THE DIMENSION OF THE PLANAR BROWNIAN FRONTIER IS 4/3

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1. Introduction

The purpose of this note is to announce and sketch the proofs of results determining the Hausdorff dimension of certain subsets of planar Brownian paths. Proofs are currently written down in a sequence of preprints [9, 10, 11, 12]. The present announcement will give an overview of some of the contents of this series. At this point, it appears that some parts of the proofs can be significantly simplified using different means (see [20]). However, it seems premature to give details about this at this time.

Let B_t be a Brownian motion taking values in $\mathbb{R}^2$ (or $\mathbb{C}$). The *hull* of the Brownian motion B at time t is the union of $B[0,t] = \{B_s : 0 \leq s \leq t\}$ with the bounded components of the complement $\mathbb{R}^2 \setminus B[0,t]$ of $B[0,t]$. In other words, the hull is the Brownian path with all the holes "filled in". The boundary of the hull is called the *frontier* or *outer boundary* of Brownian motion. Based on simulations and the analogy with self-avoiding walks, Mandelbrot [16] conjectured that the dimension of the frontier is 4/3. In this note, we announce a proof of this conjecture as well as proofs of some other related questions on exceptional subsets of planar Brownian paths.

A time $s \in (0,t)$ is called a *cut time* and B_s is called a *cut point* for $B[0,t]$ if

$$B[0,s) \cap B(s,t] = \emptyset.$$

A point B_s is called a *pioneer point* if B_s is on the frontier at time s, that is, if B_s is on the boundary of the unbounded component of the complement of $B[0,s]$.

Theorem 1.1 ([10, 12]). *Let B_t be a planar Brownian motion. With probability one, the Hausdorff dimension of the frontier of $B[0,1]$ is 4/3; the Hausdorff dimension of the set of cut points is 3/4; and the Hausdorff dimension of the set of pioneer points is 7/4.*

This theorem is a corollary of a theorem determining the values of the Brownian intersection exponents $\xi(j,\lambda)$ which we define in the next section. In [6, 5, 8],
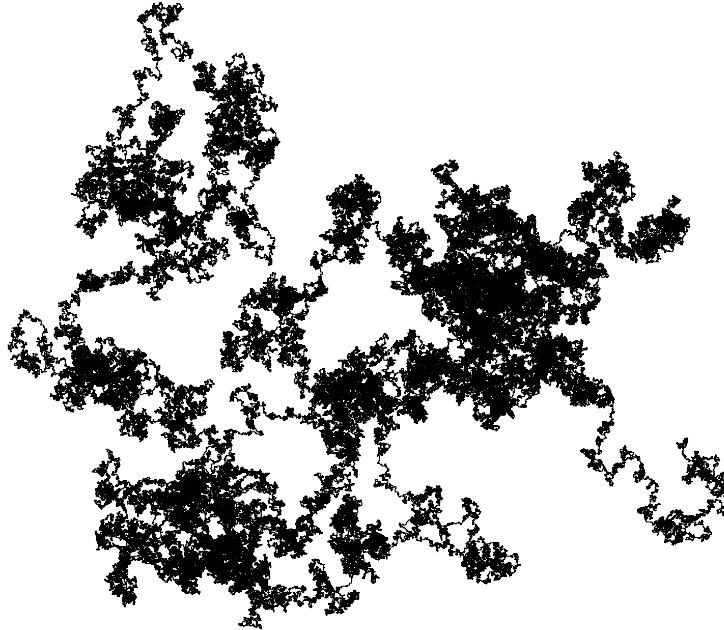
FIGURE 1. Simulation of a planar Brownian path

it had been established that the dimension of the frontier, cut points, and pioneer points are $2 - \xi(2,0), 2 - \xi(1,1)$, and $2 - \xi(1,0)$, respectively. Duplantier and Kwon [4] were the first to conjecture the values $\xi(1,1) = 5/4, \xi(1,0) = 1/4$ using ideas from conformal field theory. Duplantier has also developed another non-rigorous approach to these results based on "quantum gravity" (see e.g. [3]). For a more complete list of references and background, see e.g. [9].

