2}(M)$.

Integrating "by parts" the above inequality on M_t, one obtains

$$\begin{aligned} -\int_{\cup_i \Sigma_t^i} t\frac{\partial}{\partial t}\zeta^2 d\sigma_t \quad &\geq \quad \frac{2}{\beta}\left(\frac{1}{n}+2\beta-1\right)\int_{M_t}|\nabla\zeta|^2 - 2n\beta\int_{M_t}\zeta^2 \\ &\quad - \int_{M_t}\zeta^{-\frac{2(1-\beta)}{\beta}}\mathbb{K}_N. \end{aligned}$$

The left hand side can be estimated as follows:

$$\left|\int_{\cup_i \Sigma_t^i} t\frac{\partial}{\partial t}\zeta^2 d\sigma_t\right| \leq 2\int_{\cup_i \Sigma_t^i}\zeta^2 d\sigma_t + 2\int_{\cup_i \Sigma_t^i}|\nabla\zeta|^2 d\sigma_t.$$

Therefore, since $\zeta \in L^{1,2}(M)$, one can choose a sequence of t's tending to 0, as in [13, 14], such that

$$\left|\int_{\cup_i \Sigma_t^i} t\frac{\partial}{\partial t}\zeta^2 d\sigma_t\right| \to 0.$$

As a consequence,

$$-\int_M \zeta^{-\frac{2(1-\beta)}{\beta}}\mathbb{K}_N + \frac{2}{\beta}\left(\frac{1}{n}+2\beta-1\right)\int_M|\nabla\zeta|^2 \leq 2n\beta\int_M\zeta^2. \tag{2.2}$$

If f is not a constant map, then $\zeta \not\equiv 0$ and hence

$$\lambda_g \leq \frac{n\beta^2}{\frac{1}{n}+2\beta-1}.$$

Since $(n-1)/n > 1/2$ for $n \geq 3$, by letting $\beta = (n-1)/n$ in this case, we have

$$\lambda_g \leq n-1.$$

Therefore, if $\lambda_g > n-1$, then f must be a constant map. If $\lambda_g = n-1$, then inequality (2.2) implies that N is flat. Then the result follows from X. Wang's result [13, 14].

If $n = 2$, Cheng's result [3] implies that $\lambda_g = n-1 = 1$. Hence

$$\int_M|\nabla\zeta|^2 \geq \int_M\zeta^2.$$

Putting this into (2.2), one obtains, by taking $\beta = \frac{1}{2}+\delta$ with any $\delta > 0$,

$$-\int_M \zeta^{-\frac{2(1-\beta)}{\beta}}\mathbb{K}_N \leq \frac{8\delta^2}{1+2\delta}\int_M\zeta^2.$$

To estimate $\int_M \zeta^2$, we fix a sufficiently small $t_1 > 0$ and note that

$$\int_M \zeta^2 = \int_{M_{t_1}} \zeta^2 + \int_{\cup_i E^{(i)}_{t_1}} \zeta^2.$$

By sub-mean value property, $|\nabla f|^2$ is uniformly bounded and, hence, there is constant $C > 0$ such that

$$\int_{M_{t_1}} \zeta^2 = \int_{M_{t_1}} |\nabla f|^{1+2\delta} \leq C$$

for sufficiently small $\delta > 0$. For the second term, we apply (2.1) to each $E^{(i)}_{t_1}$ and get

$$\int_{\cup_i E^{(i)}_{t_1}} \zeta^2 \leq C_1 V^{\frac{1}{2}-\delta} \frac{1}{1-2^{-4\delta}},$$

where $V = \sup_i \left(\sup_{0<t<t_1} \mathrm{vol}_h(\Sigma^{(i)}_t)/2\right)$. Since $1/(1-2^{-4\delta}) \to +\infty$ as $\delta \to 0$, we have

$$-\int_M \zeta^{-\frac{2(1-\beta)}{\beta}} \mathbb{K}_N \leq C_2 V^{\frac{1}{2}-\delta} \frac{1}{1-2^{-2\delta}} \cdot \frac{8\delta^2}{1+2\delta},$$

for some $C_2 > 0$. Since the upper bound tends to 0 as $\delta \to 0$, we conclude again that N is flat and the result follows as above. This completes the proof of the Theorem 2. □

3. Application to asymptotically hyperbolic Einstein manifolds

As an application of our vanishing theorem, we first prove the following topological theorem.

Theorem 3.1. *Suppose that (M^{n+1}, g), $n \geq 2$, is an asymptotically hyperbolic conformally compact manifold of order C^1 such that*

$$\mathrm{Ric}_g \geq -ng \quad \textit{and} \quad \lambda_g \geq n-1$$

Then for any nonpositively curved compact manifold N, the homotopy classes in $[(M, \partial M), (N, *)]$ are trivial or M slpits as $\mathbb{R} \times \Sigma$ for some compact manifold Σ.

Proof. Since M is conformally compact, $(M, \partial M)$ is homotopy equivalent to $(M \setminus (\cup E^i_t), \cup \Sigma^i_t)$ for sufficiently small t. Therefore, in each class, we can find a smooth representative $g : M \to N$ which maps each end E^i_t into the marked point $*$. Thus g has bounded energy density and bounded image. Hence by [9], the harmonic map heat flow has a unique solution. Also the square norm of the tension field of g is in L^p for $p > 1$ and tends to 0 near the boundary. The results in [9] imply that the heat flow converges to a harmonic map f with the same boundary data as g. So, f is also a representative of the class $[g]$. (In particular, f is asymptotically constant on each end.) On the other hand, using the result in [6], g has finite total energy, and the uniqueness implies that f also has finite total energy. Therefore, Theorem 2 immediately implies that either M splits or f must be a constant and hence the homotopy class is trivial. □

The above theorem includes the following results of [13, 14].

