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NON-COMMUTATIVE SYMPLECTIC GEOMETRY, QUIVER VARIETIES, AND OPERADS

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Abstract. Quiver varieties have recently appeared in various different areas of Mathematics such as representation theory of Kac-Moody algebras and quantum groups, instantons on 4-manifolds, and resolutions of Kleinian singularities. In this paper, we show that many important affine quiver varieties, e.g., the Calogero-Moser space, can be imbedded as coadjoint orbits in the dual of an appropriate infinite dimensional Lie algebra. In particular, there is an infinitesimally transitive action of the Lie algebra in question on the quiver variety. Our construction is based on an extension of Kontsevich's formalism of 'non-commutative symplectic geometry'. We show that this formalism acquires its most adequate and natural formulation in the much more general framework of $\mathcal{P}$-geometry, a 'non-commutative geometry' for an algebra over an arbitrary cyclic Koszul operad.

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1. Introduction

We first remind the definition of quiver varieties. Let Q be a quiver, that is a finite oriented graph with vertex set I. Let $V = \{V_i\}_{i\in I}$ be a collection of finite dimensional $\mathbb{C}$-vector spaces. By a representation of Q in V we mean an assignment of a linear map: $V_i \to V_j$, for any pair $i, j \in I$ and each oriented edge of Q with tail i and head j. Let $\mathcal{R}(Q, V)$ denote the set of all representations of Q in V, which is a $\mathbb{C}$-vector space. The group $\sqcap_{i\in I} \mathtt{GL}_{\mathbb{C}}(V_i)$ acts naturally on $\mathcal{R}(Q, V)$, by conjugation. This action clearly factors through $G(V) := (\sqcap_i \mathtt{GL}(V_i))\,/\mathbb{C}^*$, the quotient by the group $\mathbb{C}^*$ imbedded diagonally, as scalar matrices, into each

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of groups $\mathtt{GL}(V_i)$. Let $\mathfrak{g}(V) = (\oplus_i \mathfrak{gl}(V_i))/\mathbb{C}$ denote the Lie algebra of the group $G(V)$.

Let $\overline{Q}$ be the *double* of Q, the quiver obtained by adding a reverse arrow a^*, for every (oriented) arrow $a \in Q$. For any $V = \{V_i\}_{i\in I}$, the vector space $\mathcal{R}(\overline{Q}, V)$ may be identified naturally with $T^*\mathcal{R}(Q,V) =$ *cotangent bundle on* $\mathcal{R}(Q,V)$. Hence, $\mathcal{R}(\overline{Q}, V)$ has a canonical symplectic structure. Furthermore, the $G(V)$-action on $\mathcal{R}(\overline{Q}, V)$ is Hamiltonian, and the corresponding moment map $\mu : \mathcal{R}(\overline{Q}, V) \longrightarrow \mathfrak{g}(V)^*$ is given by the following formula:

$$(1.1)\quad \varrho \mapsto \mu(\varrho) = \Big\{ \{\mu(\varrho)_i\}_{i\in I} \in \bigoplus_i \mathfrak{gl}(V_i) \;\Big|\; \mu(\varrho)_i = \sum_{\left\{ {a\in Q \atop \mathrm{head}(a)=i} \right\}} \varrho(a)\cdot\varrho(a^*) \;-\; \sum_{\left\{ {a\in Q \atop \mathrm{tail}(a)=i} \right\}} \varrho(a^*)\cdot\varrho(a) \Big\}.$$

Here and below, we identify $\mathfrak{g}(V)^*$ with a subspace in $\oplus_i \mathfrak{gl}(V_i)$ by means of the trace pairing: $x, y \mapsto \sum_{i\in I} \mathtt{tr}(x_i \cdot y_i)$. Specifically, we have:

$$\mathfrak{g}(V)^* \;\simeq\; \mathfrak{sg}(V) \;:=\; \{\, x = (x_i)_{i\in I} \;\in \oplus_i \mathfrak{gl}(V_i) \;\;\big|\;\; \textstyle\sum_{i\in I} \mathtt{tr}(x_i) = 0 \,\}\,.$$

Example. Let Q be the quiver consisting of a single vertex and a single edge-loop at this vertex. Thus $\overline{Q}$ is the quiver with two edge-loops at the same vertex. Clearly, giving a representation of $\overline{Q}$ in the vector space $V = \mathbb{C}^n$ amounts to giving an arbitrary pair of $n \times n$-matrices. Therefore, we have: $\mathcal{R}(\overline{Q}, \mathbb{C}^n) = \mathfrak{gl}_n \oplus \mathfrak{gl}_n$, and hence: $G(V) = \mathtt{PGL}_n$. The moment map (1.1) reduces to the map $\mu : \mathfrak{gl}_n \oplus \mathfrak{gl}_n \longrightarrow \mathfrak{g}(V)^* = \mathfrak{sl}_n$, given by the formula: $(x,y) \mapsto [x,y]$.

Next, fix $\mathsf{O} \subset \mathfrak{sg}(V)$, a closed $\mathtt{Ad}\,G(V)$-orbit, and assume that the group $G(V)$ acts freely on the subvariety $\mu^{-1}(\mathsf{O}) \subset \mathcal{R}(\overline{Q}, V)$. Then, the orbit space $\mathcal{R}_{\mathsf{O}}(\overline{Q}, V) := \mu^{-1}(\mathsf{O})/G(V)$ is an affine variety, to be called an *affine quiver variety*. Thus, by definition: $\mathcal{R}_{\mathsf{O}}(\overline{Q}, V) := \mathtt{Spec}\big(\mathbb{C}[\mathcal{R}(\overline{Q}, V)]^{G(V)}/\mathcal{J}^{G(V)}\big)$, where $\mathcal{J} \subset \mathbb{C}[\mathcal{R}(\overline{Q}, V)]$ stands for the defining ideal of the subvariety $\mu^{-1}(\mathsf{O})$, and we have used that $\mathbb{C}[\mathcal{R}(\overline{Q}, V)]^{G(V)}/\mathcal{J}^{G(V)} = (\mathbb{C}[\mathcal{R}(\overline{Q}, V)]/\mathcal{J})^{G(V)}$, due to reductivity of $G(V)$. If $\mu^{-1}(\mathsf{O})$ is smooth then $\mathcal{R}_{\mathsf{O}}(\overline{Q}, V)$ is also smooth, and the symplectic structure on $\mathcal{R}(\overline{Q}, V)$ induces, via the symplectic reduction construction, see [GS], a canonical symplectic structure on $\mathcal{R}_{\mathsf{O}}(\overline{Q}, V)$.

One of the main results of this paper is

Theorem 1.2. *In the above setting, the symplectic variety $\mathcal{R}_{\mathsf{O}}(\overline{Q}, V)$ can be imbedded as a coadjoint orbit in the dual of $\mathfrak{L}(Q)$, an infinite dimensional Lie algebra canonically attached to the quiver Q.*

It is implicit in the theorem that the symplectic structure on $\mathcal{R}_{\mathsf{O}}(\overline{Q}, V)$ goes, under the imbedding, into the canonical Kirillov-Kostant symplectic structure on the coadjoint orbit. Note also that the Lie algebra $\mathfrak{L}(Q)$ does not depend on the representation space V.

Remark. A choice of Hermitian metric on V makes $\mathcal{R}(\overline{Q}, V)$ a flat hyper-Kähler space. An equivalence: *holomorphic symplectic reduction* $\Leftrightarrow$ *hyper-Kähler reduction*, see [Hi], gives, for many orbits O, a hyper-Kähler structure on the quiver variety $\mathcal{R}_{\mathsf{O}}(\overline{Q}, V)$. Recall further that by a well-known result of Kronheimer [Kr], any coadjoint orbit in a complex reductive Lie algebra has a hyper-Kähler structure. Based on this analogy, N. Hitchin asked if the Calogero-Moser space (a special case of quiver variety, see below) is a coadjoint orbit of some infinite dimensional Lie algebra. Hitchin's question has been motivated by the recent work of Berest-Wilson [BW], who constructed a transitive action of $Aut(A_1)$, the automorphism group of the Weyl algebra, on the Calogero-Moser space. Theorem 1.2 gives a positive answer to Hitchin's question and sheds some new light on the Berest-Wilson construction.

Strategy of the proof of Theorem 1.2. The symplectic structure on $\mathcal{R}_{\mathsf{O}}(\overline{Q}, V)$ makes the coordinate ring $\mathbb{C}\big[\mathcal{R}_{\mathsf{O}}(\overline{Q}, V)\big]$ an infinite dimensional Lie algebra with respect to the Poisson bracket. We will construct a sequence of Lie algebra morphisms:

$$\mathfrak{L}(Q) \xrightarrow{\psi} \mathbb{C}\big[\mathcal{R}(\overline{Q}, V)\big]^{G(V)} \xrightarrow{\mathtt{pr}} \mathbb{C}\big[\mathcal{R}(\overline{Q}, V)\big]^{G(V)}/\mathcal{J}^{G(V)} = \mathbb{C}\big[\mathcal{R}_{\mathsf{O}}(\overline{Q}, V)\big], \tag{1.3}$$

where $\mathbb{C}\big[\mathcal{R}_{\mathsf{O}}(\overline{Q}, V)\big]$, the coordinate ring, is viewed as a Lie algebra with respect to the Poisson bracket arising from the symplectic structure on $\mathcal{R}(\overline{Q}, V)$, and the map `pr` stands for the canonical projection.

Now, for any affine symplectic manifold X and any point $x \in X$, evaluation at x gives a linear function on $\mathbb{C}[X]$, whence induces an evaluation map: $X \xrightarrow{\mathtt{ev}} \mathbb{C}[X]^\star$. Note that the vector space $\mathbb{C}[X]^\star$ is an (infinite dimensional) Poisson manifold with Kirillov-Kostant bracket. It is immediate from the definitions that the map: $X \to \mathbb{C}[X]^\star$ is a morphism of Poisson varieties, i.e., the induced map on polynomial functions is a morphism of Poisson algebras. Since X is smooth and affine, regular functions on X separate points of X and, moreover, the differentials of regular functions span tangent spaces at each point of X. This implies that the evaluation map is injective, and that the infinitesimal Hamiltonian action of the Lie algebra $\mathbb{C}[X]$ (with the Poisson bracket) on the image of the evaluation map is infinitesimally transitive. Thus, the evaluation imbedding makes X a coadjoint orbit in $\mathbb{C}[X]^\star$.

Applying the considerations above to the symplectic manifold $X = \mathcal{R}_{\mathsf{O}}(\overline{Q}, V)$, and dualizing the maps in (1.3), one gets a sequence of Poisson morphisms:

$$\mathcal{R}_{\mathsf{O}}(\overline{Q}, V) \overset{\mathtt{ev}}{\hookrightarrow} \mathbb{C}\big[\mathcal{R}_{\mathsf{O}}(\overline{Q}, V)\big]^\star \xrightarrow{\mathtt{pr}^\star} \Big(\mathbb{C}\big[\mathcal{R}(\overline{Q}, V)\big]^{G(V)}\Big)^\star \xrightarrow{\psi^\star} \mathfrak{L}(Q)^\star .$$

It will be shown later that the composite map above is injective, and the image of $\mathcal{R}_{\mathsf{O}}(\overline{Q}, V)$ is a coadjoint orbit in $\mathfrak{L}(Q)^\star$. Thus, a key step in proving Theorem 1.2 is the construction of Lie algebra map ψ in (1.3).