2. Intersection exponents

Let $B^1, B^2, \ldots, B^{j+k}$ be independent planar Brownian motions with uniformly distributed starting points on the unit circle. Let T_R^l denote the first time at which B^l reaches the circle of radius R, and let $\omega_R^j = B^j[0, T_R^j]$. The intersection exponent $\xi(j,k)$ is defined by the relation

$$\mathbf{P}\big[(\omega_R^1 \cup \cdots \cup \omega_R^j) \cap (\omega_R^{j+1} \cup \cdots \cup \omega_R^{j+k}) = \emptyset \big] \approx R^{-\xi(j,k)}, \qquad R \to \infty,$$

where $f \approx g$ means $\lim(\log f / \log g) = 1$. Using subadditivity, it is not hard to see that there are constants $\xi(j,k)$ satisfying this relation. Let

$$Z_R = Z_R(\omega_R^1, \ldots, \omega_R^j) := \mathbf{P}[(\omega_R^1 \cup \cdots \cup \omega_R^j) \cap \omega_R^{j+1} = \emptyset \mid \omega_R^1, \ldots, \omega_R^j].$$

Then

(2.1) $$\mathbf{E}[Z_R^k] \approx R^{-\xi(j,k)}, \qquad R \to \infty.$$

In the latter formulation, there is no need to restrict to integer k; this defines $\xi(j,\lambda)$ for all $\lambda > 0$. The existence of these exponents is also very easy to

establish. The disconnection exponent $\xi(j,0)$ is defined by the relation

$$\mathbf{P}[\,Z_R > 0\,] \approx R^{-\xi(j,0)}, \qquad R \to \infty.$$

Note that $Z_R > 0$ if and only if $\omega_R^1 \cup \cdots \cup \omega_R^j$ do not disconnect the circle of radius 1 from the circle of radius R. With this definition, $\xi(j,0) = \lim_{\lambda \searrow 0} \xi(j,\lambda)$ [7], see also [13].

Another family of intersection exponents are the half-plane exponents $\widetilde{\xi}$. Let $\mathbb{H}$ denote the open upper half-plane and define $\widetilde{\xi}$ by

$$\mathbf{P}[\,(\omega_R^1 \cup \cdots \cup \omega_R^j) \cap (\omega_R^{j+1} \cup \cdots \cup \omega_R^{j+k}) = \emptyset;\ \omega_R^1 \cup \cdots \cup \omega_R^{j+k} \subset \mathbb{H}\,] \approx R^{-\widetilde{\xi}(j,k)}.$$

If $j_1, \dots, j_n$ are positive integers and $\lambda_0, \lambda_1, \dots, \lambda_n \geq 0$, there is a natural way to extend the above definitions and define the exponents

$$\xi(j_1, \lambda_1, \dots, j_n, \lambda_n), \quad \widetilde{\xi}(\lambda_0, j_1, \dots, j_n, \lambda_n).$$

In fact [14], there is a unique extension of the intersection exponents

$$\widetilde{\xi}(a_1, a_2, \dots, a_n), \quad \xi(a_1, a_2, \dots, a_n)$$

to nonnegative reals $a_1, \dots, a_n$ (in the case of ξ, at least two of the arguments must be at least 1) such that:

- the exponents are symmetric functions
- they satisfy the "cascade relations"

(2.2) $\widetilde{\xi}(a_1, \dots, a_{n+m}) = \widetilde{\xi}(a_1, a_2, \dots, a_n, \widetilde{\xi}(a_{n+1}, \dots, a_{n+m})),$

(2.3) $\xi(a_1, \dots, a_{n+m}) = \xi(a_1, a_2, \dots, a_n, \widetilde{\xi}(a_{n+1}, \dots, a_{n+m})), \qquad a_1, a_2 \geq 1.$

Note that $\xi(j, \lambda)$ for $\lambda \in [0,1)$ and positive integer j is defined directly via (2.1).

