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THE BLOCH-KATO CONJECTURE FOR ADJOINT MOTIVES OF MODULAR FORMS

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Abstract. The Tamagawa number conjecture of Bloch and Kato describes the behavior at integers of the L-function associated to a motive over $\mathbf{Q}$. Let f be a newform of weight $k \geq 2$, level N with coefficients in a number field K. Let M be the motive associated to f and let A be the adjoint motive of M. Let λ be a finite prime of K. We verify the λ-part of the Bloch-Kato conjecture for $L(A, 0)$ and $L(A, 1)$ when $\lambda \nmid Nk!$ and the mod λ representation associated to f is absolutely irreducible when restricted to the Galois group over $\mathbf{Q}\left(\sqrt{(-1)^{(\ell-1)/2}\ell}\right)$ where $\lambda \mid \ell$.

1. Introduction

This is a summary of results on the Tamagawa number conjecture of Bloch and Kato [B-K] for adjoint motives of modular forms of weight $k \geq 2$. The conjecture relates the value at 0 of the associated L-function to arithmetic invariants of the motive. We prove in [D-F-G] that it holds up to powers of certain "bad primes." The strategy for achieving this is essentially due to Wiles [Wi], as completed with Taylor in [T-W]. The Taylor-Wiles construction yields a formula relating the size of a certain module measuring congruences between modular forms to that of a certain Galois cohomology group. This was carried out in [Wi] and [T-W] in the context of modular forms of weight 2, where it was used to prove results in the direction of the Fontaine-Mazur conjecture [F-M]. While it was no surprise that the method could be generalized to higher weight modular forms and that the resulting formula would be related to the Bloch-Kato conjecture, there remained many technical details to verify in order to accomplish this. In particular, the very formulation of the conjecture relies on a comparison isomorphism between the ℓ-adic and de Rham realizations of the motive provided by theorems of Faltings [Fa] or Tsuji [Ts], and verification of the conjecture requires the careful application of such a theorem. We also need to generalize results on congruences between modular forms to higher weight, and to compute certain local Tamagawa numbers.

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2. The adjoint motive

Suppose f is a newform of weight $k \geq 2$, level N and character ψ with coefficients in the ring of integers $\mathcal{O}$ of a number field K. For $* = B$, dr or λ (a prime of K), we let M_* denote the corresponding realization of the motive attached to f (see [De2]). This is a two-dimensional K_*-subspace (where $K_B = K_{\mathrm{dr}} = K$) of the corresponding cohomology group of the modular curve $X_1(N)$ with coefficients in a sheaf depending on k. We have certain additional structure on each M_* (an involution, filtration or $G_{\mathbf{Q}}$-action) and comparison isomorphisms relating certain pairs of realizations. Let S be a set of primes containing those dividing $Nk!$, and let $\mathcal{O}_B = \mathcal{O}_{\mathrm{dr}} = \mathcal{O}_S$ denote the S-integers in K. For $* = B$, dr or λ for $\lambda \notin S$, we also have natural $\mathcal{O}_*$-lattices $\mathcal{M}_*$ in M_* and integral versions of the comparison isomorphisms involving the $\mathcal{M}_\lambda$.

Let D_* denote the $*$-realization of the $(1-k)$-Tate twist of the motive associated to ψ. Poincaré duality for $X_1(N)$ gives rise to a perfect alternating pairing on M_* with values in D_*, i.e., an isomorphism

$$\mu_* : \wedge^2_{K_*} M_* \to D_*,$$

respecting the various additional structures and comparison isomorphisms. We also have natural $\mathcal{O}_*$-lattices $\mathcal{D}_*$ in D_*, but the map $\wedge^2_{\mathcal{O}_*}\mathcal{M}_* \to \mathcal{D}_*$ need not be an isomorphism; the image is $\eta\mathcal{D}_*$ for a certain ideal η of $\mathcal{O}_S$. More generally, if Σ is a finite set of primes contained in S, we analogously define $\mathcal{M}_*^\Sigma$, μ_*^Σ and η^Σ using level structure which is, roughly speaking, minimal outside Σ.