We now illustrate our construction of ψ in a very special case, where Q is the quiver consisting of a single vertex and a single edge-loop (see Example above). To define the Lie algebra $\mathfrak{L}(Q)$, it is convenient to introduce an auxiliary 2-dimensional symplectic vector space (E,ω) with basis x, y (corresponding to the two loops in $\overline{Q}$) such that $\omega(x,y)=1$. For any $p,q\geq 0$, we define a $\mathbb{C}$-bilinear map $\{\,,\,\}_\omega : E^{\otimes p}\times E^{\otimes q} \longrightarrow E^{\otimes(p+q-2)}$ by the formula:

$$(1.4)\quad \{u_1\otimes u_2\otimes\ldots\otimes u_p\ ,\ v_1\otimes v_2\otimes\ldots\otimes v_q\}_\omega =$$
$$\sum_{i=1}^{p}\sum_{j=1}^{q}\omega(u_i,v_j)\cdot u_{i+1}\otimes\ldots\otimes u_p\otimes u_1\otimes\ldots\otimes u_{i-1}\otimes v_{j+1}\otimes\ldots\otimes v_q\otimes v_1\otimes\ldots\otimes v_{j-1},$$

where $u_1,\ldots,u_p, v_1,\ldots,v_q\in E$. Assembled together, these maps give a bilinear pairing $\{-,-\}_\omega : TE\times TE \longrightarrow TE$, where $TE=\bigoplus_{i\geq 0} E^{\otimes i}$ is the tensor algebra of E. Let $[TE,TE]\subset TE$ denote the $\mathbb{C}$-linear span of the set $\{a\cdot b-b\cdot a\}_{a,b\in TE}$.

Proposition 1.5. *The pairing* $\{,\}_\omega$ *gives rise to a well-defined Lie algebra structure on the vector space* $\mathfrak{L}(Q):=TE/[TE,TE]$.

Remark. One of the goals of the paper is to give an interpretation of the Lie algebra $(TE/[TE,TE]\,,\{,\}_\omega)$ as a sort of Poisson algebra associated to an appropriate 'non-commutative' symplectic variety.

To complete our construction we must define a Lie algebra morphism $\psi : \mathfrak{L}(Q)=TE/[TE,TE]\to\mathbb{C}[\mathcal{R}(\overline{Q},V)]^{G(V)}$, see (1.3). As we know, for $V=\mathbb{C}^n$ one has: $\mathcal{R}(\overline{Q},V)\simeq\mathfrak{gl}_n(\mathbb{C})\oplus\mathfrak{gl}_n(\mathbb{C})$, and $G(V)\simeq\mathtt{PGL}_n$. It is convenient to identify the tensor algebra TE with the free associative algebra generated by x,y. We define a $\mathbb{C}$-linear map $\mathtt{tr} : TE\to\mathbb{C}[\mathfrak{gl}_n\oplus\mathfrak{gl}_n]$ by assigning to any non-commutative monomial $f=x^{k_1}\cdot y^{l_1}\cdot x^{k_2}\cdot\ldots\in TE$ a polynomial function $\mathtt{tr}f\in\mathbb{C}[\mathfrak{gl}_n\oplus\mathfrak{gl}_n]$, given by the formula:

$$(1.6)\qquad \mathtt{tr}f : (X,Y)\mapsto\mathtt{Trace}(X^{k_1}\cdot Y^{l_1}\cdot X^{k_2}\cdot\ldots)\quad,\quad X,Y\in\mathfrak{g}=\mathfrak{gl}_n\,.$$

It is clear that $\mathtt{tr}f\in\mathbb{C}[\mathfrak{gl}_n\oplus\mathfrak{gl}_n]^{\mathtt{GL}_n}$, and that $\mathtt{tr}f=0$ if $f\in[TE,TE]$, by symmetry of the trace. Thus, the assignment: $f\mapsto\mathtt{tr}f$ gives a well-defined linear map $\psi : \mathfrak{L}(Q)=TE/[TE,TE] \longrightarrow \mathbb{C}[\mathfrak{gl}_n\oplus\mathfrak{gl}_n]^{\mathtt{GL}_n}$. It turns out that this map is a Lie algebra morphism. This completes our construction, and the outline of the proof of Theorem 1.2. $\square$

Example. Calogero-Moser space. Let Q be the quiver consisting of a single vertex and a single edge-loop at this vertex, and assume $\dim V=n$, as above. Then, $\mathfrak{g}(V)=\mathfrak{pgl}_n$. We will be concerned with the coadjoint orbit $\mathsf{O}\subset\mathfrak{g}(V)^*=\mathfrak{sl}_n$, formed by all $n\times n$-matrices of the form: $s\text{-}\mathtt{Id}$, where s is a rank 1 semisimple matrix such that *Trace*$(s)=$ *Trace*$(\mathtt{Id})=n$. Thus, O is a closed $G(V)$-conjugacy class in $\mathfrak{sl}_n$, and it has been shown in [W] that

$$\mu^{-1}(\mathsf{O}) = \big\{(X,Y)\in\mathfrak{sl}_n\times\mathfrak{sl}_n\ \big|\ [X,Y]+\mathtt{Id}\text{ is a rank one semisimple matrix}\big\}\,,$$

is a smooth connected algebraic variety and the $\mathtt{Ad}G(V)$-diagonal action on $\mu^{-1}(\mathsf{O})$ is free. The reduced space $\mathbb{M} := \mu^{-1}(\mathsf{O})/G(V)$ is, according to [KKS] (see also [W]), nothing but the phase space of the (rational) Calogero-Moser integrable system. This is a smooth affine algebraic symplectic manifold. Thus, Theorem 1.2 makes $\mathbb{M}$ a coadjoint orbit in $(A/[A,A])^*$, where $A = TE = \mathbb{C}\langle x,y\rangle$. This very special case was the starting point of our analysis.

An earlier version of this paper has been greatly motivated by [BW], whose question led me to the development of non-commutative geometry in the special case of the Calogero-Moser space. The results presented in §3 below form a natural generalization of the Calogero-Moser case. This generalization has been found simultaneously and independently by L. Le Bruyn [LB1] and the author.

2. Non-commutative symplectic geometry

Throughout this paper we will be working over a ground field $\Bbbk$ of characteristic zero, and write $\otimes = \otimes_{\Bbbk}$. We fix a commutative unital $\Bbbk$-algebra B, and for any B-bimodule M, write $T^j_B M = M \otimes_{_B} \ldots \otimes_{_B} M$ (j factors M), which is a B-bimodule again.

Let A be a unital associative $\Bbbk$-algebra containing the commutative algebra B as a subalgebra. Recall that the free differential envelope of A over B is a graded vector space $\Omega^\bullet_{_B}A = \bigoplus_{j\geq 0} \Omega^j_{_B}A$, where $\Omega^j_{_B}A = A \bigotimes_B T^j_B(A/B)$ is the B-bimodule formed by linear combinations of expressions $a_0 \cdot da_1 \ldots da_j \in A \otimes T^j_B(A/B)$. Moreover, it is known, cf. [L], that there is a B-bimodule isomorphism: $\Omega^\bullet_{_B}A \simeq \bigoplus_{j\geq 0} T^j_B(\Omega^1_{_B}A)$, and there is a B-bimodule super-differential $d : \Omega^\bullet_{_B}A \to \Omega^{\bullet+1}_{_B}A$, making $\Omega^\bullet_{_B}A$ an associative differential graded algebra.

Given $\alpha \in \Omega^i_{_B}A$, $\beta \in \Omega^j_{_B}A$, we put: $[\alpha,\beta] = \alpha\cdot\beta - (-1)^{ij}\beta\cdot\alpha$, and write $[\Omega^\bullet_{_B}A, \Omega^\bullet_{_B}A]$ for the B-linear span of all such super-commutators. Following Karoubi [Ka], see also [L, §2.6], define the relative non-commutative de Rham complex of the pair (A,B) as the differential graded vector space:

$$\mathsf{DR}^\bullet_{_B}A = \Omega^\bullet_{_B}A/[\Omega^\bullet_{_B}A, \Omega^\bullet_{_B}A] \quad , \quad \mathsf{DR}^\bullet_{_B}A = \bigoplus_{j\geq 0} \mathsf{DR}^j_{_B}A\,,$$

where the differential and the grading are induced from those on $\Omega^\bullet_{_B}A$. Abusing the notation we will write: $a_0 \cdot da_1 \ldots da_j \in \mathsf{DR}^j_{_B}A$, meaning the corresponding class modulo commutators. We have: $\mathsf{DR}^0_{_B}A = A/[A,A]$, and $H^0(\mathsf{DR}^\bullet_{_B}A) = \ker(\mathsf{DR}^0_{_B}A \to \mathsf{DR}^1_{_B}A) = B$.

Let $\mathtt{Der}_{_B}A$ denote the Lie algebra of all B-linear derivations of A. Given $\theta \in \mathtt{Der}_{_B}A$ one introduces, following [K2], a Lie operator $L_\theta : \Omega^\bullet_{_B}A \to \Omega^\bullet_{_B}A$, resp. a contraction operator $i_\theta : \Omega^\bullet_{_B}A \to \Omega^{\bullet-1}_{_B}A$, as a derivation, resp. a super-derivation, of the associative algebra $\Omega^\bullet_{_B}A$ defined on generators by the formulas:

$$L_\theta(a_0) = \theta(a_0) \;,\; L_\theta(da) = d(\theta(a)) \quad \text{and} \quad i_\theta(a_0) = 0 \;,\; i_\theta(da) = \theta(a) \quad , \quad \forall a_0, a \in A.$$

It is straightforward to verify that the induced operators on $\mathsf{DR}^\bullet_B A$, satisfy the following standard commutation relations:

$$L_\theta = i_\theta\circ d + d\circ i_\theta \quad , \quad [L_\theta, i_\gamma] = i_{[\theta,\gamma]} \quad , \quad [L_\theta, L_\gamma] = L_{[\theta,\gamma]} \quad , \quad i_\theta\circ i_\gamma = -i_\gamma\circ i_\theta, \tag{2.1}$$

where all the commutation relations but the last one hold already in $\Omega^\bullet_B A$.

Fix $\omega \in \mathsf{DR}^2_B A$, and set $\mathtt{Der}_B(A,\omega) = \{\theta \in \mathtt{Der}_B A \mid L_\theta\omega = 0\}$. Clearly, $\mathtt{Der}_B(A,\omega)$ is a Lie subalgebra in $\mathtt{Der}_B A$. The assignment: $\theta \mapsto i_\theta\omega$ gives a linear map $i : \mathtt{Der}_B A \to \mathsf{DR}^1_B A$. The 2-form $\omega \in \mathsf{DR}^2_B A$ is called *non-degenerate* provided the map i is bijective.

Lemma 2.2. *Let* $\omega \in \mathsf{DR}^2_B A$ *be a non-degenerate 2-form such that* $d\omega = 0$ *in* $\mathsf{DR}^3_B A$. *Then the map*: $\theta \mapsto i_\theta\omega$ *induces a bijection* i: $\mathtt{Der}_B(A,\omega) \xrightarrow{\sim} (\mathsf{DR}^1_B A)_{\text{closed}}$, *that is*: $\theta \in \mathtt{Der}_B(A,\omega) \iff d(i_\theta\omega) = 0$ *in* $\mathsf{DR}^2_B A$.

Proof. Since, $d\omega = 0$, we have: $L_\theta\omega = i_\theta d\omega + di_\theta\omega = di_\theta\omega$. Hence, $\theta \in \mathtt{Der}_B(A,\omega) \iff 0 = L_\theta\omega = d(i_\theta\omega)$. ☐

By Lemma 2.2, one may invert the isomorphism i to obtain a linear bijection $i^{-1} : (\mathsf{DR}^1_B A)_{\text{closed}} \xrightarrow{\sim} \mathtt{Der}_B(A,\omega)$. Let: $f \mapsto \theta_f$ denote the map given by the composition:

$$A/[A,A] = \mathsf{DR}^0_B A \xrightarrow{d} (\mathsf{DR}^1_B A)_{\text{exact}} \hookrightarrow (\mathsf{DR}^1_B A)_{\text{closed}} \xrightarrow{i^{-1}} \mathtt{Der}_B(A,\omega). \tag{2.3}$$

Using the map: $f \mapsto \theta_f$, we define a Poisson bracket on $A/[A,A]$ by any of the following equivalent expressions:

$$\{f,g\}_\omega := i_{\theta_f}(i_{\theta_g}\omega) = i_{\theta_f}(dg) = -i_{\theta_g}(df) = L_{\theta_f}g = -L_{\theta_g}f. \tag{2.4}$$

Here, in the first expression for $\{f,g\}_\omega$ we have used the composite map: $i_{\theta_f}\circ i_{\theta_g}$: $\mathsf{DR}^2_B A \to \mathsf{DR}^1_B A \to \mathsf{DR}^0_B A$. Other equalities, e.g.: $i_{\theta_f}(i_{\theta_g}\omega) = L_{\theta_f}g$, follow from the equation $i_{\theta_g}\omega = dg$ (which is the definition of θ_g), the obvious identity: $i_{\theta_f}(dg) = L_{\theta_f}g$, and the last equation in (2.1).

