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INFINITE TOWERS OF TREE LATTICES

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0. Introduction

Let X be a locally finite tree and let $G = \mathrm{Aut}(X)$. Then G is naturally a locally compact group ([BL], Ch. 3). A discrete subgroup $\Gamma \leq G$ is called an *X-lattice* if

(1) $$\mathrm{Vol}\,(\Gamma\backslash\backslash X) \quad := \quad \sum_{x \in V(\Gamma\backslash X)} \frac{1}{|\Gamma_x|}$$

is finite, and a *uniform X-lattice* if $\Gamma\backslash X$ is a finite graph, *non-uniform* otherwise ([BL], Ch. 3). Bass and Kulkarni have shown ([BK], (4.10)) that $G = \mathrm{Aut}(X)$ contains a uniform X-lattice if and only if X is the universal covering of a finite connected graph, or equivalently, that G is unimodular and $G\backslash X$ is finite. In this case, we call X a *uniform* tree.

Following ([BL], (3.5)) we call X *rigid* if G itself is discrete, and we call X *minimal* if G acts minimally on X, that is, there is no proper G-invariant subtree. If X is uniform then there is always a unique minimal G-invariant subtree $X_0 \subseteq X$ ([BL] (5.7), (5.11), (9.7)). We call X *virtually rigid* if X_0 is rigid (*cf.* ([BL], (3.6)).

Let X be a locally finite tree, and let $\Gamma \leq \Gamma'$ be an inclusion of X-lattices. Then by ([BL], (1.7)) we have:

(2) $$\mathrm{Vol}\,(\Gamma'\backslash\backslash X) \quad = \quad \frac{\mathrm{Vol}\,(\Gamma\backslash\backslash X)}{[\Gamma' : \Gamma]}.$$

We call an infinite ascending chain

(3) $$\Gamma_1 < \Gamma_2 < \Gamma_3 < \dots$$

of X-lattices an *infinite tower of X-lattices*. By (0.2), the lattice inclusions of (0.3) are of finite index, and $\mathrm{Vol}\,(\Gamma_i\backslash\backslash X) \longrightarrow 0$ as $i \longrightarrow \infty$.

The Kazhdan-Margulis property for lattices in Lie groups ([KM]) states that the covolume of a lattice is bounded away from zero. Hence the existence of

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infinite towers of X-lattices in $G = \mathrm{Aut}(X)$ shows that the Kazhdan-Margulis property is violated for X-lattices.

Bass and Kulkarni have given ([BK], (Sec. 7)) several examples of uniform trees such that $G = \mathrm{Aut}(X)$ contains infinite towers of uniform X-lattices. The second author has extended the results and techniques of Bass-Kulkarni to all uniform trees that are not rigid ([R]).

Here our main result is that, with one exception (see §5), if $G = \mathrm{Aut}(X)$ contains a non-uniform X-lattice, then G contains an infinite tower of non-uniform X-lattices.

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1. The setting

An *edge-indexed graph* (A, i) consists of an underlying graph A, and an assignment of a positive integer $i(e) > 0$ to each oriented edge $e \in EA$. Our underlying graph A will always be understood to be locally finite. In [BK] and [BL] one allows $i(e)$ to be any positive cardinal, but our interest here is only in finite $i(e)$. If $i(e) > 1$, we call e *ramified* and *unramified* otherwise.

Let $\mathbb{A} = (A, \mathcal{A})$ be a graph of groups, with underlying graph A, vertex groups $(\mathcal{A}_a)_{a\in VA}$, edge groups $(\mathcal{A}_e = \mathcal{A}_{\overline{e}})_{e\in EA}$ and monomorphisms $\alpha_e\colon \mathcal{A}_e \hookrightarrow \mathcal{A}_{\partial_0 e}$. A graph of groups $\mathbb{A}$ naturally gives rise to an edge-indexed graph $I(\mathbb{A}) = (A, i)$ whose indices are the indices of the edge groups as subgroups of the adjacent vertex groups: that is, $i(e) = [\mathcal{A}_{\partial_0 e} : \alpha_e \mathcal{A}_e]$, which we assume to be finite, for all $e \in EA$.

