Mathematical Research Letters **8**, 693–701 (2001)

NON-ORIENTABLE LAGRANGIAN SURFACES WITH CONTROLLED AREA

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Abstract. We show that any closed curve in $\mathbb{R}^4$ bounds a Lagrangian Möbius band with quadratic area(i.e. area bounded by length square). And we generalize this result to flat chains mod 2 to conclude that in $\mathbb{R}^4$ any one-dimensional integral flat chain mod 2 without boundary bounds a two-dimensional Lagrangian integral flat chain mod2 with quadratic area. Moreover we prove that in $\mathbb{R}^4$ the set of Lagrangian integral flat chains mod2 is dense under the flat norm in the space of all two-dimensional integral flat chains mod2.

1. Introduction

Let $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$ be the standard symplectic form in $\mathbb{R}^4$. Let $\eta = (x_1dy_1 + x_2dy_2 - y_1dx_1 - y_2dx_2)/2$ be a primitive of ω. For a closed curve $\gamma : [0,1] \to \mathbb{R}^4$, define the *symplectic area of* γ to be $\int_\gamma \eta$. A two-dimensional plane T is called Lagrangian if $\omega|_T = 0$. And a surface is called *Lagrangian* if the tangent plane at each point is a Lagrangian plane. Stokes' Theorem tells us that if γ bounds any Lagrangian disk, it must have zero symplectic area. Conversely D.Allcock [Al] and Gromov [Gr] proved that if a closed curve has zero symplectic area, then it in deed bounds a Lagrangian disk. Moreover the Lagrangian disk has area controlled by L^2,where L is the length of γ.

In the present paper, the above result is modified to the case where the curve does not have zero symplectic area. In this case, it no longer bounds any orientable Lagrangian surface(by Stokes' theorem), but it turns out that if we allow non-orientable surfaces, the similar isoperimetric result is still true. In fact we show in Corollary 2 that any closed curve bounds a Lagrangian Möbius band with area controlled by L^2. In section 5, using the decomposition theorem of one-dimensional flat chain mod2, we extend this result and conclude that in $\mathbb{R}^4$ any one-dimensional integral flat chain mod2 without boundary bounds a two-dimensional Lagrangian integral flat chain mod2 with quadratic area.

One natural question in the study of Lagrangian surfaces is whether the space of Lagrangian surfaces is compact, i.e. ,whether the limit(in some suitable sense) of a sequence of Lagrangian surfaces is still Lagrangian. If the surfaces are orientable, this is known to be true(see R.Schoen and J. Wolfson [S-W]). But unfortunately, this is not true if we allow non-orientable surfaces. In fact in

Received March 24, 2001.

section 5 we prove that the set of all non-oriented Lagrangian surfaces is dense in the space of all non-oriented surfaces in $\mathbb{R}^4$.

I would like very much to thank my advisor Prof. Richard Schoen for his superb direction, without which the writing of this paper is certainly impossible. I would also like to thank Prof. Brian White for pointing out the construction in the proof of Theorem 4. Finally I am pleased to thank Xiaodong Wang for many helpful discussions.

Through out this paper, $c, c', c_1, c_2, \dots$, will be used to denote absolute constants.

2. Main Results

The followings are the main results of this paper:

Theorem 1. *Let $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$ be the standard symplectic form in $\mathbb{R}^4$. Let $\eta = (x_1dy_1 + x_2dy_2 - y_1dx_1 - y_2dx_2)/2$ be a primitive of ω. If γ_1 and γ_2 are two piecewise smooth closed curves in $\mathbb{R}^4$ such that*

$$\int_{\gamma_1} \eta + \int_{\gamma_2} \eta = 0$$

then there exists a piecewise smooth oriented Lagrangian surface H whose oriented boundary is γ_1 and γ_2, and which has the area bound:

$$Area(H) \le c(L^2(\gamma_1) + L^2(\gamma_2) + dist^2(\gamma_1, \gamma_2))$$

where $L(\gamma_i)$ is the length of γ_i, dist(γ_1,γ_2) is the distance between the two curves, and c is an absolute constant.

We will prove this theorem in section 4.

A immediate consequence of Theorem 1 is the following:

Corollary 2 (Isoperimetric Inequality)**.** *Let γ be any piecewise smooth closed curve in $\mathbb{R}^4$. Then γ bounds a non-oriented piecewise smooth Lagrangian surface M (and it may be chosen to be a singular Möbius band) such that*

$$Area(M) \le cL^2(\gamma)$$

where $L(\gamma)$ is the length of γ and c is an absolute constant.