Theorem 2.5. (i) *The bracket* (2.4) *makes* $A/[A,A]$ *into a Lie algebra.*

(ii) *The map:* $f \mapsto \theta_f$ *gives a Lie algebra homomorphism:* $A/[A,A] \longrightarrow \mathtt{Der}_B(A,\omega)$.

Proof. We prove (ii) first. To this end, observe that for any $\theta,\gamma \in \mathtt{Der}_B A$, using the first two identities in (2.1) we get:

$$i_{[\theta,\gamma]} = L_\theta\circ i_\gamma - i_\gamma\circ L_\theta = d\circ i_\theta\circ i_\gamma + i_\theta\circ d\circ i_\gamma - i_\gamma\circ d\circ i_\theta - i_\gamma\circ i_\theta\circ d. \tag{2.6}$$

Now, take $\theta = \theta_f$ and $\gamma = \theta_g$, for some $f, g \in A/[A,A]$. Then, $i_\theta\omega = df$, and $i_\gamma\omega = dg$. Applying both sides of (2.6) to ω and using that $d\omega = 0$ and $d^2 f = 0 = d^2 g$, we find

$$i_{[\theta_f,\theta_g]}\omega = d{\circ}i_{\theta_f}(dg) + i_{\theta_f}{\circ}d(dg) - i_{\theta_g}{\circ}d(df) = d{\circ}i_{\theta_f}(dg) + 0 - 0 = d\{f,g\}_\omega .$$

The latter equation means, by definition, that $\theta_{\{f,g\}_\omega} = [\theta_f, \theta_g]$, and part (ii) follows.

We now prove (i). Skew-symmetry of the bracket $\{-,-\}_\omega$ is clear. To prove Jacobi identity, for any $f\,,\,g\,,\,h \in \mathsf{DR}^0_B A$, we write:

$$(\theta_f{\circ}\theta_g - \theta_g{\circ}\theta_f)h = [\theta_f,\theta_g]h = \theta_{\{f,g\}}h = \{\{f,g\}\,,\,h\}\,.$$

The leftmost commutator here equals: $\{f,\{g,h\}\} - \{g,\{f,h\}\}$. Therefore, we get: $\{f,\{g,h\}\} - \{g,\{f,h\}\} = \{\{f,g\},h\}$, and the Theorem is proved. $\square$

Assume that $B = \Bbbk\oplus\Bbbk\oplus\ldots\oplus\Bbbk$ (direct sum of p copies of the ground field). For each $i \in \{1,\ldots,p\}$, let $\mathbf{1}_i \in B$ denote the idempotent corresponding to the i-th direct summand $\Bbbk$.

Further, let V be a finite dimensional left B-module. Clearly, giving such a V amounts to giving a collection of finite dimensional $\Bbbk$-vector spaces $\{V_i\}_{1\le i\le p}$, one for each i, such that $V = \bigoplus_i V_i$, and such that $\mathbf{1}_i \in B$ acts as the projector onto the i-th direct summand. We consider the algebra $\mathtt{End}V := \mathtt{End}_\Bbbk V$ of $\Bbbk$-linear endomorphisms of V. The action of B on V makes $V^* := \mathtt{Hom}_\Bbbk(V,\Bbbk)$ a right B-module, and gives an algebra imbedding: $B \hookrightarrow \mathtt{End}V$. Hence, left and right multiplication by B make $\mathtt{End}V$ a B-bimodule which is canonically isomorphic to the B-bimodule $V\otimes_\Bbbk V^*$. Further, the assignment:

$$f \mapsto \big(\mathsf{tr}(\mathbf{1}_1{\cdot}f{\cdot}\mathbf{1}_1)\ ,\ \mathsf{tr}(\mathbf{1}_2{\cdot}f{\cdot}\mathbf{1}_2)\ ,\ldots,\ \mathsf{tr}(\mathbf{1}_p{\cdot}f{\cdot}\mathbf{1}_p)\big) \in \Bbbk\oplus\Bbbk\oplus\ldots\oplus\Bbbk = B$$

gives a canonical B-bimodule trace map $\mathsf{tr} : \mathtt{End}V \longrightarrow B$.

Representation functor. Given a finitely generated associative B-algebra A, let $\mathtt{Hom}_{B\text{-alg}}(A,\mathtt{End}V)$ denote the affine algebraic variety of all associative algebra homomorphisms $\rho : A \to \mathtt{End}V$, such that $\rho\big|_B = \mathtt{Id}_B$. Let $\mathsf{Rep}(A,V) := \Bbbk[\mathtt{Hom}_{B\text{-alg}}(A,\mathtt{End}V)]$ denote the coordinate ring of $\mathtt{Hom}_{B\text{-alg}}(A,\mathtt{End}V)$. The natural action on $\mathtt{End}V$ of the group $G(V) = GL_B(V)$ (of B-linear automorphisms of V) by conjugation induces a $G(V)$-action on $\mathtt{Hom}_{B\text{-alg}}(A,\mathtt{End}V)$. This gives a $G(V)$-action on $\mathsf{Rep}(A,V)$ by algebra automorphisms.

The tautological evaluation map: $A\times\mathtt{Hom}_{B\text{-alg}}(A,\mathtt{End}V) \longrightarrow \mathtt{End}V$ assigns to any element $a \in A$ an $\mathtt{End}V$-valued function $\hat a$ on $\mathtt{Hom}_{B\text{-alg}}(A,\mathtt{End}V)$. Equivalently, this function may be viewed as an element $\hat a \in \big(\mathsf{Rep}(A,V)\bigotimes_B \mathtt{End}V\big)^{G(V)}$. Taking the trace on the second tensor factor, one obtains a $G(V)$-invariant $\Bbbk$-valued function $\mathsf{tr}(\hat a) \in \big(\mathsf{Rep}(A,V)\bigotimes_B B\big)^{G(V)} = \mathsf{Rep}(A,V)^{G(V)}$. The assignment: $a \mapsto \mathsf{tr}(\hat a)$ clearly vanishes on $[A,A]$ due to the cyclic symmetry of the trace map. Thus, it descends to a well-defined B-linear map

$$\widehat{\mathsf{tr}} : \mathsf{DR}^0_B A = A/[A,A] \longrightarrow \mathsf{Rep}(A,V)^{G(V)} \quad , \quad a \mapsto \mathsf{tr}(\hat a)\,. \tag{2.7}$$

Remark. More generally, for any $p \geq 0$, the assignment: $a_0 \cdot da_1 \dots da_p \mapsto \mathsf{tr}(\hat{a}_0 \cdot d\hat{a}_1 \dots d\hat{a}_p)$ gives a well-defined map from $\mathsf{DR}^p_B A$ to the space of $G(V)$-invariant regular p-forms (in the ordinary sense) on the algebraic variety $\mathtt{Hom}_{B\text{-alg}}(A, \mathtt{End}V)$.

3. Lie algebra associated to a quiver

Fix B, a commutative $\Bbbk$-algebra and E, a finite rank projective B-bimodule, i.e. a projective $B \otimes B^{op}$-module. The space $E^\vee := \mathtt{Hom}_{\mathsf{left}\,B\text{-}\mathsf{mod}}(E, B)$ has a canonical B-bimodule structure given by: $(b_1\varphi b_2)(e) = \varphi(e \cdot b_1) \cdot b_2$, where $b_1, b_2 \in B$, $e \in E$, and $\varphi \in E^\vee$.

A B-bimodule map $\omega : E \otimes_B E \to B$ will be referred to as a B-bilinear form on E. For such an ω, the assignment: $e \mapsto \omega(- \otimes e)$ gives a B-bimodule map $E \to E^\vee$. We call ω *non-degenerate* if the latter map is an isomorphism. If, furthermore, ω is skew-symmetric, i.e. $\omega(x, y) + \omega(y, x) = 0$, for any $x, y \in E$, we will say that ω is a symplectic B-form on E. For example, for any finite rank projective B-bimodule V, the bimodule $E = V \bigoplus V^\vee$ carries a canonical symplectic B-form.

Fix a finite dimensional B-bimodule E, and let $A = T_B E := \bigoplus_{i \geq 0} T^i_B E$ be the tensor algebra, a graded associative algebra such that $T^0_B E = B$. For each $i > 0$, let $(T^i_B E)_{\mathsf{cyclic}}$ denote the quotient of $T^i_B E$ by the B-sub-bimodule generated by the elements:

$$x_1 \otimes_B x_2 \otimes_B \dots \otimes_B x_i \,-\, x_i \otimes_B x_1 \otimes_B \dots \otimes_B x_{i-1} \quad , \quad \forall x_1, \dots, x_i \in E\,.$$

The following result was obtained independently by L. Le Bruyn [LB1] and the author.

Lemma 3.1. (i) *The de Rham complex of $A = T_B E$ is acyclic, i.e., $H^k(\mathsf{DR}^\bullet_B A) = 0$, for all $k \geq 1$. Furthermore, $H^0(\mathsf{DR}^\bullet_B A) = B$.*

(ii) *We have:* $\mathsf{DR}^0_B(T_B E) = (T_B E)_{\mathsf{cyclic}}$, *and* $\mathsf{DR}^1_B(T_B E) = (T_B E) \bigotimes_B E$.

Proof. To prove (i), we imitate, following Kontsevich [K2], the classical proof of the Poincaré lemma. To this end, introduce a (B-linear) Euler derivation $\mathsf{eu} : T_B E \to T_B E$ by letting it act on generators $x \in E = T^1_B E$ by: $\mathsf{eu}(x) = x$. The induced map $L_{\mathsf{eu}} : \mathsf{DR}^\bullet_B A \longrightarrow \mathsf{DR}^\bullet_B A$ is diagonalizable and has non-negative integral eigenvalues. Cartan's homotopy formula: $L_{\mathsf{eu}} = d \circ i_{\mathsf{eu}} + i_{\mathsf{eu}} \circ d$ shows that the de Rham complex is quasi-isomorphic to the zero eigen-space of the operator L_{eu}, which is the subspace B sitting in degree 0. Part (i) follows. Part (ii) is straightforward. $\square$

From now until the end of the section assume that $B = \Bbbk^I$, where I is a finite set, and put $A := T_B E$, where (E, ω) is a symplectic B-bimodule. Using the isomorphism: $E \xrightarrow{\sim} E^\vee = \mathtt{Hom}_{\mathsf{left}\,B\text{-}\mathsf{mod}}(E, B)$, provided by ω, one transports the symplectic structure from E to $E^\vee$. Let $\omega^\vee = \sum_r \phi_r \otimes \psi_r \in E \otimes E$ be the resulting symplectic B-form on $E^\vee$. It is straightforward to see that $\sum_r d\phi_r \otimes d\psi_r \in \Omega^2_B A$ gives a well-defined *closed* and *non-degenerate* class in

$\mathsf{DR}^2_B A$, to be dented ω_{DR}. Thus, the general construction (2.4) yields a Lie bracket $\{\,,\,\}_{\omega_{\mathsf{DR}}}$ on $A/[A,A]$.