Theorem 2.1 ([11]). *For all* $a_1, a_2, \dots, a_n \geq 0$,

(2.4)
$$\widetilde{\xi}(a_1, \dots, a_n) = \frac{\left(\sqrt{24a_1 + 1} + \sqrt{24a_2 + 1} + \cdots + \sqrt{24a_n + 1} - (n-1)\right)^2 - 1}{24}.$$

Theorem 2.2 ([10, 12]). *For all* $a_1, a_2, \dots, a_n \geq 0$ *with* $a_1, a_2 \geq 1$,

(2.5) $\xi(a_1, \dots, a_n) = \dfrac{\left(\sqrt{24a_1 + 1} + \sqrt{24a_2 + 1} + \cdots + \sqrt{24a_n + 1} - n\right)^2 - 4}{48}.$

For all positive integers j *and all* $\lambda \geq 0$,

(2.6) $\xi(j, \lambda) = \dfrac{(\sqrt{24j + 1} + \sqrt{24\lambda + 1} - 2)^2 - 4}{48}.$

Note that three particular cases of the last theorem are $\xi(2,0) = 2/3, \xi(1,1) = 5/4, \xi(1,0) = 1/4$, which, as noted above, are the values from which Theorem 1.1 follows. The link between the exponents and the Hausdorff dimensions of the frontier, the sets of cut-points and the set of pioneer points loosely speaking goes as follows (here, for instance, for frontier points): A point x in the plane is in the ϵ-neighborhood of a frontier point if the Brownian motion (before time 1) reaches

the ϵ-neighborhood of x, and if the whole path $B[0,1]$ does not disconnect the disc of radius ϵ around x from infinity. This whole path can be divided in two parts: B until it reaches the circle of radius ϵ around x, and B after it reaches this circle. Both behave roughly like independent Brownian paths, and it is easy to see that the probability that x is in the ϵ neighborhood of a frontier point decays like $\epsilon^{\xi(2,0)}$ when ϵ goes to zero. This, together with some second moment estimates, implies that the Hausdorff dimension of the frontier is $2 - \xi(2,0)$. See [6, 5, 8] for details.

3. Restriction property and universality

This section reviews some of the results in [15]. Roughly speaking, a Brownian excursion in a domain is a Brownian motion starting and ending on the boundary, conditioned to stay in the domain. To be more precise, consider the unit disk D, start a Brownian motion uniformly on the circle of radius $1 - \epsilon$ and then stop the Brownian motion when it reaches the unit circle. This gives a probability measure μ_ϵ on paths. The Brownian excursion measure is the infinite measure

$$\mu = \lim_{\epsilon \searrow 0} 2\pi \epsilon^{-1} \mu_\epsilon.$$

If A_1, A_2 are closed disjoint arcs on the circle, then the measure $\mu(A_1, A_2)$ of the set of paths starting at A_1 and ending at A_2 is finite; moreover, as A_1, A_2 shrink, $\mu(A_1, A_2)$ decays like $e^{-L(A_1, A_2)}$, where $L(A_1, A_2) = L(A_1, A_2; D)$ denotes π times the extremal distance between A_1 and A_2 (in the unit disk). A particular Brownian excursion divides the unit disk into two regions, say U^+, U^-, in such a way that the unit disk is the union of U^+, U^- and the "hull" of the excursion (we choose U^+ and U^- in such a way that the starting point of the excursion, $\partial U^+ \cap \partial D$, the end-point of the excursion and $\partial U^- \cap \partial D$ are ordered clockwise on the unit circle).

It can be shown [15] that the Brownian excursion measure is invariant under Möbius transformations of the unit disk. (Here we are considering two curves to be the same if one can be obtained from the other by an increasing reparameterization.) Hence, by transporting via a conformal map, the excursion measure can be defined on any simply connected domain. There is another important property satisfied by this measure that we call the *restriction property*. Suppose A_1, A_2 are disjoint arcs of the unit circle and D' is a simply connected subset of D whose boundary includes A_1, A_2. Consider the set of excursions $X(A_1, A_2, D')$ starting at A_1, ending at A_2, and staying in D'. The excursion measure gives two numbers associated with this set of paths: $\mu(X(A_1, A_2, D'))$ and $\mu(\phi(X(A_1, A_2, D')))$, where ϕ is an arbitrary conformal homeomorphism from D' onto the unit disk. The restriction property for the excursion measure states that these two numbers are the same, for all such D', A_1, A_2.

Let $\mathcal{X}$ denote the space of all compact connected subsets X of the closed unit disk such that the intersection of X with the unit circle has two labeled connected components u^+ and u^- and the complement of X in the plane is connected. The hull of a Brownian excursion is an example for such an X.