Proof. Let γ be parametrized by $\gamma(s) = (x_1(s), y_1(s), x_2(s), y_2(s)), 0 \le s \le 1$. By translation, we can assume $\gamma(0) = 0 = \gamma(1)$. Choose one point $P = (a_1, b_1, a_2, b_2)$, whose distance from the origin is $4L(\gamma)$. Define a new loop $\tilde{\gamma}$ by

$$\tilde{x}_i(s) = \begin{cases} (x_i(2s) - a_i)/\sqrt{2}, & \text{if } 0 \le s \le 1/2; \\ (x_i(2s-1) - a_i)/\sqrt{2}, & \text{if } 1/2 \le s \le 1; \end{cases}$$

$$\tilde{y}_i(s) = \begin{cases} (y_i(2s) - b_i)/\sqrt{2}, & \text{if } 0 \le s \le 1/2; \\ (y_i(2s-1) - b_i)/\sqrt{2}, & \text{if } 1/2 \le s \le 1; \end{cases}$$

for $i = 1, 2$. Then it is easy to check that $\tilde{\gamma}$ is a closed curve winding around twice with $\int_{\tilde{\gamma}} \eta = \int_{\gamma} \eta$. Apply Theorem 1, we get an oriented Lagrangian surface M with oriented boundary γ and $\tilde{\gamma}^{-1}$. Note that the distance between γ and $\tilde{\gamma}$ is less than $4L(\gamma)$. Moreover $L(\tilde{\gamma}) \leq 2L(\gamma)$. Thus $Area(M) \leq c_1 L^2(\gamma)$. But since $\tilde{\gamma}$ winds around twice, we see that as non-oriented surface, the boundary of M is γ. And in fact it is a Möbius band bounded by γ. □

3. A Sequence of Lemmas

In this section, we give a sequence of lemmas which will lead to the proof of Theorem 1.

The following Lemma 1 and Lemma 2 were proven by D.Allcock in [Al]. For completeness, we include the proofs here.

Lemma 1 (Allcock)**.** *Let α, β be two piecewise smooth closed curves in $\mathbb{R}^4$. If their images lie in x_1y_1-plane and x_2y_2-plane respectively, and if they are parametrized such that*

$$\omega(\alpha(s), \alpha'(s)) = \omega(\beta(s), \beta'(s)), s \in [0, 1],$$

then there exists a Lagrangian homotopy[1] *between α and β with area less than $c(A+B)(L(\alpha)+L(\beta))$, where $A = max(\|\alpha(s)\| : 0 \leq s \leq 1)$, $B = max(\|\beta(s)\| : 0 \leq s \leq 1)$, and c is an absolute constant.*

Proof. Define $H : [0, 1] \times [0, \pi/2] \to \mathbb{R}^4$ as

$$H(s, t) = \alpha(s) \cos t + \beta(s) \sin t.$$

Clearly H is a homotopy between α and β. To show it is Lagrangian, notice that

$$H_* \frac{\partial}{\partial s}(s,t) = \frac{\partial H}{\partial s}(s,t) = \alpha'(s) \cos t + \beta'(s) \sin t, \tag{1}$$
$$H_* \frac{\partial}{\partial t}(s,t) = \frac{\partial H}{\partial t}(s,t) = -\alpha(s) \sin t + \beta(s) \cos t.$$

Therefore, using the fact that α, β lie in two symplectically orthogonal planes, together with the fact that $\omega(\alpha(s), \alpha'(s)) = \omega(\beta(s), \beta'(s))$, we conclude that $\omega(\partial H/\partial s, \partial H/\partial t) = 0$. Thus H is Lagrangian.

To get the area bound, consider the general formula

$$\begin{aligned} &Area(H) \\ &= \iint_{[0,1]\times[0,\pi/2]} \sqrt{\|\partial H/\partial s\|^2 \|\partial H/\partial t\|^2 - ((\partial H/\partial s)\cdot(\partial H/\partial t))^2} dsdt. \end{aligned} \tag{2}$$

[1] A homotopy $H(s, t)$ from $[0, 1] \times [0, 1]$ to $\mathbb{R}^4$ is said to be Lagrangian if $\omega(H_*\partial/\partial s, H_*\partial/\partial t) = 0$. Therefore the image of H is a Lagrangian surface.