Example. For each $i \in I$, let $\mathbf{1}_i \in B = \Bbbk^I$ denote the idempotent corresponding to the i-th direct summand. Clearly, giving a finite rank B-bimodule amounts to giving a finite dimensional $\Bbbk$-vector space E equipped with a direct sum decomposition: $E = \bigoplus_{i,j \in I} E_{i,j}$, where $E_{i,j} = \mathbf{1}_i \cdot E \cdot \mathbf{1}_j$. Thus, one may think of the data (B,E) as an oriented graph with vertex set I and with $\dim E_{i,j}$ edges going from the vertex i to the vertex j.

Conversely, let Q denote an oriented quiver with vertex set I. Set $B = \Bbbk^I$, and let E_Q be the $\Bbbk$-vector space with basis formed by the set of edges $\{a \in Q\}$. Then E_Q has an obvious B-bimodule structure, and $T_B(E_Q)$ is known as the *path algebra* of Q. Further, the B-bimodule $E_{\overline{Q}}$ associated with $\overline{Q}$, the *double* of Q, has a natural symplectic B-form. The corresponding class in $\mathsf{DR}^2_B\big(T_B(E_{\overline{Q}})\big)$ is given by the formula: $\omega_{\mathsf{DR}} = \sum_{a \in Q} da \otimes da^*$.

In the special case $B = \Bbbk$, the Lie bracket $\{\,,\,\}_{\omega_{\mathsf{DR}}}$ on $A/[A,A]$ has been introduced by Kontsevich [K2] in a somewhat different way as follows. Let $x_1, \ldots, x_n, y_1, \ldots, y_n$ be a symplectic basis of the vector space E, i.e. a $\Bbbk$-basis such that: $\omega(x_i, y_j) = \delta_{ij}$, and $\omega(x_i, x_j) = \omega(y_i, y_j) = 0$. By Lemma 3.1(ii) , one has: $\mathsf{DR}^1_{\Bbbk} A \simeq A \otimes E$. Kontsevich exploits this isomorphism to write any 1-form $\alpha \in \mathsf{DR}^1_{\Bbbk} A$ as: $\alpha = \sum_{i=1}^n F_{x_i}(\alpha) \otimes x_i + \sum_{j=1}^n F_{y_j}(\alpha) \otimes y_j$, for certain uniquely determined elements $F_{x_i}(\alpha), F_{y_j}(\alpha) \in A$. He then introduces , for any $i = 1, \ldots n,$ the following $\Bbbk$-linear maps:

$$\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i} : \ \mathsf{DR}^0_{\Bbbk} A \longrightarrow A, \text{ given by } \ \frac{\partial f}{\partial x_i} := F_{x_i}(df) \quad \text{and} \quad \frac{\partial f}{\partial y_i} := F_{y_i}(df).$$

For instance, let $n = 2$ so that $A = \Bbbk\langle x, y\rangle$, and $\mathsf{DR}^0_{\Bbbk} A = \Bbbk\langle x, y\rangle_{\text{cyclic}}$, see Lemma 3.1(ii). Then, given a monomial $a_1 a_2 \ldots a_p \in \Bbbk\langle x, y\rangle_{\text{cyclic}}$, where each a_i equals either x or y, we have an explicit formula:

$$\frac{\partial(a_1 a_2 \ldots a_p)}{\partial x} = \sum\nolimits_{\{i \in [1,p] \;\big|\; a_i = x\}} a_{i+1} \cdot \ldots \cdot a_p \cdot a_1 \cdot a_2 \cdot \ldots \cdot a_{i-1},$$

and a similar formula holds for $\frac{\partial(a_1 a_2 \ldots a_p)}{\partial y}$.

Using the maps $\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}$, Kontsevich defines (put another way: gives a coordinate expression for) the Lie bracket $\{-,-\}_\omega$ by the familiar formula:

$$(3.2) \quad \{f, g\}_\omega := \sum_{i=1}^n \Big(\frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i} \cdot \frac{\partial g}{\partial x_i}\Big) \bmod [A,A] \in A/[A,A] = \mathsf{DR}^0_{\Bbbk} A,$$

where "dot" stands for the product in A. We leave to the reader to check that formulas (2.4), and (1.4) give rise to the same bracket on $\mathsf{DR}^0_{\Bbbk} A = A/[A,A]$ as formula (3.2).

In the general case of an arbitrary quiver Q, the analogue of Kontsevich's formula (3.2) for the Poisson bracket associated with the algebra $A = T_B(E_{\overline{Q}})$, in obvious notation, cf. (1.1), is:

$$\{f,g\}_{\omega} = \sum_{a\in Q}\Big(\frac{\partial f}{\partial a}\cdot\frac{\partial g}{\partial a^*} - \frac{\partial f}{\partial a^*}\cdot\frac{\partial g}{\partial a}\Big)\mathrm{mod}\ [A,A] \in A/[A,A] = \mathsf{DR}^0_{B} A\,. \tag{3.3}$$

The next Proposition gives a non-commutative analogue of the classical Lie algebra exact sequence:

0 → constant functions → regular functions → symplectic vector fields → 0 ,

associated with a connected and simply-connected symplectic manifold.

Proposition 3.4. *Given a B-symplectic structure on E, for $A = T_B E$, there is a natural Lie algebra central extension:*

$$0 \longrightarrow B \longrightarrow A/[A,A] \longrightarrow \mathtt{Der}_B(A,\omega) \longrightarrow 0\,.$$

Proof. It is immediate from formula (2.3) that for the map: $f \mapsto \theta_f$ we have: $\mathtt{Ker}\{A/[A,A] \longrightarrow \mathtt{Der}_B(A,\omega)\} = \mathtt{Ker}\, d$. By Lemma 3.1(i) we get: $\mathtt{Ker}\, d = B$. Further, Lemma 3.1(i) insures that every closed element in $\mathsf{DR}^1_B A$ is exact. This yields surjectivity of the map: $A/[A,A] \longrightarrow \mathtt{Der}_B(A,\omega)$. Theorem 2.5(ii) completes the proof. ☐

Representations. We now fix a finite dimensional left B-module V, as at the end of §2. Observe that if Q is a quiver, and $E = E_{\overline{Q}}$ is the symplectic B-bimodule attached, as has been explained earlier, to the double of Q, then for $A = T_B(E_{\overline{Q}})$, in the notation of the Introduction we have: $\mathtt{Hom}_{B\text{-}\mathsf{alg}}(A, \mathtt{End}V) = \mathcal{R}(\overline{Q}, V)$. In general, let E be a finite dimensional symplectic B-bimodule. Then, for $A = T_B E$, one has: $\mathtt{Hom}_{B\text{-}\mathsf{alg}}(A, \mathtt{End}V) = \mathtt{Hom}_{B\text{-bimod}}(E, \mathtt{End}V)$. The latter space can be naturally identified with $E^\vee \otimes_B \mathtt{End}V$. Note that we have the symplectic B-form $\omega^\vee$ on $E^\vee$, and a non-degenerate symmetric bilinear form $\mathsf{tr} : \mathtt{End}V \otimes_B \mathtt{End}V \longrightarrow B$, given by: $(F_1, F_2) \mapsto \mathsf{tr}(F_1 \circ F_2)$. By standard Linear Algebra, the tensor product of a skew-symmetric and symmetric non-degenerate forms gives the skew-symmetric non-degenerate bilinear form: $\omega_{\mathsf{Rep}} := \omega^\vee \otimes \mathsf{tr}$. The 2-form ω_{Rep} makes $E^\vee \otimes_B \mathtt{End}V$, hence, $\mathtt{Hom}_{B\text{-}\mathsf{alg}}(A, \mathtt{End}V)$, a symplectic B-bimodule, therefore gives rise to a $G(V)$-invariant Poisson bracket $\{\ ,\ \}_{\omega_{\mathsf{Rep}}}$ on the coordinate ring $\Bbbk[\mathtt{Hom}_{B\text{-}\mathsf{alg}}(A, \mathtt{End}V)] = \mathsf{Rep}(A, V)$. The invariants, $\mathsf{Rep}(A, V)^{G(V)}$, clearly form a Poisson subalgebra in $\mathsf{Rep}(A, V)$, and we have:

Proposition 3.5. *The map $\widehat{\mathsf{tr}} : A/[A,A] \longrightarrow \mathsf{Rep}(A, V)^{G(V)}$ defined in (2.7) is a Lie algebra homomorphism, that is, for any $f, g \in A$, one has:*

$$\{\widehat{\mathsf{tr}} f\,,\, \widehat{\mathsf{tr}} g\}_{\omega_{\mathsf{Rep}}} = \widehat{\mathsf{tr}}(\{f,g\}_{\omega_{\mathsf{DR}}})\,.$$

Proof. Straightforward calculation for f, g taken to be non-commutative monomials. ☐

We can now complete the proof of Theorem 1.2. As we have mentioned in the Introduction, the $G(V)$-action on $\mathtt{Hom}_{B\text{-alg}}(A, \mathtt{End}V) = E^\vee \otimes_B \mathtt{End}V$ turns out to be Hamiltonian, and the corresponding moment map $\mu : E^\vee \otimes_B \mathtt{End}V \longrightarrow \mathfrak{g}(V)^* = \mathfrak{sg}(V)$, cf. (1.1), is given by the following formula:

$$(3.6)\ \sum_i \phi_i \otimes F_i \ \mapsto \sum_{j<k} \omega^\vee(\phi_j, \phi_k) \cdot [F_j, F_k] \ \in \mathfrak{sg}(V)\ ,\ \phi_i \in E^\vee\ ,\ F_i \in \mathtt{End}V.$$

Fix $\mathsf{O} \subset \mathfrak{sg}(V)$, a closed $\mathtt{Ad}\, G(V)$-orbit, and assume that the group $G(V)$ acts freely on the subvariety $\mu^{-1}(\mathsf{O}) \subset E^\vee \otimes_B \mathtt{End}V$. Then, the orbit space $\mu^{-1}(\mathsf{O})/G(V)$ is a smooth affine subvariety in $\mathtt{Spec}(\mathsf{Rep}(A, V)^{G(V)})$.

Proposition 3.7. *The composite map:*

$$\mu^{-1}(\mathsf{O})/G(V) \ \hookrightarrow\ \mathtt{Spec}\big(\mathsf{Rep}(A, V)^{G(V)}\big)$$
$$\xrightarrow{\text{evaluation}} \ \big(\mathsf{Rep}(A, V)^{G(V)}\big)^* \ \xrightarrow{\mathsf{tr}^*}\ (A/[A, A])^*$$

is injective and makes $\mu^{-1}(\mathsf{O})/G(V)$ a coadjoint orbit in $(A/[A, A])^*$.

Proof. Set $X = \mu^{-1}(\mathsf{O})/G(V)$, a smooth affine variety. As we have argued in §1, proving the proposition amounts to showing that regular functions on $\mathtt{Spec}\big(\mathsf{Rep}(A, V)^{G(V)}\big)$ of the form $\mathsf{tr}(\hat{a})$, $a \in A/[A, A]$, separate points and tangents of the variety $X \subset \mathtt{Spec}\big(\mathsf{Rep}(A, V)^{G(V)}\big)$. This is clearly true for the whole algebra $\mathsf{Rep}(A, V)^{G(V)}$, since it is true for the algebra $\Bbbk[X]$, and every regular function on X is obtained from an element of $\mathsf{Rep}(A, V)^{G(V)}$, by restriction.