We let A_* denote the set of trace-zero elements in $\mathrm{End}_{K_*} M_*$. So A_* is three-dimensional over K_* and we have comparison isomorphisms, including

$$I_\infty : A_{\mathrm{dr}} \otimes \mathbf{C} \to A_B \otimes \mathbf{C},$$

induced by those for the M_*. Replacing f by a twist does not change this data, so we always assume f has minimal conductor among its twists. The L-function associated to A_λ is independent of λ; we denote it $L(A,s)$ and regard it as taking values in $K \otimes \mathbf{C}$. We know also that $L(A,s)$ extends to an entire function on the complex plane and satisfies the functional equation (see [G-J], [Sc]),

$$\Lambda(A,s) = \epsilon(A,s)\Lambda(A,1-s),$$

where

$$\Lambda(A,s) = L(A,s)\Gamma_{\mathbf{R}}(s+1)\Gamma_{\mathbf{C}}(s+k-1)$$

and $\epsilon(A,s)$ is as defined by Deligne [De1]. Moreover $L(A,0)$ and $L(A,1)$ are invertible elements of $\mathbf{R} \otimes K$, and $\epsilon(A,s) = \pm C^{1-2s}$ for some positive integer C dividing N.

3. The Bloch-Kato conjecture

We show that the conjecture of Bloch and Kato [B-K] correctly predicts the values of $L(A,0)$ and $L(A,1)$ up to an element of $\mathcal{O}_S^\times$, where S is a certain finite set of primes containing those dividing $Nk!$. We use the formulation of the conjecture given by Fontaine and Perrin-Riou [F-P], which is easily generalized to the setting of K-coefficients.

The *fundamental line* for A is the K-line defined by

$$\begin{aligned}\Delta &= \mathrm{Hom}_K(A_B^+,\ A_{\mathrm{dr}}/\mathrm{Fil}^0 A_{\mathrm{dr}})\\ &\cong \mathrm{Hom}_K(\mathrm{Fil}^{k-1}M_{\mathrm{dr}},\ M_{\mathrm{dr}}/\mathrm{Fil}^{k-1}M_{\mathrm{dr}}).\end{aligned}$$

The *Deligne period* for A is the natural basis c^+ for $\Delta\otimes\mathbf{R}$ gotten from I_∞. One finds that $c^+L(A,0)^{-1}$ is an element $\delta\in\Delta$ characterized by

$$\mu_{\mathrm{dr}}(f\wedge\delta(f)) = \frac{i^{k-\eta}((k-2)!)^2\epsilon(M)\prod_{p\in P}(1+p^{-1})}{2\epsilon(D)\epsilon(A)}\cdot G_{\mathrm{dr}}, \tag{3.1}$$

where $\eta\in\{0,1\}$ has the same parity as k, P is a certain exceptional set of primes dividing N and G_{dr} (for Gauss sum) is the usual basis for D_{dr}. This is proved by relating the Deligne period to the Petersson product, applying a method of Rankin and Shimura to relate this to $L(A,1)$ (see [St], [Hi], [Sc]) and using the functional equation.

To state the λ-part of the Bloch-Kato conjecture, one chooses a Galois-stable $\mathcal{O}_\lambda$-lattice $\mathcal{A}_\lambda$ in $A_\lambda\cong A_B\otimes_K K_\lambda$ and an $\mathcal{O}_\lambda$-lattice ω in $(A_{\mathrm{dr}}/\mathrm{Fil}^0A_{\mathrm{dr}})\otimes_K K_\lambda$. The conjecture, which turns out to be independent of these choices, states that