Apply this to (1), we get

$$\begin{aligned}
& Area(H) \\
\leq\ & \iint_{[0,1]\times[0,\pi/2]} \|\partial H/\partial s\| \|\partial H/\partial t\| ds dt \\
\leq\ & \iint_{[0,1]\times[0,\pi/2]} (\|\alpha(s)\| + \|\beta(s)\|)(\|\alpha'(s)\| + \|\beta'(s)\|) ds dt \\
\leq\ & (\pi/2)(A+B)(L(\alpha)+L(\beta))
\end{aligned}$$

□

Lemma 2 (Allcock)**.** *Let $\gamma : [0,1] \to \mathbb{R}^4$ be a piecewise smooth closed curve passing through the origin, with $\gamma(s) = \gamma_1(s) + \gamma_2(s)$, where γ_i is the orthogonal projection of γ on to the $x_i y_i$-plane. Then there is a piecewise smooth Lagrangian homotopy between γ and the loop obtained by first traversing γ_1 and then γ_2. Moreover, the area of the homotopy is less than $cL^2(\gamma)$.*

Proof. By extending its domain, we may assume that γ is a piecewise smooth map from $\mathbb{R}^1$ to $\mathbb{R}^4$ which vanishes outside $[0,1]$. Define $H(s,t) = \gamma_1(s+t) + \gamma_2(s)$, where $t \in [0,1]$ and $s \in [-1,1]$. Then it is easy to check H is a Lagrangian homotopy between γ and the loop obtained by first traversing γ_1 and then γ_2(it is in fact parametrized by $\gamma_1(s+1)+\gamma_2(s)$). The area bound is also straightforward using (2). □

The next lemma tells us that we can translate curves Lagrangianly in certain directions.

Lemma 3. *Let $\gamma : [0,1] \to \mathbb{R}^4$ be a piecewise smooth closed curve such that $\gamma(s) - \gamma(0) \in x_2y_2$-plane, $\forall s \in [0,1]$. Then γ is Lagrangianly homotopic to a closed curve entirely lying in x_2y_2-plane, and the area of the homotopy is less than*

$$4(L^2(\gamma) + dist^2(0,\gamma)).$$

Proof. Assume $\gamma(0) = (a_1, b_1, a_2, b_2)$. Let

$$H(s,t) = \gamma(s) - t(a_1, b_1, 0, 0), \qquad (s,t) \in [0,1] \times [0,1].$$

It is easy to check $H(s,1)$ lies in x_2y_2-plane. Since $\gamma(s) = (a_1, b_1, x(s), y(s))$, $\gamma'(s) = (0,0,x'(s),y'(s))$. Also note that

$$\partial H/\partial s = \gamma'(s), \quad \partial H/\partial t = (a_1, b_1, 0, 0).$$

Then $\omega(\partial H/\partial s, \partial H/\partial t) = 0$. So H is Lagrangian. Moreover, by (2), we have

$$\begin{aligned} Area(H) &\leq \iint_{[0,1]\times[0,1]} \|\partial H/\partial s\| \|\partial H/\partial t\| ds dt \\ &\leq \iint_{[0,1]\times[0,1]} \sqrt{a_1^2 + b_1^2}\ \|\gamma'(s)\| ds dt \\ &= \sqrt{a_1^2 + b_1^2}\ L(\gamma) \\ &\leq \|\gamma(0)\| L(\gamma) \\ &\leq 4(L^2(\gamma) + dist^2(0,\gamma)) \end{aligned}$$

The last inequality follows from the fact that $\|\gamma(0)\| \leq L(\gamma) + dist(0,\gamma)$. □

Lemma 4. *Let $\gamma : [0,1] \to \mathbb{R}^4$ be a piecewise smooth closed curve lying entirely in x_2y_2-plane. Then there exists a piecewise smooth closed curve $\tilde{\gamma}$, which lies entirely in x_1y_1-plane , which passes through the origin, and which is Lagrangianly homotopic to γ with homotopy area less than $c(L^2(\gamma) + dist^2(0,\gamma))$, where c is an absolute constant.*

Proof. By means of Lemma 1, we want to find a curve $\tilde{\gamma} : [0,1] \to \mathbb{R}^4$, parametrized by $\tilde{\gamma}(s) = (\tilde{x}(s), \tilde{y}(s), 0, 0)$, such that

$$\tilde{\gamma}(0) = \tilde{\gamma}(1) = 0, \qquad (3)$$
$$\omega(\gamma(s), \gamma'(s)) = \omega(\tilde{\gamma}(s), \tilde{\gamma}'(s))$$