We now use the result of Le Bruyn- Procesi [LP], saying that the algebra $\mathsf{Rep}(A, V)^{G(V)}$ is generated by elements of the form:

$$\mathsf{tr}(\hat{\mathbf{1}}_i \cdot \hat{x}_1 \cdot \hat{x}_2 \cdot \ldots \cdot \hat{x}_k \cdot \hat{\mathbf{1}}_i) \qquad , \qquad i = 1, \ldots, p\ ,\ \ x_j \in E\ ,\ \ k \geq 1\,.$$

The expression above is nothing but $\mathsf{tr}(\hat{a})$, for $a = \mathbf{1}_i \cdot (x_1 \otimes \ldots \otimes x_k) \cdot \mathbf{1}_i \in T^k_B E \subset A$. It follows that, although the map $\widehat{\mathsf{tr}} : A/[A, A] \longrightarrow \mathsf{Rep}(A, V)^{G(V)}$ is not itself surjective, the algebra $\mathsf{Rep}(A, V)^{G(V)}$ is generated by its image. Thus, elements of the image separate points and tangents of the variety X. □

Quantization. Fix a quiver Q, and set $A = T_B(E_{\overline{Q}})$. Consider the $\Bbbk$-vector space, $\mathcal{R}(Q, V)$, of representations of Q in a B-bimodule V. Recall the canonical identification: $\mathcal{R}(\overline{Q}, V) = T^*\mathcal{R}(Q, V)$. There is an interesting "quantization" of the map $\widehat{\mathsf{tr}} : A/[A, A] \longrightarrow \mathbb{C}[\mathcal{R}(\overline{Q}, V)]^{G(V)}$, in which the Poisson algebra $\mathbb{C}[\mathcal{R}(\overline{Q}, V)]^{G(V)} = \mathbb{C}[T^*\mathcal{R}(Q, V)]^{G(V)}$ gets replaced by the Lie algebra $\mathcal{D}\big(\mathcal{R}(Q, V)\big)^{G(V)}$ of $G(V)$-invariant polynomial differential operators (with respect to the commutator bracket) on the vector space $\mathcal{R}(Q, V)$.

To construct this quantization, recall that $V = \oplus_{i \in I} V_i$. For each vertex $i \in I$, we choose and fix a basis $\{v_\mu\}_{\mu=1,2,\ldots,\dim V_i}$ in V_i, and let $\{\check{v}_\mu\}$ be the dual basis of V_i^*. By definition we have: $\mathcal{R}(Q, V) \simeq$

$\bigoplus_{a\in Q}$ $\mathtt{Hom}(V_{\mathrm{tail}(a)}, V_{\mathrm{head}(a)})$, and we equip each $\mathtt{Hom}$-space above with the induced basis: $\Big\{\delta_{\mu\nu} = \check{v}_\nu \otimes v_\mu \in V^*_{\mathrm{tail}(a)} \otimes V_{\mathrm{head}(a)}\Big\}$. Any edge $a \in Q$ gives a linear map $\hat{a} \in \mathtt{Hom}(V_{\mathrm{tail}(a)}, V_{\mathrm{head}(a)})$, which we can write, using the basis, in the matrix form: $\hat{a} = \|a_{\mu\nu}\|$, that is: $\hat{a} = \sum_{\mu\nu} a_{\mu\nu} \cdot \delta_{\mu\nu}$. Further, write $\Big\{\delta^*_{\mu\nu} \in \mathtt{Hom}\big(V_{\mathrm{tail}(a)}, V_{\mathrm{head}(a)}\big)^*\Big\}$ for the dual basis in the dual space. To a^*, the edge of $\overline{Q}$ reverse to an edge $a \in Q$, we associate the element $\hat{a}^* := \sum_{\mu\nu} a^*_{\mu\nu} \cdot \delta^*_{\mu\nu} \in \mathtt{Hom}\big(V_{\mathrm{tail}(a)}, V_{\mathrm{head}(a)}\big)^*$, whose matrix $\|\hat{a}^*_{\mu\nu}\| := \|\hat{a}_{\nu\mu}\|$ is transposed to that of a. Equivalently, one may view $\hat{a}^*$ as a linear map $\hat{a}^* : V_{\mathrm{head}(a)} \to V_{\mathrm{tail}(a)}$, given by the transposed matrix, and identify this map with a linear function on $\mathtt{Hom}(V_{\mathrm{tail}(a)}, V_{\mathrm{head}(a)})$ via the trace pairing. In what follows, we will treat a base vector $\delta_{\mu\nu}$ as a first order constant coefficient differential operator on the vector space $\mathtt{Hom}(V_{\mathrm{tail}(a)}, V_{\mathrm{head}(a)})$ (or on the larger space, $\mathcal{R}(Q,V)$, containing it as a direct summand), and the dual base vector, $\delta^*_{\mu\nu}$, as a linear function on that vector space.

Recall next that by Lemma 3.1(ii) one has an isomorphism: $A/[A,A] \simeq A_{\mathsf{cyclic}}$. One may further identify A_{cyclic} with the space $A^{\mathsf{cyclic}} \subset A$ formed by the closed paths in $\overline{Q}$ (modulo cyclic permutations). Given such a closed path $\mathbf{x} = x^{(1)} \cdot \ldots \cdot x^{(p)} \in A^{\mathsf{cyclic}}$, where each $x^{(p)}$ is an edge of $\overline{Q}$ and, moreover, $\mathrm{head}(x^{(p)}) = \mathrm{tail}(x^{(1)})$, we define a differential operator $\mathsf{D}(X) \in \mathcal{D}\big(\mathcal{R}(Q,V)\big)$ by the formula:

$$\text{(3.8)}\quad \mathsf{D}(\mathbf{x}) = \sum_{\mu_1,\mu_2,\ldots,\mu_p} (x^{(1)})_{\mu_1\mu_2} \cdot (x^{(2)})_{\mu_2\mu_3} \cdot \ldots \cdot (x^{(p-1)})_{\mu_{p-1}\mu_p} \cdot$$
$$\cdot (x^{(p)})_{\mu_p\mu_1} \cdot \frac{1}{p} \sum_{\circlearrowleft} \Big(\delta^{(1)}_{\mu_1\mu_2} \cdot \delta^{(2)}_{\mu_2\mu_3} \cdot \ldots \cdot \delta^{(p-1)}_{\mu_{p-1}\mu_p} \cdot \delta^{(p)}_{\mu_p\mu_1} \Big) .$$

In this formula, each $x^{(k)}$ stands for either a, an edge of Q, or a^*, the reverse edge. Accordingly, the symbol $\delta^{(k)}_{\mu\nu}$ is understood as follows

$$\delta^{(k)}_{\mu\nu} = \begin{cases} \delta_{\mu\nu} & = \text{1-st order constant differential operator} \quad \text{if} \quad x^{(k)} = a \in Q \\ \delta^*_{\mu\nu} & = \text{multiplication by linear function} \quad \text{if} \quad x^{(k)} = a^*, \text{for } a \in Q. \end{cases}$$

Finally, $\sum_{\circlearrowleft} (X \cdot Y \cdot \ldots \cdot Z)$ denotes the sum over cyclic permutations of factors $X, Y, \ldots, Z$.

The assignment: $\mathbf{x} \mapsto \mathsf{D}(\mathbf{x})$ of formula (3.8) gives a well-defined map $\mathsf{D} : A^{\mathsf{cyclic}} \longrightarrow \mathcal{D}\big(\mathcal{R}(Q,V)\big)$, independent of the choices of bases. Moreover, it is clear that the image of this map is contained in $\mathcal{D}\big(\mathcal{R}(Q,V)\big)^{G(V)}$, the space of $G(V)$-invariant differential operators. In the Proposition below we treat the algebra $\mathcal{D}\big(\mathcal{R}(Q,V)\big)^{G(V)}$ as a Lie algebra with respect to the commutator bracket. A direct calculation, left out for the reader, yields the following analogue of Proposition 3.5(ii):

Proposition 3.9. *The composite map* $\widehat{\mathsf{tr}}_{\text{quantum}} : A/[A,A] \xrightarrow{\sim} A^{\mathsf{cyclic}} \xrightarrow{\mathsf{D}} \mathcal{D}\big(\mathcal{R}(Q,V)\big)^{G(V)}$ *is a Lie algebra homomorphism.* □

Remark. The map $\widehat{\mathsf{tr}}_{\text{quantum}}$ is closely related to the construction of M. Holland [Ho].

4. Stabilization: infinite dimensional limit

We keep the setup of §2; in particular, we let $B = \Bbbk^I$ and fix A, a finitely generated associative B-algebra. Any imbedding: $V \hookrightarrow V'$ of finite rank left B-modules induces a map: $\mathtt{Hom}_{B\text{-alg}}(A, \mathtt{End}V) \hookrightarrow \mathtt{Hom}_{B\text{-alg}}(A, \mathtt{End}V')$, which is a closed imbedding of affine algebraic varieties. The latter imbedding gives rise to the restriction homomorphism of coordinate rings

$$r_{V',V} : \ \mathsf{Rep}(A, V')^{G(V')} \longrightarrow \mathsf{Rep}(A, V)^{G(V)}. \tag{4.1}$$

Observe that the collection of all finite rank B-modules, V, forms a direct system with respect to B-module imbeddings, and we set $V_\infty := \varinjlim_V V$, and let $G_\infty := \varinjlim_V G(V)$ be the corresponding ind-group. By definition we put: $\mathsf{Rep}(A, V_\infty)^{G_\infty} := \varprojlim_V \mathsf{Rep}(A, V)^{G(V)}$.

There is a standard way to introduce a cocomutative coproduct $\Delta : \mathsf{Rep}(A, V_\infty)^{G_\infty} \longrightarrow \mathsf{Rep}(A, V_\infty)^{G_\infty} \bigotimes_B \mathsf{Rep}(A, V_\infty)^{G_\infty}$. To see this, it is convenient to think of $\mathsf{Rep}(A, V_\infty)$ as some sort of coordinate ring $\Bbbk[\mathtt{Hom}_{B\text{-alg}}(A, \mathtt{End}V_\infty)]$. Then any choice of B-module isomorphism: $V_\infty \overset{\varpi}{\simeq} V_\infty \oplus V_\infty$ gives a morphism of ind-schemes:

$$\begin{aligned} &\mathtt{Hom}_{B\text{-alg}}(A, \mathtt{End}V_\infty) \times \mathtt{Hom}_{B\text{-alg}}(A, \mathtt{End}V_\infty) \\ &\qquad \hookrightarrow \mathtt{Hom}_{B\text{-alg}}\big(A, \mathtt{End}(V_\infty \oplus V_\infty)\big) \overset{\varpi}{\xrightarrow{\sim}} \mathtt{Hom}_{B\text{-alg}}(A, \mathtt{End}V_\infty). \end{aligned}$$

The coproduct Δ on $\mathsf{Rep}(A, V_\infty)^{G_\infty}$ is the one induced by the corresponding algebra map:

$$\begin{aligned} \Delta : \ &\Bbbk[\mathtt{Hom}_{B\text{-alg}}(A, \mathtt{End}V_\infty)] \\ &\longrightarrow \ \Bbbk[\mathtt{Hom}_{B\text{-alg}}(A, \mathtt{End}V_\infty)] \bigotimes_B \Bbbk[\mathtt{Hom}_{B\text{-alg}}(A, \mathtt{End}V_\infty)]\,. \end{aligned}$$

Let $\mathtt{prim}\big(\mathsf{Rep}(A, V_\infty)^{G_\infty}\big)$ denote the B-module of primitive elements in $\mathsf{Rep}(A, V_\infty)^{G_\infty}$, i.e., the elements $f \in \mathsf{Rep}(A, V_\infty)^{G_\infty}$ such that $\Delta(f) = f \otimes 1 + 1 \otimes f$.

Observe further that the map $\widehat{\mathsf{tr}}_V : A/[A, A] \longrightarrow \mathsf{Rep}(A, V)^{G(V)}$ given by (2.7) is compatible with restriction morphisms $r_{V',V}$, see (4.1), that is, for any imbedding $V \hookrightarrow V'$, one has a commutative triangle: $r_{V',V} \circ \widehat{\mathsf{tr}}_{V'} = \widehat{\mathsf{tr}}_V$. Therefore, the maps $\{\widehat{\mathsf{tr}}_V\}$ give rise to a well-defined limit map $\mathsf{tr}_\infty : A/[A, A] \longrightarrow \mathsf{Rep}(A, V_\infty)^{G_\infty}$.

We now specialize to the setup of §3 and assume that $A = T_B E$, for a certain finite rank projective B-bimodule E. The following result is, in a sense, dual to the well-known relationship, see [LQ], [L], between cyclic homology of an associative algebra A and primitive homology of the Lie algebra $\mathfrak{gl}_\infty(A)$.