Conformally invariant measures on $\mathcal{X}$ that satisfy the restriction property and have a well-defined crossing exponent (i.e., a number α such that the measure of the set of paths from A_1 to A_2 decays like $e^{-\alpha L(A_1, A_2)}$ as A_1, A_2 shrink to distinct points) are called *completely conformally invariant (CCI)*. The Brownian excursion measure μ gives a CCI measure with $\alpha = \alpha(\mu) = 1$. Another CCI measure can be obtained by taking j independent excursions and considering the hull formed by their union; this measure has crossing exponent $\alpha = j$. Given a CCI measure ν, the intersection exponent $\widetilde{\xi}(\lambda_1, \nu, \lambda_2)$ can be defined by saying that as A_1, A_2 shrink (i.e. as $L(A_1, A_2) \to 0$)

$$\int \exp\Big(-\lambda_1 L(A_1, A_2; U^+) - \lambda_2 L(A_1, A_2; U^-)\Big)\, d\nu(X)$$
$$\approx \exp\Big(-\widetilde{\xi}(\lambda_1, \nu, \lambda_2) L(A_1, A_2)\Big).$$

Here U^+ and U^- denote the two components of X in the unit disk that have respectively u^+ and u^- as parts of their boundary. In the case of the Brownian excursion measure μ, $\alpha = 1$ and $\widetilde{\xi}(\lambda_1, \mu, \lambda_2) = \widetilde{\xi}(\lambda_1, 1, \lambda_2)$ where the latter denotes the Brownian intersection exponent as in the previous section. In [15] it is shown that any CCI measure ν with crossing exponent α acts like the union of "α Brownian excursions" at least in the sense that for all $\lambda_1, \lambda_2 \geq 0$,

$$\widetilde{\xi}(\lambda_1, \nu, \lambda_2) = \widetilde{\xi}(\lambda_1, \alpha, \lambda_2).$$

The measure ν also induces naturally a family of measures on excursions on annuli bounded by the circles of radius r and 1. For the Brownian excursion measure, it corresponds to those Brownian excursions (in the unit disc) stopped at their hitting time of the circle of radius r in case their reach it. We can define a similar exponent $\xi(\nu, \lambda)$, and it can be shown that $\xi(\nu, \lambda) = \xi(\alpha, \lambda)$ for all $\lambda \geq 1$. In particular, if the intersection exponents for one CCI measure could be determined, then we would have the exponents for all CCI measures.

It is conjectured that the continuum limit of percolation cluster boundaries and self-avoiding walks give CCI measures (or, at least, there are CCI measures in the same universality class as these measures). For example, the well-known conjectures on self-avoiding walk exponents can be reinterpreted as the conjecture that a self-avoiding walk acts like 5/8 Brownian motion. (More precisely, two nonintersecting self-avoiding walks act like two intersecting Brownian paths. This interpretation is similar to the conjecture of Mandelbrot). The restriction property appears to be equivalent to the property of having "central charge zero" from conformal field theory.

In terms of computing the Brownian intersection exponents, the result in [15] does not appear promising. If one could compute percolation or self-avoiding walk exponents, one might be able to get the Brownian exponents; however, percolation and self-avoiding walks exponents are even harder to compute directly than Brownian intersection exponents. It is in fact not even rigorously known that they exist. What was needed was a conformally invariant process on

paths for which one can prove the restriction property and compute intersection exponents. As we shall now point out, such a process exists — the stochastic Löwner evolution at the parameter $\kappa = 6$.

4. Stochastic Löwner evolution

The stochastic Löwner evolution (SLE_κ) was first developed in [18] as a model for conformally invariant growth (κ denotes a positive real). There, two versions of SLE_κ were introduced, which we later called *radial* and *chordal* SLE. Moreover, it was shown [18] that radial SLE_2 is the scaling limit of loop-erased random walk (or Laplacian random walk), if the scaling limit exists and is conformally invariant, and it was conjectured that chordal SLE_6 is the scaling limit of percolation cluster boundaries. We now give the definition of chordal SLE, explain more precisely the conjecture relating chordal SLE_6 and percolation, and then define radial SLE.