Given an edge-indexed graph (A, i), a graph of groups $\mathbb{A}$ such that $I(\mathbb{A}) = (A, i)$ is called a *grouping* of (A, i). We call $\mathbb{A}$ a *finite grouping* if the vertex groups $\mathcal{A}_a$ are finite and a *faithful grouping* if $\mathbb{A}$ is a faithful graph of groups, that is, if $\pi_1(\mathbb{A}, a)$ acts faithfully on $X = \widetilde{(\mathbb{A}, a)}$.

Let $\mathbb{A}'$ and $\mathbb{A}$ be groupings of (A, i). Then $\mathbb{A}' = (A, \mathcal{A}')$ is called a *full graph of subgroups* of $\mathbb{A} = (A, \mathcal{A})$ (as in ([B], (1.14)) if $\mathcal{A}'_a \le \mathcal{A}_a$ for $a \in A'$, and for $e \in EA'$, $\mathcal{A}'_e \le \mathcal{A}_e$, and $\alpha'_e = \alpha_e|_{\mathcal{A}'_e}$. We further assume that for $e \in EA'$, with $\partial_0 e = a$, $\mathcal{A}'_a \cap \alpha_e \mathcal{A}_e = \alpha_e \mathcal{A}'_e$, that is $\mathcal{A}'_a/\alpha_e \mathcal{A}'_e \longrightarrow \mathcal{A}_a/\alpha_e \mathcal{A}_e$ is injective, and hence bijective. This assumption implies that $I(\mathbb{A}') = (A, i)$, and that $\pi_1(\mathbb{A}', a') \le \pi_1(\mathbb{A}, a)$ ([B], (1.14)).

Let (A, i) be an edge-indexed graph. A *tower of groupings* on (A, i) is a semi-infinite sequence $(\mathbb{A}_i)_{i\in\mathbb{Z}_{>0}}$ of groupings of (A, i) such that each $\mathbb{A}_i$ is a full graph of proper subgroups of $\mathbb{A}_{i+1}$. A tower of faithful groupings induces an infinite ascending chain of fundamental groups:

$$\pi_1(\mathbb{A}_1, a_0) \le \pi_1(\mathbb{A}_2, a_0) \le \pi_1(\mathbb{A}_3, a_0) \le \dots \tag{1}$$

For an edge $e \in EA$, define:

$$\Delta(e) \quad := \quad \frac{i(\overline{e})}{i(e)}. \tag{2}$$

If $\gamma = (e_1, \ldots, e_n)$ is a path, set:

$$\Delta(\gamma) \quad := \quad \Delta(e_1) \ldots \Delta(e_n).$$

Definition. *An edge-indexed graph (A, i) is called unimodular if $\Delta(\gamma) = 1$ for all closed paths γ in A.*

Now assume that (A, i) is unimodular. Pick a base point $a_0 \in VA$, and define, for $a \in VA$,

$$(3) \qquad N_{a_0}(a) := \frac{\Delta a}{\Delta a_0} (= \Delta(\gamma) \text{ for any path } \gamma \text{ from } a_0 \text{ to } a) \in \mathbb{Q}_{>0}.$$

For $e \in EA$, put

$$N_{a_0}(e) \quad := \quad \frac{N_{a_0}(\partial_0(e))}{i(e)}.$$

Following ([BL], (2.6)), we say that (A, i) has *bounded denominators* if

$$\{N_{a_0}(e) \mid e \in EA\}$$

has bounded denominators, that is, if for some integer $D > 0$, $D \cdot N_{a_0}$ takes only integer values on edges. Since

$$N_{a_1} \quad = \quad \frac{\Delta a_0}{\Delta a_1} N_{a_0},$$

this condition is independant of $a_0 \in VA$.

Theorem ([BK], (2.4)). *An indexed graph (A, i) admits a finite grouping if and only if (A, i) is unimodular and has bounded denominators. The grouping can further be taken to be faithful.*

As in ([BL], Ch. 2) we define the *volume* of an indexed graph (A, i) at a basepoint $a_0 \in VA$:

$$(4) \qquad \mathrm{Vol}_{a_0}(A, i) \quad := \quad \sum_{a \in VA} \frac{1}{(\frac{\Delta a}{\Delta a_0})} \quad = \quad \sum_{a \in VA} (\frac{\Delta a_0}{\Delta a}).$$

Then

$$\mathrm{Vol}_{a_1}(A, i) \quad = \quad \frac{\Delta a_0}{\Delta a_1} \mathrm{Vol}_{a_0}(A, i),$$

([BL], Ch. 2) We write $\mathrm{Vol}\,(A, i) < \infty$ if $\mathrm{Vol}_a(A, i) < \infty$ for some, and hence every $a \in VA$.