- $H^1_f(G_{\mathbf{Q}},A_\lambda/\mathcal{A}_\lambda)$ is finite;
- δ generates the $\mathcal{O}_\lambda$-submodule

$$\frac{\mathrm{Fitt}_{\mathcal{O}_\lambda}H^0(G_{\mathbf{Q}},A_\lambda/\mathcal{A}_\lambda)\cdot\mathrm{Fitt}_{\mathcal{O}_\lambda}H^0(G_{\mathbf{Q}},B_\lambda/\mathcal{B}_\lambda)}{\mathrm{Tam}^0_\omega(\mathcal{A}_\lambda)\cdot\mathrm{Fitt}_{\mathcal{O}_\lambda}H^1_f(G_{\mathbf{Q}},A_\lambda/\mathcal{A}_\lambda)}\cdot\mathrm{Hom}_{\mathcal{O}_\lambda}(\mathcal{A}_\lambda^+,\omega)$$

  of $\Delta\otimes_K K_\lambda$, where $\mathcal{B}_\lambda=\mathrm{Hom}_{\mathbf{Z}_\ell}(\mathcal{A}_\lambda,\mathbf{Z}_\ell(1))$, $B_\lambda=\mathcal{B}_\lambda\otimes\mathbf{Q}$ and $\mathrm{Tam}^0_\omega(\mathcal{A}_\lambda)$ is the Tamagawa ideal of $\mathcal{A}_\lambda$ relative to ω.

4. Methods

The two main problems are to compute the local contribution at ℓ to the Tamagawa ideal, $\mathrm{Tam}_{\ell,\omega}(\mathcal{A}_\lambda)$, and the length of $H^1_f(G_{\mathbf{Q}},A_\lambda/\mathcal{A}_\lambda)$. We restrict our attention to primes λ not in S, and let $\mathcal{A}_*$ denote the set of elements of $\mathrm{End}_{\mathcal{O}_*}(\mathcal{M}_*)$ of trace zero. One can then use existing machinery to show that $\mathrm{Tam}_{\ell,\omega}(\mathcal{A}_\lambda)=\mathcal{O}_\lambda$, where $\omega=(\mathcal{A}_{\mathrm{dr}}/\mathrm{Fil}^0\mathcal{A}_{\mathrm{dr}})\otimes_{\mathcal{O}_{\mathrm{dr}}}\mathcal{O}_\lambda$. The proof requires only that λ not divide $Nk!$ and uses the integral version of Faltings' comparison theorem [Fa]. For primes between k and $2k$, the argument is slightly delicate since the filtration length of A_{dr} is greater than $\ell-2$.

The computation of the H^1_f relies on the methods of Wiles [Wi] and Taylor-Wiles [T-W], as modified in [Di1] and [Di2]. Now we have to impose another hypothesis on λ:

(2) $\bar\rho: G_{\mathbf{Q}}\to\mathrm{Aut}_{\mathcal{O}_\lambda/\lambda}(\mathcal{M}_\lambda/\lambda\mathcal{M}_\lambda)$ has absolutely irreducible restriction to G_L, where $L=\mathbf{Q}(\sqrt{(-1)^{(\ell-1)/2}\ell})$.

We give an axiomatic formulation of the method of the following nature: Suppose

$$\mathcal{R} = \{\rho : G_{\mathbf{Q}} \to \mathrm{Aut}_{K_\rho} V_\rho\}$$

is a set of liftings of $\bar{\rho}$, where for each ρ in $\mathcal{R}$, K_ρ is a finite extension of K_λ in $\bar{K}_\lambda$ and the restriction of ρ to $G_{\mathbf{Q}_\ell}$ is crystalline of Hodge-Tate type $(0, k-1)$ (see [Fo]). For any finite set of primes Σ not containing ℓ, let $\mathcal{R}^\Sigma$ denote the set of $\rho \in \mathcal{R}$ such that ρ is minimally ramified outside Σ. Suppose that for each $\rho \in \mathcal{R}^\Sigma$ we are given an isomorphism

$$\mu_\rho^\Sigma : \wedge^2_{K_\rho} V_\rho \to D_\lambda \otimes_{K_\lambda} K_\rho.$$