The Löwner equation relates a curve (or an increasing family of sets) in a complex domain to a continuous real-valued function. SLE is obtained by choosing the real-valued function to be a one-dimensional Brownian motion.

Let β_t denote a standard one-dimensional Brownian motion and let $W_t = \sqrt{\kappa}\beta_t$ where $\kappa > 0$. Let $\mathbb{H}$ denote the upper half-plane, and for $z \in \mathbb{H}$ consider the differential equation

$$(4.1) \qquad \partial_t g_t(z) = \frac{2}{g_t(z) - W_t}$$

with initial condition $g_0(z) = z$. The solution is well-defined up to a (possibly infinite) time T_z at which $\lim_{t \nearrow T_z} g_t(z) - W_{T_z} = 0$. Let D_t be the set of z such that $T_z > t$. Then g_t is the conformal transformation of D_t onto $\mathbb{H}$ with

$$g_t(z) = z + \frac{2t}{z} + o(\frac{1}{z}), \qquad z \to \infty.$$

The set $K_t = \mathbb{H} \setminus D_t$ is called the SLE_κ hull. The behavior of SLE_κ depends strongly on the value of κ. One way to see this is to fix a $z \in \mathbb{H}$ (or $z \in \partial\mathbb{H}$) and let $Y_t = [g_t(z) - W_t]/\sqrt{\kappa}$. Then Y_t satisfies the stochastic differential equation

$$dY_t = \frac{2}{\kappa Y_t}\, dt - d\beta_t$$

defining the so-called Bessel process of dimension $1 + 4/\kappa$. If $\kappa < 4$, the Bessel process never hits the origin and K_t is a simple curve; if $\kappa > 4$, every point is eventually contained in the hull. If D is a simply connected domain and w, z are on the boundary, we define SLE_κ starting at w and ending at z by applying a conformal map from $\mathbb{H}$ onto D which takes 0 and ∞ to w and z, respectively. (This assumes that w and z are "nice" boundary points; otherwise, one has to consider prime ends instead.) This defines a process uniquely only up to increasing time change. This is *chordal SLE*.

Consider critical percolation in $\mathbb{H}$ obtained by taking each hexagon in a hexagonal grid of mesh ϵ to be white or black with probability $1/2$, independently.
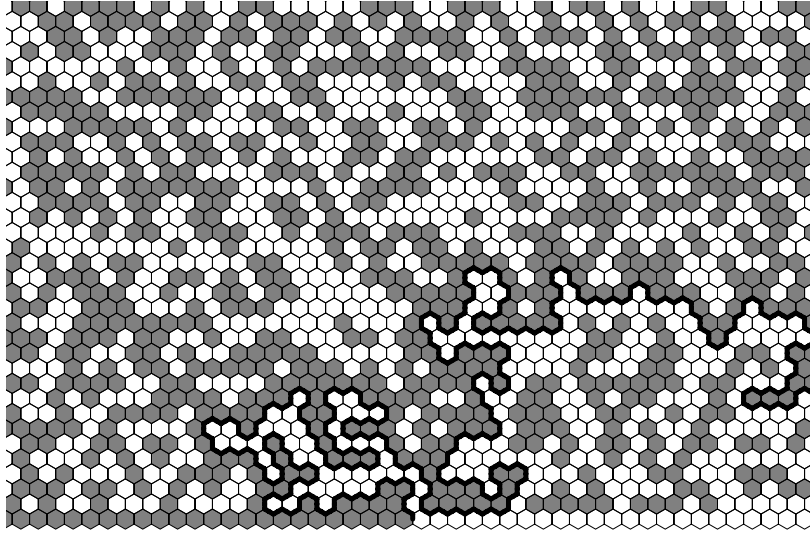
FIGURE 2. The percolation boundary path

This is in fact critical site percolation on the triangular lattice. Let H_W be the connected component of the union of the white hexagons and the positive real ray which contains the positive real ray, and let H_B be the connected component of the union of the black hexagons and the negative real ray which contains the negative real ray. Then $\partial H_B \cap \partial H_W$ is a simple path γ_ϵ connecting 0 and ∞ in the closure of $\mathbb{H}$. See figure 4.