Let $\gamma(s) = (0, 0, x(s), y(s))$, then (3) becomes

$$\tilde{x}'(s)\tilde{y}(s) - \tilde{x}(s)\tilde{y}'(s) = x'(s)y(s) - x(s)y'(s) \qquad (4)$$
$$\tilde{x}(0) = \tilde{x}(1) = 0$$
$$\tilde{y}(0) = \tilde{y}(1) = 0$$

It is not difficult to solve this boundary value ordinary differential equation. Let

$$\tilde{y}_0(s) = \begin{cases} \sqrt[4]{s}, & \text{if } 0 \leq s \leq 1/2, \\ \sqrt[4]{1-s}, & \text{if } 1/2 \leq s \leq 1. \end{cases} \qquad (5)$$

Let $k = \sqrt{L^2(\gamma) + dist^2(0,\gamma)}$. Define $\tilde{y}(s) = k\tilde{y}_0(s)$. Then define

$$\tilde{x}(s) = \tilde{y}(s) \int_0^s \frac{f(t)}{\tilde{y}^2(t)} dt \qquad (6)$$

where $f(t) = x'(t)y(t) - x(t)y'(t)$. It is easy to check that $\tilde{x}(s)$ is well-defined and that $\tilde{x}(s)$, $\tilde{y}(s)$ satisfy (4). Therefore, by Lemma 1, there exists a Lagrangian homotopy H between γ and $\tilde{\gamma}$ with area less than $c_1(A+B)(L(\gamma) + L(\tilde{\gamma}))$, where $A = max(\|\gamma(s)\| : 0 \leq s \leq 1)$, $B = max(\|\tilde{\gamma}(s)\| : 0 \leq s \leq 1)$, and c_1 is an absolute constant.

Now we need to get an estimate on the area of H. First note that by a reparametrization of γ, we may assume that $\|\gamma'(s)\| = L(\gamma)$ (s ranges from 0 to 1). Then we have

$$\begin{aligned} |f(t)| &= |x'(t)y(t) - x(t)y'(t)| \\ &\leq \sqrt{x^2(t)+y^2(t)}\sqrt{(x'(t))^2+(y'(t))^2} \\ &= \sqrt{x^2(t)+y^2(t)}\|\gamma'(s)\| \\ &\leq (dist(0,\gamma)+L(\gamma))L(\gamma) \\ &\leq 4k^2. \end{aligned}$$

Then combining (6) we have $B = max(\|\tilde{\gamma}(s)\| : 0 \leq s \leq 1)) \leq c_2 k$, where c_2 is an absolute constant.

Similarly we can control $L(\tilde{\gamma})$. In fact,

$$\begin{aligned} L(\tilde{\gamma}) &= \int_0^1 \sqrt{(\tilde{x}'(s))^2+(\tilde{y}'(s))^2}ds \\ &= \int_0^1 \sqrt{[\tilde{y}'(s)\int_0^s \frac{f(t)}{\tilde{y}^2(t)}dt + \frac{f(s)}{\tilde{y}(s)}]^2 + (\tilde{y}'(s))^2}ds \\ &\leq c_3 k \int_0^1 \sqrt{[|\tilde{y}_0'(s)|\int_0^s \frac{1}{\tilde{y}_0^2(s)}dt + \frac{1}{\tilde{y}_0(s)}]^2 + (\tilde{y}_0'(s))^2}ds \\ &\leq c_4 k \end{aligned}$$

where c_3 and $c_4 = \int_0^1 \sqrt{[|\tilde{y}_0'(s)|\int_0^s \frac{1}{\tilde{y}_0^2(s)}dt + \frac{1}{\tilde{y}_0(s)}]^2 + (\tilde{y}_0'(s))^2}ds$ are absolute constants(the finiteness of the integral is easy to check using (5)). Therefore we get $L(\tilde{\gamma}) \leq c_4 k$.