Proposition 4.2. *For $A = T_{\scriptscriptstyle B}E$, the map tr_∞ sets up a bijection:*

$$\mathsf{tr}_\infty : A/(B + [A,A]) \xrightarrow{\sim} \mathtt{prim}\big(\mathsf{Rep}(A, V_\infty)^{G_\infty}\big)\ .$$

Notice next that, for $A = T_{\scriptscriptstyle B}E$, we have: $\mathsf{Rep}(A, V_\infty) = \Bbbk[\mathtt{Hom}_{B\text{-bimod}}(E, \mathtt{End}V_\infty)]$, is a polynomial algebra with a natural grading, that also induces a grading on $\mathsf{Rep}(A, V_\infty)^{G_\infty}$. Furthermore, the coproduct Δ is compatible with the (graded) algebra structure, hence makes $\mathsf{Rep}(A, V_\infty)^{G_\infty}$ a commutative and cocomutative graded Hopf B-algebra. The structure theorem for commutative and cocommutative graded Hopf algebras implies that $\mathsf{Rep}(A, V_\infty)^{G_\infty}$ must be the symmetric algebra (over B) on the B-bimodule of its primitive elements. Therefore, Proposition 4.2 yields

Corollary 4.3. *For $A = T_{\scriptscriptstyle B}E$, the map tr_∞ extends, by multiplicativity, to a graded* isomorphism *of Poisson algebras:*

$$\mathtt{Sym}^\bullet\big(A/(B + [A,A])\big) \quad \xrightarrow{\sim} \quad \mathsf{Rep}(A, V_\infty)^{G_\infty}\ .$$

$\square$

Remark. It is interesting to note that, for any finite dimensional V such that $\dim V > 1$, the variety $\mathtt{Spec}\big(\mathsf{Rep}(A, V)^{G(V)}\big)$ is quite complicated, e.g., in the Calogero-Moser case. Nonetheless, Corollary 4.3 says that the 'limiting' variety $\mathtt{Spec}\big(\mathsf{Rep}(A, V_\infty)^{G_\infty}\big)$ is always a vector space.

Proof of Proposition 4.2. It is clear from definitions, that $\mathsf{tr}_\infty(f) \in \mathsf{Rep}(A, V_\infty)^{G_\infty}$ is a primitive element, for any homogeneous element $f \in A$ such that $\deg f > 0$. Furthermore, one verifies that any element not contained in the image of the map tr_∞ cannot satisfy the equation $\Delta(f) = f \otimes 1 + 1 \otimes f$, hence, is not primitive. Thus, the map tr_∞ is surjective, and it suffices to prove it is injective.

In order to avoid complicated notation, we restrict ourselves to proving injectivity in the special case of the quiver Q consisting of a single vertex and a single edge-loop, that is the Calogero-Moser quiver (the general case goes in a similar fashion with minor modifications). Thus, we assume that $A = \Bbbk\langle x, y\rangle$, and therefore, $\mathsf{Rep}(A, V_\infty) = \Bbbk[\mathfrak{gl}_\infty \oplus \mathfrak{gl}_\infty]$, where $\mathfrak{gl}_\infty := \varinjlim \mathfrak{gl}_n(\Bbbk)$. We must show that, given $f \in A$, the equation: $\mathsf{tr}_\infty(f) = 0$ implies: $f \in [A,A]$. This is proved as follows (the argument below seems to be standard, but we could not find an appropriate reference in the literature).

Let $\mathbb{A} = \Bbbk\langle x_1, x_2, \ldots, y_1, y_2 \ldots\rangle$ be the free associative algebra on countably many variables, and $[\mathbb{A}, \mathbb{A}]$ the $\Bbbk$-linear subspace of $\mathbb{A}$ spanned by the commutators. Similarly to formula (1.6), to any element $F \in \mathbb{A}$ one assignes a polynomial function $\mathsf{tr}F$ in infinitely many matrix variables: $X_1, X_2, \ldots, Y_1, Y_2 \ldots \in \mathfrak{gl}_\infty$, by inserting matrices instead of formal variables. We claim that: *if F is multilinear in all its variables, and the polynomial function $\mathsf{tr}F$ is identically zero on $\mathfrak{gl}_\infty$, then $F \in [\mathbb{A}, \mathbb{A}]$.* To prove this, note that modulo $[\mathbb{A}, \mathbb{A}]$ we can write: $F(x_1, x_2, \ldots, y_1, y_2 \ldots) = x_1 \cdot \mathsf{F}(x_2, \ldots, y_1, y_2 \ldots)$. Hence, equation: $0 = \mathsf{tr} F(X_1, X_2, \ldots, Y_1, Y_2 \ldots) = \mathsf{tr}\big(X_1 \cdot \mathsf{F}(X_2, \ldots, Y_1, Y_2 \ldots)\big)$ implies, since the trace pairing on $\mathfrak{gl}_\infty$ is nondegenerate, that the function $\mathsf{F}(x_2, \ldots, y_1, y_2 \ldots)$ is identically zero on $\mathfrak{gl}_\infty$. Furthermore, since $\mathfrak{gl}_\infty$ (viewed as an associative algebra) is known to be an algebra without polynomial identities, we conclude that $\mathsf{F} = 0$. Thus, $F \in [\mathbb{A}, \mathbb{A}]$, and our claim is proved.

We can now complete the proof of the Proposition. Fix $f \in A$ such that $\mathsf{tr} f(X, Y) = 0$ identically on $\mathfrak{gl}_\infty \oplus \mathfrak{gl}_\infty$. Rescaling transformations: $X \mapsto t \cdot X$, $Y \mapsto s \cdot Y$, $\forall t, s \in \Bbbk^\times$, show that we may reduce to the case where f is homogeneous in X and Y of degrees, say p, q, respectively. We now use the standard polarisation trick, and formally substitute: $x = t_1 \cdot x_1 + \ldots t_p \cdot x_p$, $y = s_1 \cdot y_1 + \ldots s_q \cdot y_q$ into f, and then take the term multilinear in $t_1, \ldots, s_q$. This way we get from $f \in A$ a multilinear element $F \in \mathbb{A}$ such that $\mathsf{tr} F = 0$ identically on $\mathfrak{gl}_\infty$. By the claim of the preceeding paragraph we conclude that $F \in [\mathbb{A}, \mathbb{A}]$. Observe now that sending all the x_i's to x, and all the y_i's to y yields an algebra homomorphism $\pi : \mathbb{A} \to A$ such that $\pi(F) = p!q! \cdot f$. Applying this homomorphism to F we get: $f = \frac{1}{p!q!}\pi(F) \in \pi([\mathbb{A}, \mathbb{A}]) = [A, A]$. $\square$

5. The basics of $\mathcal{P}$-geometry.

Let $\mathcal{P} = \{\mathcal{P}(n),\ n = 1, 2, \ldots\}$ be a $\Bbbk$-linear quadratic operad with $\mathcal{P}(1) = \Bbbk$, see [GiK]. Let $\mathbb{S}_n$ denote the Symmetric group on n letters. Given $\mu \in \mathcal{P}(n)$ and a $\mathcal{P}$-algebra A, we will write: $\mu_A(a_1, \ldots, a_n)$ for the image of $\mu \otimes a_1 \otimes \ldots \otimes a_n$ under the structure map: $\mathcal{P}(n) \otimes_{\mathbb{S}_n} A^{\otimes n} \longrightarrow A$. Following [GiK, §1.6.4], we introduce an *enveloping algebra* $\mathcal{U}^{\mathcal{P}}A$, the associative unital $\Bbbk$-algebra such that the abelian category of (left) A-modules is equivalent to the category of left modules over $\mathcal{U}^{\mathcal{P}}A$, see [GiK, Thm. 1.6.6]. The algebra $\mathcal{U}^{\mathcal{P}}A$ is generated by the symbols: $u(\mu, a)$, $\mu \in \mathcal{P}(2), a \in A$, subject to certain relations, see [Ba, §1.7].

An ideal I in a $\mathcal{P}$-algebra A will be called N-nilpotent if, for any $n \geq N$, $\mu \in \mathcal{P}(n)$, and $a_1, \ldots, a_n \in A$, one has: $\mu_A(a_1, \ldots, a_n) = 0$, whenever at least N among the elements $a_1, \ldots, a_n$ belong to I. The following useful reformulation of the notion of a left A-module is essentially well-known, see e.g., [Ba, 1.2]:

Lemma 5.1. *Giving a left A-module structure on a vector space M is equivalent to giving a $\mathcal{P}$-algebra structure on $A \sharp M := A \oplus M$ such that the following conditions hold:*

(i) *The imbedding: $a \mapsto a \oplus 0$ makes A a $\mathcal{P}$-subalgebra in $A \sharp M$.*

(ii) *M is a 2-nilpotent ideal in $A \sharp M$.* $\square$

A $\mathcal{P}$-algebra in the monoidal category of $\mathbb{Z}/2$-graded, (resp. $\mathbb{Z}$-graded) super-vector spaces, see [GiK, §1.3.17-1.3.18], will be referred to as a $\mathcal{P}$-*superalgebra*, (resp. graded superalgebra). Any $\mathcal{P}$-algebra may be regarded as a $\mathcal{P}$-superalgebra concentrated in degree zero. Given a finite dimensional (super-) vector space V, write $\overline{V}$ for the same vector space with reversed parity. Let

$$\mathsf{T}^\bullet_{\mathcal{P}} V := \bigoplus\nolimits_{i \geq 1}\ \mathcal{P}(i) \otimes_{\mathbb{S}_i} V^{\otimes i} \quad \textit{and} \quad \check{\mathsf{T}}^\bullet_{\mathcal{P}} V := \bigoplus\nolimits_{i \geq 1}\ \mathcal{P}(i) \otimes_{\mathbb{S}_i} \overline{V}^{\otimes i}$$

be the free graded $\mathcal{P}$-algebra (resp. super-algebra) generated by V.

Fix a $\mathcal{P}$-algebra A, and consider the category of A-algebras, i.e. of pairs (B,p), where B is a $\mathcal{P}$-algebra and $p : A \to B$ is a $\mathcal{P}$-algebra morphism. Note that such a morphism makes B an A-module. Thus, we get an obvious forgetful functor: *A-algebras $\longrightarrow$ A-modules.* The result below says that this functor has a right adjoint:

Lemma 5.2. (i) *Given a $\mathcal{P}$-algebra A, there is a functor: $M \mapsto T^\bullet_A M$, (resp. $M \mapsto \check{T}^\bullet_A M$) assigning to a left A-module M a graded $\mathcal{P}$-algebra $T^\bullet_A M = \bigoplus_{i\geq 0} T^i_A M$ (resp. graded $\mathcal{P}$-superalgebra $\check{T}^\bullet_A M = \bigoplus_{i\geq 0} T^i_A \overline{M}$), such that $T^0_A M = A$.*

(ii) *For any $\mathcal{P}$-algebra map: $A \to B$, one has a natural adjunction isomorphism:*

$$\mathtt{Hom}_{A\text{-mod}}(M,B) \xrightarrow{\sim} \mathtt{Hom}_{\mathcal{P}\text{-alg}}(T^\bullet_A M, B)\,.$$

Proof. If A is a $\mathcal{P}$-subalgebra in a $\mathcal{P}$-algebra $\widetilde{A}$, we define a $\mathcal{P}$-algebra $T_{A}\widetilde{A}$ as the quotient of $\mathsf{T}^\bullet_{\mathcal{P}}\widetilde{A}$, a free $\mathcal{P}$-algebra, modulo two-sided ideal generated by all relations of the form:

$$\mu\otimes a\otimes\tilde{a} = \mu_{\tilde{A}}(a,\tilde{a}) \quad , \quad \mu\otimes\tilde{a}\otimes a = \mu_{\tilde{A}}(\tilde{a},a) \quad , \quad \forall \mu\in\mathcal{P}(2),\, a\in A\subset\widetilde{A},\, \tilde{a}\in\widetilde{A},$$

where $\mu\otimes\tilde{a}\otimes a\,,\, \mu\otimes a\otimes\tilde{a} \in \mathcal{P}(2)\otimes\widetilde{A}^{\otimes 2} = \mathsf{T}^2_{\mathcal{P}}\widetilde{A}$, and $\mu_{\tilde{A}}(a,\tilde{a}),\, \mu_{\tilde{A}}(\tilde{a},a) \in \mathcal{P}(1)\otimes\widetilde{A} = \mathsf{T}^1_{\mathcal{P}}\widetilde{A}$. We now apply this construction to the algebra $\widetilde{A} = A\sharp M$, and put $T^\bullet_A M := T_{A}\widetilde{A}$, where the grading on the left accounts for the number of occurrences of elements of M, which is well-defined since the relations involved in the definition of $T_{A}\widetilde{A}$ are 'homogeneous in M'.

