A GRADED SYZYGY THEOREM
IN MIXED CHARACTERISTIC

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About twenty years ago we gave the original proof of the so-called "syzygy theorem" [6], at least in the case of a local integral domain containing a field. In the intervening years several different styles of proofs and generalizations have appeared (e.g., see our monograph [7, Chapter 3]). Perhaps the most notable of these were the characteristic p proof of Hochster and Huneke [12] and the generalization by Bruns [2] in which the minimal free complex is allowed some positive homology (see [3, 9.5.5]). In addition Dutta [5] and Hochster [11] combined to show that the crucial argument in [6] which rested upon the existence of maximal Cohen-Macaulay modules could be reduced to an application of the weaker canonical element theorem which holds in the equicharacteristic setting (see also Ogoma [13]).

Our original proof [6] relied heavily on the fact we were able to show order ideals of minimal generators for kth syzygies of finite projective dimension must have grade at least k. We have long known that a single minimal generator with this property would suffice. Until now we remained unaware of any context in which one could actually apply this (no doubt well-known) fact. However, a recent examination of a graded version of the syzygy theorem in mixed characteristic produced just such a situation. To be specific, we are able to show (see Theorem 8 and Corollary 9), under suitable conditions, a nonfree graded kth syzygy of finite projective dimension over a graded ring $R = R_0 \oplus R_1 \oplus R_2 \oplus \cdots$ must have rank at least k, where the ring R_0 is a discrete valuation ring. Other than the observation described above, our techniques are simply a straightforward blend of those from our original manuscript and standard facts about graded rings and modules. In order to keep the exposition somewhat self contained, we supply a few elementary observations which can be found in some form in our articles [6], [7], [8] or Bruns-Herzog [3, Chapter 9]. These observations or facts about syzygies have been tailored to the context of graded modules over $R = R_0 \oplus R_1 \oplus R_2 \oplus \cdots$ as described above.

Preliminaries. Let R be a commutative Noetherian ring and let M be a finitely generated R-module. For $k \geq 0$, we say that M is a kth syzygy if there is a projective acyclic complex $\cdots \rightarrow F_s \rightarrow \cdots \rightarrow F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} 0$ such that $M \cong \operatorname{coker} \varphi_{k+1}$. In case R is local or graded (and $H_0(F_\bullet)$ is graded), then it suffices to consider acyclic free resolutions. The notation $\operatorname{pd}_R M < \infty$ indicates that M has finite projective dimension. We say that M satisfies the Serre condition S_k if

$$\operatorname{depth}_{R_P} M_P \geq \min(k, \operatorname{codim} P)$$

for each prime ideal $P \in \operatorname{Spec} R$. The R-dual of M is denoted by $M^* = \operatorname{Hom}_R(M, R)$. If $e \in M$, then the order ideal of e, $O_M(e)$, is defined by

$$O_M(e) = \{f(e) \mid f \in M^*\}.$$

We observe that the element e induces an R-homomorphism $e : M^* \rightarrow R$ defined by $e(f) = f(e)$. Moreover, the image of this homomorphism is $O_M(e)$. When M has finite projective dimension and, in addition, M has a finite free resolution (as will always be the case in this article) then the rank of M may be defined as the alternating sum of the ranks in any finite free resolution of M.

For the final portion of this discussion we assume that $R = R_0 \oplus R_1 \oplus R_2 \oplus \cdots$ is an $\mathbb{N}$-graded ring ($\mathbb{N}$ = nonnegative integers) in which R_0 is a local ring. Actually, our main interest here will concentrate on the case where R_0 is a DVR (i.e., discrete valuation ring). For such an R, there is a unique graded maximal ideal $m = m_0 + R_+$ where m_0 is the maximal ideal of R_0 and $R_+ = \oplus_{i>0} R_i$. If $M = M_0 \oplus M_1 \oplus \cdots$ is a nonzero graded R-module, we may assume $M_0 \neq 0$ (by "twisting" M) as long as the actual grading is not important. We refer to an element $e \in M - mM$ as a minimal generator of M. This is equivalent to the statement that the element e is part of a minimal set of homogenous generators. This is consistent with conventional terminology in the local case (see [7, p. 2]). We remark that, if $m \in M$ is homogenous, then its order ideal $O_M(m)$ is graded as well (although the elements of degree zero may all be zero in this case).

Finally, any unexplained terminology may be found in Bruns-Herzog [3] or our monograph [7].