We assume that for fixed ρ and varying Σ, the ratios of the μ_ρ^Σ are determined by Euler factors. We assume also that $\oplus_{\rho \in \mathcal{R}} (V_\rho \otimes_{K_\rho} \bar{K}_\lambda)$ contains a lattice $\mathcal{V}$ having a certain self-duality property with respect to the pairings. We show that if the numbers $\#\mathcal{R}^\Sigma$ satisfy a certain numerical criterion, then

- every lifting of $\bar{\rho}$ which is crystalline of Hodge-Tate type $(0, k-1)$ is isomorphic to an element of $\mathcal{R}$;
- if $\rho \in \mathcal{R}^\Sigma$ and $K_\rho = K_\lambda$, then

$$\mathrm{Fitt}_{\mathcal{O}_\lambda} H^1_\Sigma(G_{\mathbf{Q}}, \mathcal{W}_\rho \otimes_{\mathcal{O}_\lambda} (K_\lambda/\mathcal{O}_\lambda)) = \mathrm{Fitt}_{\mathcal{O}_\lambda} \mathcal{D}_\lambda / \mu_\rho^\Sigma (\wedge^2_{\mathcal{O}_\lambda} \mathcal{V}_\rho),$$

  where $\mathcal{V}_\rho = \mathcal{V} \cap V_\rho$ and $\mathcal{W}_\rho$ is the set of elements of $\mathrm{End}_{\mathcal{O}_\lambda} \mathcal{V}_\rho$ of trace zero.

5. Main results

Generalizing results and methods of Ribet, Wiles and others ([Ri], [D-T], [Wi], [T-W], [Di1]), we verify that these hypotheses are satisfied by the set $\mathcal{R}$ of liftings of $\bar{\rho}$ associated to newforms of weight k, level prime to ℓ and character ψ. We thus obtain the following result in the direction of the Fontaine-Mazur conjecture [F-M]:

Theorem 1. *Suppose $\rho : G_{\mathbf{Q}} \to \mathrm{Aut}_{K_\lambda} V_\lambda$ is continuous, two-dimensional, unramified outside a finite set of primes and has crystalline restriction to $G_{\mathbf{Q}_\ell}$ of Hodge-Tate type $(0, d)$ with $0 < d < \ell - 1$. If $\bar{\rho}$ is modular and has absolutely irreducible restriction to G_L, then ρ is modular.*

We also find that $\mathrm{Fitt}_{\mathcal{O}_\lambda} H^1_\Sigma(G_{\mathbf{Q}}, A_\lambda/\mathcal{A}_\lambda) = \eta^\Sigma \mathcal{O}_\lambda$. Using the modified Poitou-Tate sequence (see [Fl2]), we can then compute the length of $H^1_f(G_{\mathbf{Q}}, A_\lambda/\mathcal{A}_\lambda)$, which is dual to $H^1_f(G_{\mathbf{Q}}, B_\lambda/\mathcal{B}_\lambda)$ [Fl1]. Note that (2) implies that $H^0(G_{\mathbf{Q}}, A_\lambda/\mathcal{A}_\lambda)$ and $H^0(G_{\mathbf{Q}}, B_\lambda/\mathcal{B}_\lambda)$ both vanish. Applying equation (1) and its analogue for $L(A, 1)$, we conclude:

Theorem 2. *Let f be a newform of weight $k \geq 2$ and level N with coefficients in a number field K. Let S denote the set of primes λ such that λ divides $Nk!$ or (2) fails. For λ not in S, the λ-part of the Bloch-Kato conjecture holds for $L(A, 0)$ and $L(A, 1)$.*

Finally we remark that the set S is finite. In fact, if λ does not divide $Nk!$ and (2) fails, then $\ell = 2k-1$, $\ell = 2k-3$ or $\mathcal{M}_\lambda/\lambda\mathcal{M}_\lambda$ defines a reducible representation of $G_{\mathbf{Q}}$, and reducibility for infinitely many λ would violate the Ramanujan conjecture.

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