A closer look at the construction above shows that

$$T^\bullet_A M \;=\; A \;\bigoplus\; (\mathsf{T}^\bullet_{\mathcal{P}}M)/\langle\!\langle \mu^{(12)}(a,m_1)\,\otimes\, m_2 - m_1\,\otimes\mu(a,m_2)\rangle\!\rangle\,, \tag{5.3}$$

where $\langle\!\langle\ldots\rangle\!\rangle$ denotes the two-sided ideal generated by the indicated subset of $\mathcal{P}(2)\otimes M^{\otimes 2} = \mathsf{T}^2_{\mathcal{P}}M$, for all $\mu\in\mathcal{P}(2), a\in A, m_1, m_2\in M$, and where $\mu^{(12)}$ stands for the action of the transposition $(12)\in\mathbb{S}_2$ on μ. In particular, we have: $T^0_A M = A$ and $T^1_A M = M$. $\square$

Let A be a $\mathcal{P}$-algebra and M a left A-module. By Lemma 5.1, we may (and will) regard $A\sharp M$ as a $\mathcal{P}$-algebra.

Definition 5.4. *A $\Bbbk$-linear map $\theta : A \to M$ is called a derivation if the map: $a \bigoplus m \mapsto a \bigoplus \theta(a) + m$, is an automorphism of the $\mathcal{P}$-algebra $A\sharp M$.*

Equivalently, following [Ba, Definition 3.2.6], extend θ to a $\Bbbk$-linear map $\theta^\sharp : A\sharp M \to A\sharp M$, given by $\theta^\sharp : a\oplus m \;\mapsto\; 0\oplus\theta a$. Then, θ is a derivation if and only if, for any $\mu \in \mathcal{P}(n)$, we have:

$$\theta^\sharp\big(\mu_{_{A\sharp M}}(b_1, \dots, b_n)\big) = \\ \sum_{i=1}^n \mu_{_{A\sharp M}}(b_1, \dots, b_{i-1}, \theta^\sharp b_i, b_{i+1}, \dots, b_n) \quad, \ \forall b_1, \dots, b_n \in A\sharp M.$$

Let $\mathtt{Der}_{_\mathcal{P}}(A, M)$ denote the $\Bbbk$-vector space of all derivations from A to M. It is straightforward to see that the ordinary commutator makes $\mathtt{Der}_{_\mathcal{P}}(A, A)$ a Lie algebra.

Next we define, following [Ba, Def. 4.5.2], an A-module of *Kähler differentials* as the left $\mathcal{U}^\mathcal{P}A$-module, $\Omega^1_{_\mathcal{P}}A$, generated by the symbols da, for $a \in A$, subject to the relations:

(i) $d(\lambda_1 a_1 + \lambda_2 a_2) = \lambda_1 da_1 + \lambda_2 da_2 \quad , \ \forall \lambda_1, \lambda_2 \in \Bbbk$;

(ii) $d(\mu(a_1, a_2)) = u(\mu, a_1) \otimes da_2 + u(\mu^{(12)}, a_2) \otimes da_1$, $\forall \mu \in \mathcal{P}(2), a_1, a_2 \in A$, where $u(\mu, a)$ denote the standard generators of $\mathcal{U}^\mathcal{P}A$, see [Ba].

By construction, $\Omega^1_{_\mathcal{P}}A$ is a left A-module, and the assignment $a \mapsto da$ gives a derivation $d \in \mathtt{Der}_{_\mathcal{P}}(A\,,\,\Omega^1_{_\mathcal{P}}A)$. Moreover, this derivation is universal in the following sense. Given any left A-module M and a derivation $\theta : A \to M$, there exists an A-module morphism $\Omega^1\theta : \Omega^1_{_\mathcal{P}}A \to M$, uniquely determined by the condition that $(\Omega^1\theta)(da) = \theta(a)$. It follows that the A-module of Kähler differentials represents the functor $\mathtt{Der}_{_\mathcal{P}}(A, -)$, i.e., we have (see [Ba, Remark 4.5.4]):

Lemma 5.5. *For any left A-module M there is a natural isomorphism:*

$$\mathtt{Der}_{_\mathcal{P}}(A, M) \simeq \mathtt{Hom}_{_{A\text{-mod}}}(\Omega^1_{_\mathcal{P}}A, M)\,. \quad \Box$$

In particular, for $M = A$, we get an isomorphism: $\mathtt{Der}_{_\mathcal{P}}(A, A) \xrightarrow{\sim} \mathtt{Hom}_{_{A\text{-mod}}}(\Omega^1_{_\mathcal{P}}A, A)$. We let $i_\theta \in \mathtt{Hom}_{_{A\text{-mod}}}(\Omega^1_{_\mathcal{P}}A, A)$ denote the morphism: $\Omega^1_{_\mathcal{P}}A \to A$, corresponding to $\theta \in \mathtt{Der}_{_\mathcal{P}}(A, A)$ under the isomorphism above.

We set $\Omega^\bullet_{_\mathcal{P}}A := \check{T}^\bullet_A(\Omega^1_{_\mathcal{P}}A)$, a graded $\mathcal{P}$-superalgebra generated by the A-module $\Omega^1_{_\mathcal{P}}A$. Recall that the *differential envelope* of a $\mathcal{P}$-algebra A is a differential graded $\mathcal{P}$-super-algebra $D^\bullet(A) = \bigoplus_{i\geq 0} D^i(A)$, such that $D^0(A) = A$, and such that the following universal property holds: *For any differential graded $\mathcal{P}$-superalgebra $\widetilde{D}^\bullet = \bigoplus_{i\geq 0} \widetilde{D}^i$, and a $\mathcal{P}$-algebra morphism $\rho : A \to \widetilde{D}^0$ there exists a unique DG-superalgebra morphism $D(\rho) : D^\bullet(A) \to \widetilde{D}^\bullet$ such that $D(\rho)\big|_{_{D^0(A)}} = \rho$.*

Proposition 5.6. (i) *On $\Omega^\bullet_{_\mathcal{P}}A$, there exists a natural super-differential $d : \Omega^\bullet_{_\mathcal{P}}A \longrightarrow \Omega^{\bullet+1}_{_\mathcal{P}}A$, $d^2 = 0$, such that its restriction: $A = \Omega^0_{_\mathcal{P}}A \longrightarrow \Omega^1_{_\mathcal{P}}A$ coincides with the canonical A-module derivation $d : A \to \Omega^1_{_\mathcal{P}}A$.*

(ii) *The differential graded $\mathcal{P}$-superalgebra $(\Omega^\bullet_{_\mathcal{P}}A\,,\,d)$ is the differential envelope of A.*

Proof. We first give a direct construction of the differential envelope $D^\bullet(A)$ of a $\mathcal{P}$-algebra A, as follows. Let $\overline{A}$ denote a second copy of A viewed as a $\Bbbk$-vector space, and write $\overline{a}$ for the element of $\overline{A}$ corresponding to an element $a \in A$. We form the graded super-vector space $A \oplus \overline{A}$, where A is placed in grade degree zero, and $\overline{A}$ is placed in grade degree 1. Let $\check{\mathsf{T}}^\bullet_{\mathcal{P}}(A \oplus \overline{A}) := \bigoplus_{i \geq 1} \ \mathcal{P}(i) \otimes_{\mathbb{S}_i} (A \oplus \overline{A})^{\otimes i}$ be the free $\mathcal{P}$-superalgebra generated by $A \oplus \overline{A}$, viewed as a graded superalgebra with respect to the total grading coming from both the grading on $A \oplus \overline{A}$ and the grading on the tensor algebra. We put: $D^\bullet(A) := \check{\mathsf{T}}^\bullet_{\mathcal{P}}(A \oplus \overline{A})/I$, where I is the two-sided ideal generated by the following set:

$$\{\mu \otimes a_1 \otimes a_2 - \mu(a_1, a_2) \ , \qquad \mu \otimes \overline{a}_1 \otimes a_2 + \mu \otimes a_1 \otimes \overline{a}_2 - \overline{\mu(a_1, a_2)}\}_{\mu \in \mathcal{P}(2)\,,\, a_1, a_2 \in A}. \tag{5.7}$$

Thus, $D^\bullet(A)$ is a graded $\mathcal{P}$-superalgebra.

The $\Bbbk$-linear endomorphism of $A \oplus \overline{A}$ given by the assignment: $a \oplus \overline{a}_1 \mapsto 0 \oplus \overline{a}$ extends uniquely to a super-derivation: $\check{\mathsf{T}}^\bullet_{\mathcal{P}}(A \oplus \overline{A}) \longrightarrow \check{\mathsf{T}}^\bullet_{\mathcal{P}}(A \oplus \overline{A})$. This derivation descends to a well-defined derivation d on $D^\bullet(A)$. Note that, for any $x \in A \oplus \overline{A}$ we have: $d^2(x) = 0$. This implies, since the subspace $A \oplus \overline{A}$ generates the algebra $D^\bullet(A)$, that $d^2 = 0$ identically on $D^\bullet(A)$. Thus, d makes $D^\bullet(A)$ a differential graded $\mathcal{P}$- superalgebra.

The zero-degree component, $D^0(A)$, of the super-algebra $D^\bullet(A)$ is by construction a $\mathcal{P}$-subalgebra isomorphic to A, i.e., there is a canonical superalgebra imbedding $j : A = D^0(A) \hookrightarrow D^\bullet(A)$. Hence, $D^\bullet(A)$ may be regarded as an A-module, and the assignment: $a \mapsto \overline{a}$ gives a derivation $d \in \mathtt{Der}_{\mathcal{P}}(A, D^\bullet(A))$. This derivation is universal in the sense explained above (for uniqueness property use that the superalgebra $D^\bullet(A)$ is generated by the subspace $A \oplus \overline{A}$). Hence $D^\bullet(A)$ is the differential envelope of A.

Observe next that the degree 1 component of $D^\bullet(A)$ is isomorphic, by definition of $D^\bullet(A)$, to the quotient of $\mathcal{U}^{\mathcal{P}}A \otimes \overline{A}$ by the relations (i)–(ii) defining the module $\Omega^1_{\mathcal{P}}A$ of Kähler differentials. Therefore, $D^1(A)$, the degree 1 component of $D^\bullet(A)$, is isomorphic to $\Omega^1_{\mathcal{P}}A$ and, moreover, the canonical derivation $d : A \to \Omega^1_{\mathcal{P}}A$ may be identified with the map: $a \mapsto \overline{a} \in D^\bullet(A)$.