The Graded Syzygy Theorem in Mixed Characteristic. Throughout the remainder of this article the ring $R = R_0 \oplus R_1 \oplus R_2 \oplus \cdots$ is a Noetherian graded ring where $R_0 = V$ is a DVR with uniformizing parameter p. Further, we assume p is a regular element on R, that is, the V-modules R_i, for $i > 0$, are finitely generated, torsion free modules (we discuss the situation where p is a zero divisor at the end of the article). Although our central point of interest concerns the situation V has mixed characteristic, actually any DVR will do. The graded maximal ideal of R is denoted by $m = pV + R_+$. For a graded module M, the notation $\bar{M}$ denotes M/pM; so also $\bar{R} = R/pR$. Of course $\bar{M}$ is a graded $\bar{R}$-module.

We begin with a sequence of elementary lemmas concerning grade and codimension of order ideals of homogenous elements. While our statements are for the graded case, it goes without saying that these observations hold in the local case as well. The first of these lemmas is a simple fact about colon ideals. We omit its proof.

Lemma 1. *Let (R, m) be as above and let I and J be homogenous ideals such that $I \subseteq J$. If $\operatorname{codim}(I : J) \geq k$, then $\operatorname{codim} I \geq \min(k, \operatorname{codim} J)$.*

Lemma 2. *Let $M = M_0 \oplus M_1 \oplus \cdots$ be a graded kth syzygy of finite projective dimension and let $e \in M_0 - pM_0$. Let $\bar{e} = e + pM \in \bar{M}$. Then one has the following inequalities*

$$\operatorname{codim}_R O_M(e) \geq \operatorname{codim}_{\bar{R}}(O_M(e) + (p)/(p)) \geq \min(k, \operatorname{codim}_{\bar{R}} O_{\bar{M}}(\bar{e}))$$

Proof. The element e induces an R-homomorphism $e : M^* \to R$ and similarly $\bar{e}$ induces a homomorphism $\bar{e} : \bar{M}^* \to \bar{R}$. Moreover, one has the following commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M^* & \xrightarrow{p} & M^* & \longrightarrow & \bar{M}^* & \longrightarrow & T & \longrightarrow & 0 \\
& & {\scriptstyle e}\downarrow & & {\scriptstyle e}\downarrow & & {\scriptstyle \bar{e}}\downarrow & & & & \\
0 & \longrightarrow & R & \xrightarrow{p} & R & \longrightarrow & \bar{R} & \longrightarrow & 0. & &
\end{array}
$$

We note that the module T is a result of the fact $\overline{M^*} \hookrightarrow \bar{M}^*$ is not always an isomorphism; T represents the cokernel. Moreover, T fits into the left exact sequence

$$0 \to T \to \operatorname{Ext}^1_R(M, R) \xrightarrow{p} \operatorname{Ext}^1_R(M, R).$$

Since M is a kth syzygy of finite projective dimension the R-module $\operatorname{Ext}^1_R(M, R)$ has no support in codimension $\leq k$. Therefore, $\operatorname{codimension}_R(\operatorname{ann} \operatorname{Ext}^1_R(M, R)) \geq k + 1$. Since $T \subseteq \operatorname{Ext}^1_R(M, R)$ and $pT = 0$ it follows $\operatorname{codim}_{\bar{R}}(\operatorname{ann} T) \geq k$. Let $\mathfrak{a} = \operatorname{ann}_{\bar{R}} T$. Then

$$\mathfrak{a} O_{\bar{M}}(\bar{e}) \subseteq O_M(e) + (p)/(p).$$

From Lemma 1 we have

$$\operatorname{codim}_{\bar{R}}(O_M(e) + (p)/(p)) \geq \min(k, \operatorname{codim}_{\bar{R}} O_{\bar{M}}(\bar{e})).$$

Finally an application of Krull's principal ideal theorem yields

$$\operatorname{codim}_R O_M(e) \geq \operatorname{codim}_{\bar{R}}(O_M(e) + (p)/(p)).$$

$\square$

Corollary 3 (Notation as above). *If p belongs to a prime divisor of $O_M(e)$ of minimal codimension, then*

$$\operatorname{codim}_R O_M(e) \geq 1 + \min(k, \operatorname{codim}_{\bar{R}} O_{\bar{M}}(\bar{e})).$$

Proof. This statement follows from Lemma 2 and the observation

$$\operatorname{codim}_R O_M(e) = 1 + \operatorname{codim}_{\bar{R}}(O_M(e) + (p)/(p)).$$

$\square$

Corollary 4. *If M satisfies the Serre condition S_k, then so does R. Moreover, the notation "codimension" in (3) may be replaced by "grade", that is,*

$$\operatorname{grade}_R O_M(e) \geq 1 + \min(k, \operatorname{grade}_{\bar{R}} O_{\bar{M}}(\bar{e})).$$