It is conjectured that when $\epsilon \searrow 0$ the law of the path γ_ϵ tends to a conformally invariant measure. Here, conformal invariance means that if we take an analogous construction in another simply connected proper subdomain of $\mathbb{R}^2$ in place of $\mathbb{H}$, the resulting limit will be the same as the image of the limit in $\mathbb{H}$ under a conformal homeomorphism between the domains. Assuming this conjecture, it can be shown [19] that the scaling limit of γ_ϵ is given by chordal SLE_6. More precisely, the limit path, appropriately parameterized, is given by $\gamma(t) = g_t^{-1}(W_t)$, where g_t and W_t are the SLE_6 maps and driving parameter.

The restriction property for SLE_6 is an easy consequence of the conjectured conformal invariance and the independence properties of critical percolation. However, one can actually prove [9] the restriction property for SLE_6 without appealing to the conjecture. In contrast to the proof of the restriction property for Brownian excursions, the proof of the restriction property for SLE_6 is quite involved. It is not hard to see that the restriction property fails for SLE_κ when $\kappa \neq 6$.

To understand the framework of the restriction property proof, consider an infinitesimal deformation of the domain (say, a small vertical slit α) away from the starting point of an SLE, and then considers the conformal map taking the slit domain to the entire half-space, the SLE path is transformed as well.

Essentially, it is mapped to a path doing a Löwner evolution with a different "driving function" than W_t. It is necessary to see how this new driving function evolves as the slit α grows. With the aid of stochastic calculus, it can be shown that the new driving function is a martingale only when $\kappa = 6$. This argument is still rather mysterious. An explicit calculation is done and the drift term disappears exactly when $\kappa = 6$.

The definition of radial SLE is similar to chordal SLE. Again, take β_t to be a standard Brownian motion on the real line. Define $\zeta_t := \exp(i\sqrt{\kappa}\beta_t)$, which is a Brownian motion on the unit circle. Take $g_0(z) = z$ for $z \in \mathbb{U}$, and let $g_t(z)$ satisfy the Löwner differential equation

$$\partial_t g_t(z) = g_t(z)\frac{g_t(z) + \zeta_t}{g_t(z) - \zeta_t},$$

up to the first time T_z where $g_t(z)$ hits ζ_t. Let $D_t := \{z \in \mathbb{U} : T_z > t\}$ and $K_t := \mathbb{U} \setminus D_t$. This defines radial SLE_κ from 1 to 0 in $\mathbb{U}$. Note that the main difference from chordal SLE is that g_t is normalized at an interior point 0 and the set K_t "grows towards" this interior point.

5. Exponents for SLE_6

Intersection exponents for SLE_κ can be computed for any κ. We will consider only the case $\kappa = 6$ here. We will briefly describe how to compute the exponents $\widetilde{\xi}(\lambda_1, SLE_6, \lambda_2)$ and $\xi(SLE_6, \lambda)$, but we first focus on another family of exponents $\widehat{\xi}$ that can be described as follows. Start a chordal SLE_6 at the top left corner $i\pi$ of the rectangle $R := [0, L] \times [0, \pi]$ going towards the lower right corner L. Let K be the hull of the SLE_6 at the first time that it hits the right edge $\{L\} \times [0, \pi]$. Let $\mathcal{L}_+$ be π times the extremal distance between the vertical edges of R in $R \setminus K$, where we take $\mathcal{L}_+ = \infty$ if there is no path in $R \setminus K$ connecting the vertical edges of R (that is, if K intersects the lower edge). Then the intersection exponent $\widehat{\xi}(SLE_6, \lambda)$ can be defined by the relation

$$\mathbf{E}[e^{-\lambda \mathcal{L}_+}] \approx e^{-\hat{\xi}(SLE_6, \lambda)L}, \quad L \to \infty.$$

The exponent $\widehat{\xi}(SLE_6, \lambda)$ will turn out to be closely related to the Brownian exponent $\widetilde{\xi}(1/3, \lambda)$ and therefore also to the Brownian exponent $\widetilde{\xi}(1, \lambda)$. However, observe that the SLE is here allowed to touch one horizontal edge of the rectangle. For this reason, we use the notation $\widehat{\xi}$, instead of $\widetilde{\xi}$.