If $\mathbb{A}$ is a finite grouping of (A, i), then we have ([BL], (2.6.15)):

$$(5) \qquad \mathrm{Vol}\,(\mathbb{A}) \quad = \quad \frac{1}{|\mathcal{A}_a|} \mathrm{Vol}_a(A, i),$$

which is automatically finite if $\mathrm{Vol}\,(A,i) < \infty$.

We now describe a method for constructing X-lattices which follows naturally from the fundamental theory of Bass-Serre ([B], [S]), and was first suggested in ([BK]). We begin with an edge-indexed graph (A,i). Then (A,i) determines $X = \widetilde{(A,i,a_0)}$ up to isomorphism ([BL], Ch. 2).

We say that (A,i) *admits a lattice* if (A,i) admits a grouping $\mathbb{A}$ such that $\pi_1(\mathbb{A},a_0)$ is an X-lattice. This happens if and only if (A,i) satisfies:

(U) (A,i) is unimodular, and
(BD) (A,i) has bounded denominators, and
(FV) (A,i) has finite volume.

Assume that (A,i) is unimodular and has bounded denominators (which is automatic if A if finite). By ([BK], (2.4)) we can find a finite faithful grouping $\mathbb{A}$ of (A,i) and a group $\Gamma = \pi_1(\mathbb{A},a_0)$ acting faithfully on X. Then

(a) Γ is *discrete*, since $\mathbb{A}$ is a graph of *finite* groups.
(b) Γ is a *uniform X-lattice* if and only if A is finite.
(c) Γ is a *non-uniform X-lattice* if and only if A is infinite, and

$$\mathrm{Vol}\,(\Gamma\backslash\backslash X) = \mathrm{Vol}\,(\mathbb{A})(:= \sum_{a\in VA} \frac{1}{|\mathcal{A}_a|} = \frac{1}{|\mathcal{A}_a|}\mathrm{Vol}_a(A,i)) < \infty. \tag{6}$$

Our task is the following: *given an edge-indexed graph* (A,i) *of finite volume, construct an infinite tower of finite faithful groupings of* (A,i). This induces an infinite tower

$$\Gamma_1 < \Gamma_2 < \Gamma_3 < \dots$$

of X-lattices in $\mathrm{Aut}(X)$, for $X = \widetilde{(A,i)}$, with $\Gamma_i\backslash X = A$, $i = 1,2,\dots$.

An edge $e \in EA$ is called *separating* if $A - \{e,\overline{e}\}$ has two connected components $A_0(e)$ and $A_1(e)$, where $A_0(e)$ and $A_1(e)$ contain $\partial_0(e)$ and $\partial_1(e)$ respectively.

Let (A,i) be any connected edge-indexed graph. A subset $\beta \subset EA$ of $n \geq 2$ (oriented) edges is called an *arithmetic bridge* for (A,i) (as in ([C1], Sec. 4)) if:

(1) $\beta \cap \overline{\beta} = \varnothing$, $A - (\beta \cup \overline{\beta})$ has two connected components, A_0 and A_1,
(2) For every $e \in \beta$, $\partial_0 e \in A_0$ and $\partial_1 e \in A_1$,
(3) There exists an integer $d > 1$ such that $d \mid i(e)$ for every $e \in \beta$.

Following [BT] we say that (A,i) is *discretely ramified* if for $e \in EA$

$$i(e) > 1 \implies i(e) = 2,\ e \text{ is separating, and } (A_0(e),i) \text{ is an unramified tree.}$$

We call (A,i) a *dominant-rooted edge-indexed tree* if A is a tree and if there exists an $a \in VA$ such that $i(e) = 1$ for all edges $e \in EA$ directed towards a. Let (A,i) be an edge-indexed graph. We say that (A,i) is *restricted* if (A,i) satisfies any one of the following conditions:

(DR) (A,i) is discretely ramified, or
(F) (A,i) is a dominant-rooted edge-indexed tree, or
(GS) A is a tree, and (A,i) contains a prime-prime interval (see [R]) and no other ramified edges.