Proof. That R satisfies S_k follows directly from the Auslander-Buchsbaum formula $\operatorname{pd} M + \operatorname{depth} M = \operatorname{depth} R$. Then since R is S_k one has $\operatorname{grade}_R I = \operatorname{codim}_R I$ for ideals I up to codimension $\leq k$. The ring $\bar{R}$ satisfies these assumptions for ideals of codimension $\leq k - 1$. $\square$

Our next result gets to the heart of the connection between order ideals and kth syzygies of finite projective dimension. We recall from [7, 3.8] that kth syzygies of finite projective dimension coincide with S_k-modules of finite projective dimension in case the ring R satisfies the Serre condition S_k.

Proposition 5. *Let M be a finitely generated graded S_k-module of finite projective dimension over R. If $e \in M_0 - pM_0$ is such that*

$$\operatorname{grade} O_M(e) \geq k > \operatorname{rank} M,$$

then M/Re is an S_{k-1} module of finite projective dimension and $\operatorname{rank}(M/Re) = \operatorname{rank} M - 1$.

Proof. If $\mathfrak{p} \in \operatorname{Spec} R$ and $\mathfrak{p} \not\supseteq O_M(e)$, then $M_{\mathfrak{p}} \cong R_{\mathfrak{p}} e \oplus N_{\mathfrak{p}}$ where $N = M/Re$. Hence $N_{\mathfrak{p}}$ is an S_k-module (so also S_{k-1}) in this case. If $\mathfrak{p} \supseteq O_M(e)$, then codim $\mathfrak{p} \geq k$. The exactness of

$$0 \to R_{\mathfrak{p}} e \to M_{\mathfrak{p}} \to N_{\mathfrak{p}} \to 0$$

together with the Depth Lemma [7, p. 13] will give

$$\operatorname{depth} N_{\mathfrak{p}} \geq \operatorname{depth} M_{\mathfrak{p}} - 1 \geq k - 1,$$

once we have established that the R-homomorphism $R \to Re$, where $1 \mapsto e$, is injective. However, if I denotes the kernel of this homomorphism, then $IO_M(e) = 0$ which means $I = 0$ since grade $O_M(e) > 0$. $\square$

Theorem 6. *Let $M = M_0 \oplus M_1 \oplus \cdots$ be a finitely generated nonfree R-module such that M has finite projective dimension and satisfies the Serre condition S_k. If $e \in M_0 - pM_0$ is such that p is contained in a prime divisor of $O_M(e)$ of minimal codimension, then rank $M \geq k$.*

Proof. We assume the contrary, i.e., rank $M < k$. Further, we may assume our "k" is the smallest positive integer for such an occurrence. In this circumstance we note $\bar{M} = M/pM$ is an S_{k-1} module of finite projective dimension over $\bar{R}$. Moreover, since the graded ring $\bar{R}$ must contain a field, we may apply the original syzygy theorem [7, 3.15] after making a straightforward translation from the local to the graded case. What we obtain here is that $\text{grade}_{\bar{R}} O_{\bar{M}}(\bar{e}) \geq k-1$ since $\bar{M}$ is necessarily nonfree as an $\bar{R}$-module. Therefore $\text{grade}_R O_M(e) \geq k$ follows from Corollary 4 above. We invoke Proposition 5 to get that $N = M/Re$ is an S_{k-1}-module having rank $< k - 1$. This contradicts our original choice of k. $\square$

Our next move will be to demonstrate that Theorem 6 comes into play in the context we described at the outset. Before doing so we make a few observations about what occurs when one inverts "p". Let K be the fraction field of V. Then $R[p^{-1}] = K \oplus R_1[p^{-1}] \oplus R_2[p^{-1}] \oplus \cdots$ is an $\mathbb{N}$-graded ring with $R[p^{-1}]_0 = K$. Moreover, $M[p^{-1}] = M_0[p^{-1}] \oplus M_1[p^{-1}] \oplus \cdots$ is a graded $R[p^{-1}]$-module. The following technical lemma is key to our final argument.