To calculate $\widehat{\xi}(SLE_6, \lambda)$, we map the rectangle to the upper half-plane and consider the exponent there. Let ϕ be the conformal map taking R onto $\mathbb{H}$ which satisfies $\phi(L) = \infty$, $\phi(0) = 1$, $\phi(L + i\pi) = 0$, and set $x := \phi(i\pi)$. Start a chordal SLE_6 in $\mathbb{H}$ from x to ∞. Let $K = K_T$ be the hull of the SLE at the first time, $T = T_R$, such that K_t hits $(-\infty, 0] \cup [1, \infty)$ (one can verify that $T_R < \infty$). Let $\mathcal{L} = \mathcal{L}_R$ be π times the extremal distance between $(-\infty, 0)$ and $(0, 1]$ in $\mathbb{H} \setminus K$. Then,

$$\mathbf{E}\left[e^{-\lambda \mathcal{L}}\right] \approx e^{-L\hat{\xi}(SLE_6, \lambda)}, \quad L \to \infty.$$

Let

$$f_t(z) = \frac{g_t(z) - g_t(0)}{g_t(1) - g_t(0)},$$

i.e., g_t normalized to fix 0 and 1. It is not hard to verify that $e^{\mathcal{L}} \approx 1 - f_T(\sup(K_T \cap [0,1]))$. In fact, we show in [9] that

(5.1)
$$\mathbf{E}\left[e^{-\lambda \mathcal{L}} \right] \approx \mathbf{E}\left[\left((1-x)f'_T(1) \right)^{\lambda} \right].$$

Define

$$w_t := \frac{W_t - g_t(0)}{g_t(1) - g_t(0)}, \quad u_t = f'_t(1).$$

The couple (w_t, u_t) is solution of a stochastic differential equation, and it is easy to check that T is in fact the first time at which w_t hits 0 or 1. Define $F(w, u) = \mathbf{E}[u_T^{\lambda} | u_0 = u, w_0 = w]$. The right hand side of (5.1) is then calculated via writing a PDE for F. The PDE becomes an ODE when symmetries are accounted for. The solution of the ODE turns out to be a hypergeometric function (the hypergeometric function predicted by Cardy [2] for the crossing probability of a rectangle for critical percolation is recovered as the special case $\lambda = 0$). This leads to

$$\widehat{\xi}(SLE_6, \lambda) = \frac{6\lambda + 1 + \sqrt{24\lambda + 1}}{6}.$$

In particular, $\widetilde{\xi}(SLE_6, 0) = 1/3$ which can very loosely be interpreted as "SLE_6 allowed to bounce off of one side equals a third of a Brownian motion." This leads directly, using the universality arguments developed in [15], to the fact that $\widetilde{\xi}(1/3, \lambda) = \widehat{\xi}(SLE_6, \lambda)$ for all $\lambda > 0$. In particular, $\widetilde{\xi}(1/3, 1/3) = 1$, and the cascade relation then gives $\widetilde{\xi}(1, \lambda) = \widehat{\xi}(SLE_6, \widehat{\xi}(SLE_6, \lambda))$ and the value of these exponents [9].

By considering the regions above and below K in the rectangle R, and disallowing K to touch the horizontal sides, we can also define the two-sided exponents $\widetilde{\xi}(\lambda_1, SLE_6, \lambda_2)$. As above, the calculation of this exponent was translated to a question about a PDE. However, in this case the PDE did not become an ODE, and was not solved explicitly. But some eigenfunctions for the PDE were found, which proved sufficient. The leading eigenvalue was shown to be equal to the sought exponent [11]. The result is

$$\widetilde{\xi}(\lambda_1, SLE_6, \lambda_2) = \frac{\left(\sqrt{24\lambda_1 + 1} + 3 + \sqrt{24\lambda_2 + 1} \right)^2 - 1}{24}.$$

The universality ideas show that $\widetilde{\xi}(\lambda_1, 1, \lambda_2) = \widetilde{\xi}(\lambda_1, SLE_6, \lambda_2)$, so that we get the value of $\widetilde{\xi}(\lambda_1, 1, \lambda_2)$ for all $\lambda_1, \lambda_2 \geq 0$ and therefore, using the cascade relations (2.2), (2.4) follows.