We say that (A, i) is *permissible* if (A, i) admits a lattice and if (A, i) is not restricted. We note that an infinite edge-indexed graph (A, i) with finite volume is automatically non-discretely ramified, is not a dominant-rooted tree ([CR2]), is obviously not (GS) as above, and hence is not restricted.

2. Rooted products of graphs of groups

Given rooted graphs of groups $\mathbb{A} = (A, \mathcal{A}, a_0)$, $a_0 \in VA$, and $\mathbb{B} = (B, \mathcal{B}, b_0)$, $b_0 \in VB$, we construct a rooted graph of groups $\mathbb{C} = (C, \mathcal{C}, c_0) = \mathbb{A} \times_{a_0=b_0} \mathbb{B}$ as follows: we set

$$C \quad := \quad A \sqcup B/(a_0 = b_0 = c_0).$$

For $a \in VA$, $e \in EA$, we set

$$\mathcal{C}_a := \mathcal{A}_a \times \mathcal{B}_{b_0}, \quad \mathcal{C}_e := \mathcal{A}_e \times \mathcal{B}_{b_0},$$

and if $\alpha_e : \mathcal{A}_e \hookrightarrow \mathcal{A}_{\partial_0 e}$, we set

$$\gamma_e \quad := \quad \alpha_e \times Id_{B_{b_0}}.$$

Similarly, for $b \in VB$, $e \in EB$, we set

$$\mathcal{C}_b := \mathcal{A}_{a_0} \times \mathcal{B}_b, \quad \mathcal{C}_e := \mathcal{A}_{a_0} \times \mathcal{B}_e,$$

and if $\beta_e : \mathcal{B}_e \hookrightarrow \mathcal{B}_{\partial_0 e}$, we set

$$\gamma_e \quad := \quad Id_{A_{a_0}} \times \beta_e.$$

If we set $(A, i^A) = I(\mathbb{A})$ and $(B, i^B) = I(\mathbb{B})$, then we have the 'rooted union of edge-indexed graphs':

$$(C, i^C) \quad := \quad (A, i^A) \sqcup (B, i^B)/(a_0 = b_0 = c_0),$$

for $c_0 \in VC$, and clearly $\mathbb{C}$ is a grouping of (C, i^C).

(2.1) Remarks.

(1) The graph of groups $\mathbb{C}$ is faithful if and only if $\mathbb{A}$ and $\mathbb{B}$ are faithful. In fact, if $N_{\mathbb{A}}$ is the maximal normal subgroup of $\mathbb{A}$, and $N_{\mathbb{B}}$ is the maximal normal subgroup of $\mathbb{B}$, then the maximal normal subgroup of $\mathbb{C}$ is $N_{\mathbb{A}} \times N_{\mathbb{B}}$.

(2) We have

$$\pi_1(\mathbb{C}, c_0) \quad = \quad (\pi_1(\mathbb{A}, a_0) \times \mathcal{B}_{b_0}) \quad *_{(\mathcal{A}_{a_0} \times \mathcal{B}_{b_0})} \quad (\mathcal{A}_{a_0} \times \pi_1(\mathbb{B}, b_0)).$$

(3) If $\mathbb{A}$ and $\mathbb{B}$ are graphs of finite groups, then so also is $\mathbb{C}$, and

$$\mathrm{Vol}\,(\mathbb{C}) \quad = \quad \frac{1}{|\mathcal{B}_{b_0}|}\mathrm{Vol}\,(\mathbb{A}) \quad + \quad \frac{1}{|\mathcal{A}_{a_0}|}\mathrm{Vol}\,(\mathbb{B}) \quad - \quad \frac{1}{|\mathcal{A}_{a_0}||\mathcal{B}_{b_0}|}.$$

(2.2) Functoriality.