Now combine the estimates foy B and $L(\tilde{\gamma})$, we have

$$\begin{aligned} Area(H) &\leq c_1(A+B)(L(\gamma)+L(\tilde{\gamma})) \\ &\leq c_5 k \\ &\leq c_6(L^2(\gamma)+dist^2(0,\gamma)) \end{aligned}$$

□

4. The Proof of Theorem 1

Now we are in the position to prove Theorem 1. It is a modification of the approach used by D.Allcock in [Al]. Without loss of generality, we assume that $\gamma_1(0) = (0,0,0,0)$ and $|\gamma_2(0) - \gamma_1(0)| = dist(\gamma_1,\gamma_2)$. By lemma 2, we might assume that γ_1 lies entirely in x_1y_1-plane and $\gamma_2 - \gamma_2(0)$ lies entirely in x_2y_2-plane. Apply Lemma 3 to γ_2, we get a closed curve β_1 entirely lying in x_2y_2-plane, which is Lagrangianly homotopic to γ_2 with quadratic homotopy area. Then apply Lemma 4, there is a closed curve β_2 entirely lying in x_1y_1-plane and passing through the origin, which is Lagrangianly homotopic to β_1. Since all the homotopies are Lagrangian, by Stokes' Theorem, we get $\int_{\gamma_2}\eta = \int_{\beta_1}\eta = \int_{\beta_2}\eta$. Now let ρ be the curve obtained by joining γ_1 and β_2. Then ρ lies entirely on

x_1y_1-plane with zero symplectitc area. Let ρ be parametrized as $(x(s), y(s), 0, 0)$, for $0 \le s \le 1$. Define a new curve σ in x_2y_2-plane by

$$\sigma(s) = (0, 0, L(\rho), \frac{\int_0^s [x(t)y'(t) - y(t)x'(t)]dt}{L(\rho)}), \ s \in [0, 1].$$

It is easy to check that $\omega(\rho(s), \rho'(s)) = \omega(\sigma(s), \sigma'(s))$ and $L(\sigma) \le 4L(\rho)$. Moreover since $\int_\rho \eta = 0$, we get $\int_0^1 [x(t)y'(t) - y(t)x'(t)]dt = 0$. Thus, σ is a closed curve. By lemma 1, there exists a Lagrangian homotopy between σ and ρ with area less than $cL(\rho)^2$.

Finally since the image of σ is actually on a straight line segment, we can easily use a linear homotopy(which is obviously Lagrangian and which has zero area) to deform it to a point in x_2y_2-plane.

Now combine all the Lagrangian homotopies constructed above , we get an oriented Lagrangian surface with boundary γ_1 and γ_2. The area bound of the total homotopy is easily obtained by keeping track of the area of each individual step.

5. Lagrangian Flat Chains mod2

In this section we extend our result about non-orientable Lagragian surfaces to flat chains mod2. Let $\mathcal{F}_k^2(\mathbb{R}^4)$ denote the set of k-dimensional flat chains mod2(See Federer [Fe] section 4.2.26 and Fleming [Fl] for a detailed discussion on flat chains mod2). We will use $M(\tau)$ to denote the mass of τ in $\mathcal{F}_k^2(\mathbb{R}^4)$. Then the *flat norm* of τ in $\mathcal{F}_k^2(\mathbb{R}^4)$ is defined to be

$$W(\tau) = inf\{M(\rho) + M(\pi) : \tau = \rho + \partial\pi, \rho \in \mathcal{F}_k^2(\mathbb{R}^4), \pi \in \mathcal{F}_{k+1}^2(\mathbb{R}^4)\} \tag{7}$$

And the *flat distance* $W(\tau_1, \tau_2)$ between τ_1 and τ_2 is define to be $W(\tau_1 - \tau_2)$.

A flat chain mod2 is said to be *rectifiable* if it could be approximated under M-norm by Lipschitz chains(the image of polyhedral chains under Lipschitz maps). A flat chain mod2 is said to be *integral* if both itself and its boundary are rectifiable. We will use $I_k^2(\mathbb{R}^4)$ to denote the set of all k-dimensional integral flat chains mod2.

To any integral flat chain mod2 τ we can associate a varifold $V(\tau)$(see Federer [Fe] section 4.2.26 for definition). $\tau \in I_2^2(\mathbb{R}^4)$ is said to be *Lagrangian* if the approximate tangent plane of $V(\tau)$ is Lagrangian μ_τ-a.e, where μ_τ is the underlying measure of τ.

Flat chains mod2 are generalization of non-oriented surfaces. For example , any Möbius band in $\mathbb{R}^4$ is a 2-dimensional integral flat chain mod2 in $\mathbb{R}^4$(See F.Morgan [Mo], Chapter 11).

We have the following generalization of Corollary 2.