The exponent $\xi(SLE_6, \lambda)$ is defined by considering the hull K of radial SLE_6 started on the unit circle stopped at the first time it reaches the circle of radius

r (when $r \to 0$). Let $\mathcal{L}$ be π times the extremal distance between the two circles in the annulus with K removed. Then, when $r \to 0$,

$$\mathbf{E}[e^{-\lambda\mathcal{L}}] \approx r^{\xi(SLE_6,\lambda)}.$$

In a similar way the evaluation of this exponent can be reduced to analyzing $\mathbf{E}[\,|g_t'(z)|^\lambda\,]$. The calculation yields

$$\xi(SLE_6, \lambda) = \frac{4\lambda + 1 + \sqrt{24\lambda + 1}}{8},$$

at least for $\lambda \geq 1$. From this and the ideas in [15], one can identify $\xi(1, \lambda)$ with $\xi(SLE_6, \lambda)$ for all $\lambda \geq 1$. From this and the cascade relations (2.3), (2.6) for all $\lambda \geq 1$ and (2.4) follow.

6. Analyticity

Unfortunately, the universality argument from [15] does not show that $\xi(SLE_6, \lambda) = \xi(1, \lambda)$ when $\lambda < 1$. To derive the values of $\xi(j, \lambda)$ for all $\lambda > 0$, we show that the function $\lambda \mapsto \xi(j, \lambda)$ is real analytic for $\lambda > 0$ [12]. Then, the formula for $\xi(j, \lambda)$ where $\lambda < 1$ follows by analytic continuation.

To prove analyticity, we interpret $\xi(j, \lambda)$ as an eigenvalue for an analytic function $\lambda \mapsto T_\lambda$, taking values in the space of bounded operators on a Banach space. For notational ease, let use assume $j = 1$. Consider the set of continuous functions $\gamma : [0, t] \to \mathbb{C}$ with $\gamma(0) = 0, |\gamma(t)| = 1$ and $0 < |\gamma(s)| < 1$ for $0 < s < t$. Let $\mathcal{C}$ be the subset of these functions with the property that there is a connected component D of $\mathbb{U} \setminus \gamma$ whose boundary includes both the origin and a subarc of the unit circle (here we write γ for $\gamma[0, t]$). Consider the following transformation on γ. If $\gamma \in \mathcal{C}$, start a Brownian motion at the endpoint of γ and let it run until it hits the circle of radius e, producing a curve β. Let γ_1 be $\gamma \cup \beta$ rescaled by $1/e$, so that its endpoint is once again on the unit circle. Let $\Phi(\gamma_1)$ be the difference in the π-extremal distance between 0 and the unit circle in $\mathbb{U} \setminus \gamma_1$ and the π-extremal distance between 0 and the circle $(1/e)\partial\mathbb{U}$ in $\mathbb{U} \setminus (1/e)\gamma$. (These distance are both infinite but the difference can easily be defined.) If f is a bounded function on $\mathcal{C}$ and $z \in \mathbb{C}$ we define $T_z f$ by

$$T_z f(\gamma) = \mathbf{E}[e^{-z\Phi(\gamma_1)} f(\gamma_1)].$$

This is well-defined if $\mathrm{Re}(z) > 0$, and if $\lambda > 0$, then $e^{-\xi(1,\lambda)}$ is an eigenvalue for T_λ. In fact, by appropriate choice of a Banach space norm, it can be shown that for any $\lambda > 0$, there is a complex neighborhood about λ such that $z \to T_z$ is holomorphic and such that $e^{-\xi(1,\lambda)}$ is an isolated point in the spectrum of T_λ. The choice of the norm is similar to that used in [17] to show that the free energy of a one-dimensional Ising model with exponentially decaying interaction is analytic in the temperature. A coupling argument is used in establishing exponential convergence that is necessary to show that the eigenvalue is isolated. Since the eigenvalue is isolated, it follows that it depends analytically on λ.

Note from editors. This paper originally appeared in Mathematical Research Letters, Volume 8, Number 1-2, pages 13-23 (2001). Because of a printing error, several lines of the article were accidentally omitted. We regret any inconvenience this has caused, and are reprinting this article in its entirety in this issue.

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