Suppose that we have groupings $\mathbb{A} \leq \mathbb{A}'$ of an edge-indexed graph (A, i^A) and $\mathbb{B} \leq \mathbb{B}'$ of an edge-indexed graph (B, i^B), then we get groupings

$$\mathbb{C} = \mathbb{A} \times_{a_0=b_0} \mathbb{B} \quad \leq \quad \mathbb{C}' = \mathbb{A}' \times_{a_0=b_0} \mathbb{B}'$$

for $a_0 \in VA$, $b_0 \in VB$ of the edge-indexed graph

$$(C, i^C) = (A, i^A) \sqcup (B, i^B)/(a_0 = b_0 = c_0),$$

for $c_0 \in VC$. In particular, for an edge $e \in VA$ with initial vertex $a \in VA$,

$$\begin{array}{ccc} \mathcal{A}'_e \times \mathcal{B}'_{b_0} & \overset{\alpha'_e \times Id_{B'_{b_0}}}{\longrightarrow} & \mathcal{A}'_a \times \mathcal{B}'_{b_0} \\ \leq \quad \leq & & \leq \quad \leq \\ \mathcal{A}_e \times \mathcal{B}_{b_0} & \overset{\alpha_e \times Id_{B_{b_0}}}{\longrightarrow} & \mathcal{A}_a \times \mathcal{B}_{b_0} \end{array}$$

commutes, and similarly in B.

(2.3) Corollary. *A tower* $\mathbb{A}_1 \leq \mathbb{A}_2 \leq \mathbb{A}_3 \leq \ldots$ *yields a tower*

$$\mathbb{A}_1 \times_{a_0=b_0} \mathbb{B} \quad \leq \quad \mathbb{A}_2 \times_{a_0=b_0} \mathbb{B} \quad \leq \quad \mathbb{A}_3 \times_{a_0=b_0} \mathbb{B} \quad \leq \quad \ldots$$

Since a unimodular edge-indexed graph with bounded denominators admits a finite faithful grouping, we can apply the above corollary repeatedly to obtain the following lemma.

(2.4) Lemma. *Let (A, i) be an edge-indexed graph and let (A_0, i) be a core subgraph such that (A, i) is obtained from (A_0, i) by attaching to finitely many vertices $a_1, \ldots, a_n \in VA_0$, rooted edge-indexed graphs (A_j, i_j, a_j), $j = 1, \ldots, n$ respectively. Suppose that (A_0, i) admits an infinite ascending chain of finite faithful groupings of finite volume. Suppose that each of the (A_j, i_j), $j = 1, \ldots, n$ are unimodular, have finite volume and bounded denominators. Then (A, i) admits an infinite tower of finite faithful groupings of finite volume.*

3. Infinite towers of uniform tree lattices

In [R], the second author proved the following:

(3.1) Theorem ([R]). *Let (A, i) be a finite permissible edge-indexed graph. Then (A, i) admits an infinite tower of finite faithful groupings.*

The proof of Theorem (3.1) generalizes the techniques of Bass-Kulkarni ([BK]) for constructing towers of groupings on certain fundamental examples, and uses certain constructions with edge-indexed graphs to extend to a more general setting.

Theorem (3.1) yields the following:

(3.2) Theorem ([R]). *Let X be a locally finite tree. The following conditions are equivalent:*

(a) *X is uniform and not rigid.*
(b) *X is the universal cover of a finite permissible edge-indexed graph.*
(c) *$Aut(X)$ contains an infinite ascending chain*

$$\Gamma_1 < \Gamma_2 < \Gamma_3 < \cdots$$

of uniform X-lattices.
(d) *The set of uniform covolumes*

$$\{ Vol(\Gamma \backslash\backslash X) \mid \Gamma \text{ is a uniform } X\text{-lattice}\} \subset \mathbb{Q}_{>0}$$

is not bounded away from zero.

This generalizes Theorem 7.1(a) of [BK] which states the result for homogeneous trees.

4. Infinite towers of non-uniform X-lattices with quotient a tree

The techniques described in §3 extend to certain infinite edge-indexed graphs. We have the following:

(4.1) Theorem. *Let (A, i) be an edge-indexed graph that admits a lattice, and which is infinite, hence permissible. Assume that (A, i) is a tree, but is not dominant-end-rooted. Then (A, i) admits an infinite tower of finite faithful groupings.*

Except for the case that (A, i) is a *dominant-end-rooted edge-indexed tree* (similar to a dominant-rooted edge-indexed tree as defined on page 4, except that an end takes the role given to the root vertex before, that is, i(e)=1 for all edges directed towards that end as opposed to the root vertex (see [CR2] for more details)), the assumption that (A, i) is an infinite permissible tree implies the existence of a *finite* permissible 'core' graph (A_0, i) which is an edge-indexed path of length $n \geq 1$. By Theorem (3.1), (A_0, i) admits an infinite tower of finite faithful groupings, and we may then apply Lemma (2.4) to extend the tower of groupings to (A, i).