Lemma 7. *Suppose the graded R-module $M = M_0 \oplus M_1 \oplus \cdots$ is such that $M_0 \neq 0$ and M is torsion free as a $V = R_0$-module, that is, assume p is regular on M. Further suppose $M[p^{-1}]$ is a projective $R[p^{-1}]$-module. Then $M[p^{-1}]$ is in fact a free $R[p^{-1}]$-module. Moreover each $e \in M_0 - pM_0$ is a part of some homogenous, free $R[p^{-1}]$-basis of $M[p^{-1}]$.*

Proof. Since $M[p^{-1}]$ is a graded $R[p^{-1}]$-module, we may identify a finite set of homogeneous elements which become a free basis after localizing at the irrelevant maximal ideal $R[p^{-1}]_+$. To form such a homogenous set we may start with a set of minimal generators of M_0 which form a basis for the $K = R[p^{-1}]$ vector space $M_0[p^{-1}]$. It is well known (see [9, Lemma 10.4]) that such a homogenous set is actually a free basis for $M[p^{-1}]$. $\square$

Our main result follows.

Theorem 8. *Let $R = V \oplus R_1 \oplus R_2 \oplus \cdots$ be a graded V-algebra for which the uniformizing parameter p is a regular element. If the graded module M is a nonfree S_k-module of finite projective dimension, then rank $M \geq k$.*

Proof. It suffices to consider $k > 0$ and $M_0 \neq 0$. We suppose to the contrary that rank $M < k$. Since $R[p^{-1}]$ contains a field, it follows from the original syzygy theorem [7, 3.15] that $M[p^{-1}]$ is a locally free module. In particular, $M[p^{-1}]$ is projective. So Lemma 7 applies and we obtain an element $e \in M_0 - pM_0$ which forms part of a free basis of $M[p^{-1}]$. Hence, $1 \in O_{M[p^{-1}]}(e)$ which is

equivalent to $p^v \in O_M(e)$ for some $v > 0$. We reach a contradiction after invoking Theorem 6. □

Corollary 9. *We assume the graded ring $R = V \oplus R_1 \oplus R_2 \oplus \cdots$ satisfies the Serre condition S_k and that M is a kth syzygy of finite projective dimension. Then rank $M \geq k$ in case M is not free.*

Proof. We may assume that $k > 0$. The result of Auslander-Bridger [1] (see [7, 3.8]) gives that M is an S_k-module. The result now follows from Theorem 8. □

Corollary 10. *Let $R = V[X_1, \ldots, X_n]$ be the polynomial ring in n variables over V with the standard grading. If M is a graded nonfree kth syzygy over R, then rank $M \geq k$.*

Proof. Since $pd\ N \leq n$ for each R-module N, it follows $k < n$ since M is nonfree. Of course R is Cohen-Macaulay; so the claim follows from Corollary 9. □

We wish to comment on the case p is a zero divisor on R. Two such situations are easily resolved. If $\dim R = \dim R/(p)$ then the "Improved New Intersection Theorem" (see introduction) holds for R and, thus the syzygy theorem holds as well. If p is contained in an embedded prime component then R cannot satisfy the Serre condition S_1. So the possibilities of S_k-modules of finite projective dimension are of little interest.

We close with some remarks on Bruns' generalization of the syzygy theorem (see [2, 9.5]). Bruns considers finite free complexes $F_\bullet$ which satisfy his condition $\mathrm{codim}\ F_\bullet \geq 0$. To be more specific, if

$$0 \to F_s \xrightarrow{\varphi_{s-1}} F_{s-1} \to \cdots \to F_1 \xrightarrow{\varphi_1} F_0 \to 0$$

is such a complex, then Bruns defines

$$\mathrm{codim}\ F_\bullet = \inf\{\mathrm{codim}\ I_{r_i}(\varphi_i) - i \mid 1 \leq i \leq s - 1\}$$

where $I_{r_i}(\varphi_i)$ denotes the ideal generated by the $r_i \times r_i$ minors of a matrix representation of the map φ_i. Bruns refers to "r_i" as the "expected rank" of φ_i, that is, the rank predicted by the Buchsbaum-Eisenbud Theorem [4] in case $F_\bullet$ is acyclic. Within this framework Bruns [2] shows, if $F_\bullet$ is a minimal free complex (over a local ring) such that $\mathrm{codim}\ F_\bullet \geq 0$, then $r_i \geq \mathrm{codim}\ F_\bullet + i$, for $i = 1, \ldots, s - 1$. It would be interesting to know if the Bruns result could be extended to the setting of this article or perhaps to the following situation: let X be a nonsingular projective scheme over V having very ample line bundle $\mathcal{O}(1)$. Suppose $\mathcal{F}_\bullet$ is a finite acyclic complex of sheaves on X where each $\mathcal{F}_i$ is a direct sum of twists "$\mathcal{O}(n_{ij})$" for each i. Does the syzygy theorem hold for such complexes?

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