Corollary 3.2. *Let (M, g) as in theorem 3.1, then $H_n(M, \mathbb{Z}) = 0$ and in particular the conformal infinity is connected.*

As we mentioned in the introduction, all of the above application and study are motivated by the results of [15]. We are now ready to prove the promised generalization (Theorem 1) which is an immediate consequence of Theorem 3.1

Proof of theorem 1. By Lee's theorem [7], M has eigenvalue $\lambda = n^2/4$ which is strictly greater than $n-1$ if $n \geq 3$. Therefore, by Theorem 3.1, the claim follows. If $n = 2$, Theorem 3.1 implies that either the claim is true or the manifold splits. As the boundary is assumed to have positive Yamabe invariant, the warped product metric cannot be negative Einstien. Hence the theorem is also true for this case. $\square$

Corollary 3.3. *Let (M^{n+1}, g), $n \geq 2$, be a conformally compact Einstein manifold with $\mathrm{Ric}_g = -ng$. Suppose that the conformal infinity of M has positive Yamabe invariant, then $H_n(M, \mathbb{Z}) = 0$ and, in particular, the conformal infinity is connected.*

Proof. This follows immediately from Theorem 1 and basic results in algebraic topology. $\square$

Finally, we apply Theorem 1 to prove a nonexistence theorem for Einstein manifolds in the situation of AdS/CFT correspondence.

Theorem 3.4. *Suppose that N^{n+1}, $n \geq 2$, is a compact manifold which support a nonpositively curved metric. Then for any embedded disc $D \subset N$, the manifold $M = N \backslash D$ has no asymptotically hyperbolic conformally compact metric of order C^1 satisfying $\mathrm{Ric} \geq -n$ and $\lambda \geq n-1$.*

Proof. It is obvious that $[(M, \partial M), (N, *)]$ is nontrivial for any point $* \in D$ and M does not split. Therefore, the existence of such metric contradicts Theorem 1. $\square$

Theorem 3.5. *Suppose that N^{n+1}, $n \geq 2$, is a compact manifold which supports a nonpositively curved metric. Then for any embedded disc $D \subset N$, the manifold $M = N \setminus D$ has no conformally compact Einstein metric of order $C^{3,\alpha}$ satisfying $\mathrm{Ric} = -n$.*

Proof. Since $\partial M = \partial D$ has positive Yamabe invariant, if there exists a conformally compact Einstein metric of order $C^{3,\alpha}$ with $\mathrm{Ric} = -n$, then Lee's theorem [7] implies that $\lambda = n^2/4$, which is greater than $n-1$ for $n \geq 3$. This is impossible, by Theorem 3.4. If $n = 2$, then the manifold splits with the given form of warped product metric. The negativity of this Einstein metric contradicts the fact that the ∂M has positive Yamabe invariant. $\square$

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References

[1] M. Anderson, *Boundary regularity, uniqueness and non-uniqueness for AH Einstein metrics on 4-manifolds*, math.DG/0104171.

[2] M. Cai & G.J. Galloway, *Boundaries of zero scalar curvature in the AdS/CFT correspondence,* hep-th/0003046.

[3] S. Y. Cheng, *Eigenvalue comparison theorems and its geometric applications,* Math. Z. **143** (1975), no. 3, 289–297.

[4] C. R. Graham & J. M. Lee, *Einstein metrics with prescribed conformal infinity on the ball*, Adv. Math. **87** (1991), no. 2, 196–225.

[5] J. Jost & S.-T. Yau, *Harmonic maps and superrigidity.* Tsing Hua lectures on geometry & analysis (Hsinchu, 1990–1991), 213–246, Internat. Press, Cambridge, MA, 1997.

[6] G. G. Liao & L.-F. Tam, *On the heat equation for harmonic maps from non-compact manifolds*, Pacific J. Math. **153** (1992), no. 1, 129–145.

[7] J. M. Lee, *The spectrum of an asymptotically hyperbolic Einstein manifold*, Comm. Anal. Geom. **3**, no. 2 (1995), 253–271.

[8] P. Li & J. Wang, *Complete manifolds with positive spectrum*, preprint (2001).

[9] P. Li & L.-F. Tam, *The heat equation and harmonic maps of complete manifolds*, Invent. Math. **105**, (1991) 1-46.

[10] R. R. Mazzeo, *The Hodge cohomology of a conformally compact metric,* J. Differential Geom. **28** (1988), no. 2, 309–339.

[11] N. Mok, Y.T. Siu & S.-K. Yeung, *Geometric superrigidity*, Invent. Math. **113** (1993), no. 1, 57–83.

[12] R. Schoen & S.-T. Yau, *Lectures on Differential Geometry,* Conference Proceedings and Lecture Notes in Geometry and Topology, I. International Press, Cambridge, MA, 1994.

[13] X. Wang, *On the geometry of conformally compact Einstein manifolds*, PhD. thesis, Stanford University, 2001.

[14] X. Wang, *On conformally compact Einstein manifolds*, preprint 2001.

[15] E. Witten & S.-T. Yau *Connectedness of the boundary in the ADS/CFT correspondence*, Adv. Theor. Math. Phys. **3** (1999), no. 6, 1635-1655 (2000); hep-th/9910245.

School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA.
E-mail address: `leung@math.umn.edu`

Department of Mathematics, The Chinese University of Hong Kong, Shatin, N.T., Hong Kong.
E-mail address: `tomwan@math.cuhk.edu.hk`