If (A, i) is a dominant-end-rooted edge-indexed tree, then (A, i) does not contain a finite permissible core. We know that in this case, the set of covolumes of non-uniform lattices in Aut(X), $X = \widetilde{(A, i)}$, is not bounded away from zero, however our techniques do not suffice to produce a tower of groupings on (A, i).

(4.2) Theorem. *Let (A, i) be as in Theorem (4.1). Let $X = \widetilde{(A, i)}$. Then there is an infinite ascending chain*

$$\Gamma_1 < \Gamma_2 < \Gamma_3 < \cdots$$

of non-uniform X-lattices in $Aut(X)$. Hence $Vol(\Gamma_i \backslash\backslash X) \longrightarrow 0$ as $i \longrightarrow \infty$.

In Theorems (4.1), and (4.2), the covering tree $X = \widetilde{(A, i)}$ may be uniform or not.

5. Infinite towers of non-uniform X-lattices

We have the following:

(5.1) Theorem ([R]). *Let (A, i) be a permissible edge-indexed graph. Suppose (A, i) contains an arithmetic bridge with $n \geq 2$ edges. Then (A, i) admits an infinite tower of finite faithful groupings.*

Concerning existence of arithmetic bridges, we have the following:

(5.2) Theorem ([C1], [CR1]). *Let (A, i) be a unimodular edge-indexed graph. Let $e \in EA$ be a ramified edge such that $\Delta(e)$ is not an integer. If e is not separating, then e is contained in an arithmetic bridge with $n \geq 2$ edges.*

Combining the results of §2, §4 and the above, we have:

(5.3) Theorem. *Let (A, i) be a permissible edge-indexed graph that is not a dominant-end-rooted edge-indexed tree. Then (A, i) admits an infinite tower of finite faithful groupings.*

A corollary of Theorem (5.3) is the following:

(5.4) Theorem. *Let X be a locally finite tree. If $Aut(X)$ contains a non-uniform X-lattice Γ, and X is not the universal cover of a dominant-end-rooted edge-indexed tree, then $Aut(X)$ contains an infinite tower*

$$\Gamma_1 < \Gamma_2 < \Gamma_3 < \cdots$$

of non-uniform X-lattices. Hence $Vol(\Gamma_i \backslash\backslash X) \longrightarrow 0$ as $i \longrightarrow \infty$.

6. Existence of non-uniform X-lattices

By Theorem (5.4), the question of existence of infinite towers of non-uniform X-lattices reduces to the question of existence of non-uniform X-lattices.

To outline the results on existence of non-uniform X-lattices, we make the following definition. Let X be a locally finite tree, $G = \mathrm{Aut}(X)$, and let μ be a (left) Haar measure on G. Suppose that G is unimodular. Then $\mu(G_x)$ is constant on G-orbits, so we can define ([BL], (1.5)):

$$\mu(G\backslash\backslash X) := \sum_{x \in V(G\backslash X)} \frac{1}{\mu(G_x)}.$$

We have the 'Lattice existence theorem':

(6.1) Theorem ([BCR], (0.2)). *Let X be a locally finite tree, let $G = Aut(X)$, and let μ be a (left) Haar measure on G. The following conditions are equivalent:*

(a) *G contains an X-lattice Γ.*
(b) *(U) G is unimodular, and*
(FV) $\mu(G\backslash\backslash X) < \infty$.

In particular, we have the following theorem, which together with Bass-Kulkarni's 'Uniform existence theorem' ([BK], (4.10)) gives Theorem (6.1):

(6.2) Theorem ([BCR], (0.5)). *Let X be a locally finite tree, let $G = Aut(X)$, and let μ be a (left) Haar measure on G. Assume that:*

(U) G *is unimodular,*

(FV) $\mu(G\backslash\backslash X) < \infty$, *and*

(INF) $G\backslash X$ *is infinite.*

Then G contains a (necessarily non-uniform) X-lattice Γ.

For uniform trees, we have the following:

(6.3) Theorem ([C1], [C2]). *If X is uniform and not virtually rigid then G contains a non-uniform X-lattice Γ.*

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