Corollary 3 (Isoperimetric Inequality)**.** *Let $\rho \in I_1^2(\mathbb{R}^4)$ be any one-dimensional integral flat chain* mod 2 *in $\mathbb{R}^4$ such that $\partial\rho = 0$ and $M(\rho) < \infty$. Then there exists a two-dimensional Lagrangian integral flat chain* mod2 *τ such that $\partial\tau = \rho$ and $M(\tau) \le cM(\rho)^2$, where c is an absolute constant.*

Proof. By the decomposition theorem of one-dimensional boundariless integral flat chains mod2(see Federer [Fe] section 4.2.25 and section 4.2.26) ρ may be written as $\rho = \sum_{i=1}^{\infty} \iota_i$, where ι_i are closed Lipschitz curves, viewed as flat chains mod 2. Moreover, the mass adds up, i.e., $M(\rho) = \sum_{i=1}^{\infty} M(\iota_i)$. By Corollary 2, we can fill in each ι_i with a Lagrangian flat chain mod 2 τ_i with $M(\tau_i) \leq cM(\iota_i)^2$. Let $\tau = \sum_{i=1}^{\infty} \tau_i$. Then we see that $\partial\tau = \sum_{i=1}^{\infty} \partial\tau_i = \sum_{i=1}^{\infty} \iota_i = \rho$. Moreover,

$$
\begin{aligned}
M(\tau) &\leq \sum_{i=1}^{\infty} M(\tau_i) \\
&\leq \sum_{i=1}^{\infty} cM(\iota_i)^2 \\
&\leq c(\sum_{i=1}^{\infty} M(\iota_i))^2 \\
&= cM(\rho)^2
\end{aligned}
\tag{8}
$$

□

Let $I_{2,L}^2(\mathbb{R}^4)$ denote the set of all integral Lagrangian flat chains mod2. In the study of the regularity of minimal Lagrangian currents, we would like to know whether or not $I_{2,L}^2(\mathbb{R}^4)$ is closed under the flat norm. Unfortunately, the answer is no.

Theorem 4. *The set of all integral Lagrangian flat chains* mod2 *in* $\mathbb{R}^4$*, denoted as* $I_{2,L}^2(\mathbb{R}^4)$*, is dense under the flat norm in the space* $I_2^2(\mathbb{R}^4)$ *of all integral flat chains* mod2 *in* $\mathbb{R}^4$.

Proof. Since the set of two-dimensional polyhedral chains is dense under flat norm in $I_2^2(\mathbb{R}^4)$ (see Fleming [Fl]), it suffices to prove that any polyhedral cell P can be approximated by elements in $I_{2,L}^2(\mathbb{R}^4)$. Furthermore, by approximating polyhedral cell with squares, we might without loss of generality assume P is a unit square in $\mathbb{R}^4$. Divide P evenly into N^2 small square $P_1, \ldots, P_{N^2}$, each with width $1/N$. By Corollary 3, for each P_i, there exists a Lagrangian flat chains mod2 τ_i such that $\partial\tau_i = \partial P_i$, and $M(\tau_i) \leq c_1 M(\partial P_i)^2 = c_1(4/N)^2$. Let $\tau = \sum_{i=1}^{N^2} \tau_i$ Notice that the interior boundaries of P_i cancel when they are added together, hence we get $\partial P = \sum_{i=1}^{N^2} \partial P_i$. Therefore, $\partial\tau = \partial P$. Moreover $M(\tau) \leq \sum_{i=1}^{N^2} M(\tau_i) \leq N^2 c_1(4/N^2) = c_2$. On the other hand, since $\partial(\tau_i - P_i) = 0$, the standard isoperimetric inequality(See Fleming [Fl]) says that there exists a three dimensional integral flat chain mod2 ϕ_i such that $\partial\phi = P_i - \tau_i$ and $M(\phi_i) \leq c_3(M(P_i - \tau_i))^{3/2}$. Since $M(P_i - \tau_i) \leq M(P_i) + M(\tau_i) \leq c_4/N^2$, we have $M(\phi_i) \leq c_5/N^3$. Let $\phi = \sum_{i=1}^{N^2} \phi_i$. It follows that $\partial\phi = P - \tau$ and $M(\phi) \leq c_5 N^2/N^3 \leq c_5/N$. By the definition of flat norm, $W(P, \tau) \leq M(\phi) \leq c_5/N$. Let N go to infinity we get the W-approximation of P. □

Finally we make a remark that these results can be generalized to isotropic surfaces in $\mathbb{R}^{2n}$. If we replace $\mathbb{R}^4$ by $\mathbb{R}^{2n}$ and replace Lagrangian by isotropic, then Theorem 1, Corollary 2, Corollary 3 and Theorem 4 still hold, the proofs of which are exactly the same.

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