By the universal property of the tensor algebra, the A-module imbedding $\Omega^1 A \hookrightarrow D^\bullet(A)$ can be extended uniquely to a graded super-algebra morphism $f : \check{T}^\bullet_A(\Omega^1_{\mathcal{P}}A) \longrightarrow D^\bullet(A)$. To show that f is an isomorphism we construct its inverse, a map $g : D^\bullet(A) \longrightarrow \check{T}^\bullet_A(\Omega^1_{\mathcal{P}}A)$, as follows. We have an obvious imbedding of $\Bbbk$-vector spaces: $A \oplus \overline{A} \hookrightarrow A \oplus \Omega^1_{\mathcal{P}}A$, given by: $a \oplus \overline{a}_1 \mapsto a \oplus da_1$. This imbedding extends, by the universal property of a free $\mathcal{P}$-algebra, to a $\mathcal{P}$-superalgebra morphism $\tilde{g} : \check{\mathsf{T}}^\bullet_{\mathcal{P}}(A \oplus \overline{A}) \longrightarrow \check{T}^\bullet_A(\Omega^1_{\mathcal{P}}A)$. The relations defining the ideal I in formula (5.7) are designed in such a way that the morphism $\tilde{g}$ descends to a well-defined super-algebra morphism $g : D^\bullet(A) \to \check{T}^\bullet_A(\Omega^1_{\mathcal{P}}A)$. It is straightforward to verify that $g = f^{-1}$. $\square$

Remark. Our construction agrees with the notion of non-commutative differential forms for an algebra over the associative operad, as defined e.g. in [L] and used in §2 above.

From now on we assume, in addition, that $\mathcal{P}$ is a *cyclic* Koszul operad, see [GeK], with $\mathcal{P}(1) = \Bbbk$. In particular, for each $n \geq 1$, the space $\mathcal{P}(n)$ is equipped with an $\mathbb{S}_{n+1}$-action that extends the $\mathbb{S}_n$-module structure on $\mathcal{P}(n)$ arising from the operad structure. Write $\mathtt{Sym}^2_{\Bbbk} A$ for the symmetric square of A. Following an idea of Kontsevich, Getzler and Kapranov introduce a functor $\mathsf{R} : \mathcal{P}$-*algebras* $\longrightarrow$ $\Bbbk$-*vector spaces* ,

$$\mathsf{R}: \ A \quad \mapsto \quad \mathsf{R}(A) := \frac{\mathtt{Sym}^2_{\Bbbk} A}{\big\langle a_0 \cdot \mu(a_1, a_2) - \mu(a_0, a_1) \cdot a_2 \big\rangle \Big|_{\, a_0, a_1, a_2 \in A \, , \, \mu \in \mathcal{P}(2)}} \, .$$

Generalizing the Karoubi's construction [Ka] in the associative case, define de Rham complex of A as the graded vector space $\mathsf{DR}^\bullet A := \mathsf{R}(\Omega^\bullet_{\mathcal{P}} A)$. The differential d on $\Omega^\bullet_{\mathcal{P}} A$ induces a differential on $\mathsf{DR}^\bullet A$.

For any $\theta \in \mathtt{Der}_{\mathcal{P}} A$, the morphism $i_\theta : \Omega^1_{\mathcal{P}} A \to A$, introduced after Lemma 5.5, extends to a super-derivation $i_\theta : \Omega^\bullet_{\mathcal{P}} A \to \Omega^{\bullet - 1}_{\mathcal{P}} A$, called the *contraction* operator. Further, the derivation θ induces, by a standard argument, a derivation L_θ of the associative algebra $\mathcal{U}^{\mathcal{P}} A$, and a map $L_\theta : \Omega^1_{\mathcal{P}} A \to \Omega^1_{\mathcal{P}} A$. The latter one extends to a derivation $L_\theta : \Omega^\bullet_{\mathcal{P}} A \to \Omega^\bullet_{\mathcal{P}} A$, called the *Lie operator*. The maps i_θ and L_θ descend naturally to the corresponding operators on $\mathsf{DR}^\bullet A$. It is straightforward to verify that these latter operators satisfy all the standard commutation relations (2.1).

6. Symplectic geometry of a free $\mathcal{P}$-algebra.

We keep the assumption that $\mathcal{P}$ is a cyclic Koszul operad. In this section which is a generalization of §3, inspired by works of Drinfeld [Dr, Proposition 6.1 and above it] and Kontsevich (private communication, 1994), we consider the case of a free $\mathcal{P}$-algebra. To avoid unnecessary repetitions and to simplify notation we only consider the 'absolute' case, i.e., the case of the ground ring $B = \Bbbk$.

Fix a finite dimensional $\Bbbk$-vector space E, and write $A = \mathsf{T}_{\mathcal{P}} E$ for the free $\mathcal{P}$-algebra (note that $\mathcal{P}$-algebras are algebras without unit, in general). We have:

$$(6.1) \quad \mathsf{R}(A) = \mathsf{DR}^0(A) = \bigoplus\nolimits_{i \geq 1} \mathcal{P}_i \otimes_{\mathbb{S}_{i+1}} E^{\otimes (i+1)} \quad , \quad \mathsf{DR}^1(A) \ = \ A \bigotimes E \, .$$

Let $\widehat{A} = \prod_{i \geq 0} \mathsf{T}^i_{\mathcal{P}} E$ denote the completion of A with respect to the augmentation, and let $\mathtt{Aut}(\widehat{A})$ denote the group of continuous algebra automorphisms of $\widehat{A}$. Any such automorphism Φ is determined by its restriction to $E = \mathsf{T}^1_{\mathcal{P}} E$, a $\Bbbk$-linear map $\phi : E \to \widehat{A}$. We have an expansion: $\phi(v) = \sum_{i=1}^{\infty} \phi_i(v)$, where $\phi_i(v) \in \mathsf{T}^i_{\mathcal{P}} E$. We write $d\Phi : E \to E$, for the map: $v \mapsto \phi_1(v)$; and we let $\mathtt{Aut}_\circ(\widehat{A})$ be the subgroup of $\mathtt{Aut}(\widehat{A})$ formed by all automorphisms Φ such that $d\Phi = \mathtt{Id}_E$.

Observe further that the obvious grading on the free algebra $A = \mathsf{T}_{\mathcal{P}}E$ induces a natural grading $\mathsf{R}^{\bullet}(A) = \bigoplus_i \mathsf{R}(A)_{(i)}$, and, for each $p \geq 0$, a similar grading $\mathsf{DR}^p(A) = \bigoplus_i \mathsf{DR}^p(A)_{(i)}$. Fix a closed 2-form $\omega \in \mathsf{DR}^2A$, and let $\omega = \omega_0 + \omega_1 + \ldots, \omega_i \in \mathsf{DR}^2(A)_{(i)}$, be its expansion into graded components. We see, in particular, that ω_0 may be viewed as an ordinary skew-symmetric $\Bbbk$-bilinear form: $E \times E \to \Bbbk$.

Theorem 6.2. (Darboux theorem). (i) *A closed 2-form* $\omega = \omega_0 + \omega_1 + \ldots \in \mathsf{DR}^2A$ *is non-degenerate if and only if so is the associated bilinear form* $\omega_0 : E \times E \to \Bbbk$.

(ii) *If* ω *is non-degenerate then there exists an automorphism* $\Phi \in \mathtt{Aut}_{\circ}(\widehat{A})$ *such that:* $\Phi^*\omega = \omega_0$.

Proof. Part (i) is clear. Part (ii) is proved by the standard 'homotopy argument'. Specifically, we consider a 1-parameter 'family': $\omega_t = \omega_0 + t \cdot \omega' \in \mathsf{DR}^2A[[t]]$, where $\omega' = \omega - \omega_0 = \omega_1 + \omega_2 + \ldots \in \mathsf{DR}^2A$. The 2-form ω' being closed, there exists $\alpha \in \bigoplus_{p\geq 1} \mathsf{DR}^1(A)_{(p)}$, such that $\omega' = -d\alpha$. Since ω_0 is non-degenerate, there exists a 1-parameter family $\theta_t \in \Bbbk[[t]] \widehat{\otimes} \mathtt{Der}_{\mathcal{P}}A = \mathtt{Der}_{\mathcal{P}}A[[t]]$ determined uniquely from the equation: $i_{\theta_t}\omega_t = \alpha$. We define $\Phi(t) \in \mathtt{Aut}(\widehat{A}[[t]])$, a formal one-parameter family of automorphisms of A, to be the solution of the differential equation: $\frac{d\Phi(t)}{dt} = L_{\theta_t}\Phi(t)$ of the form: $\Phi(t) = \mathtt{Id}_A + t \cdot \Phi_1 + t^2 \cdot \Phi_2 + \ldots$. It follows from the construction that $\Phi(t)^*\omega_t = \omega_0$, see e.g. [GS] for more details. Note further that the series $\Phi(t)$ above has only finitely many terms in any given grade degree $p \geq 0$, i.e. terms that shift the grading on A by p. In particular, setting $t = 1$ in this series gives a well-defined element of $\Phi(1) \in \mathtt{Aut}_{\circ}(\widehat{A})$ and we get: $\Phi(1)^*\omega_{t=1} = \omega_0$. But $\omega_{t=1} = \omega$, and part (ii) follows. ☐

Because of this result, there is no loss of generality in considering only degree zero symplectic 2-forms $\omega \in \mathsf{DR}^2A$, i.e., such that $\omega = \omega_0$. Fix such an ω, that is fix (E, ω), a symplectic vector space. Imitating the strategy used in §2 it is possible to define a Lie bracket on the vector space $\mathsf{R}(A)$. We prefer however to give the following direct explicit construction of this bracket similar to formula (1.4) in the associative case.

For each $i, j \geq 1$, let $\star : \mu \otimes \nu \mapsto \mu(1, \ldots, 1, \nu)$, denote the operad-composition map: $\mathcal{P}(i) \otimes \mathcal{P}(j) \simeq \mathcal{P}(1) \otimes \ldots \otimes \mathcal{P}(1) \otimes \mathcal{P}(i) \otimes \mathcal{P}(j) \longrightarrow \mathcal{P}(i + j - 1)$, where $1 \in \Bbbk = \mathcal{P}(1)$, see [GeK, Theorem 2.2(2)]. We now change the notation and write: $\mathsf{R}^{\bullet}(A) = \bigoplus_i \mathsf{R}^i(A)$, where $\mathsf{R}^i(A)$, previously denoted by $\mathsf{R}(A)_{(i)}$, is the graded component with respect to the grading induced by one on A. Also, let Sym be the 'symmetrisation map', the projection to $\mathbb{S}_n$-coinvariants. For each $i, j \geq 1$, we define a bilinear pairing $\{-,-\}_{\omega} : \mathsf{R}^i(A) \otimes \mathsf{R}^j(A) \longrightarrow \mathsf{R}^{i+j-1}(A)$

as the following composition

$$\begin{aligned}\mathsf{R}^i(A)\otimes\mathsf{R}^j(A) = \big(\mathcal{P}(i)\otimes_{\mathbb{S}_{i+1}} E^{\otimes i+1}\big) \bigotimes \big(\mathcal{P}(j)\otimes_{\mathbb{S}_{j+1}} E^{\otimes j+1}\big) &\longrightarrow \\ \big(\mathcal{P}(i)\otimes\mathcal{P}(j)\otimes E^{\otimes i+j+2}\big)_{\mathbb{S}_{i+1}\times\mathbb{S}_{j+1}} &\xrightarrow{\star} \\ \big(\mathcal{P}(i+j-1)\otimes E^{\otimes i+j+2}\big)_{\mathbb{S}_{i+1}\times\mathbb{S}_{j+1}} &\xrightarrow{\mathrm{Sym}} \\ \big(\mathcal{P}(i+j-1)\otimes_{\mathbb{S}_{i+j}} E^{\otimes i+j}\big) \bigotimes E^{\otimes 2} &\xrightarrow{\mathrm{id}\otimes\omega} \\ \mathcal{P}(i+j-1)\otimes_{\mathbb{S}_{i+j}} E^{\otimes i+j} = \mathsf{R}^{i+j-1}(A)\,.&\end{aligned}$$

An appropriate modification of the proof of Theorem 2.5, or a direct calculation, yields

Proposition 6.3. *The bracket* $\{-,-\}$ *makes* $\mathsf{R}^{\bullet-1}(A)$ *into a graded Lie algebra.* $\square$

Remark. It is likely, cf. [GeS], that there is a graded Lie *super*-algebra structure on $\mathsf{DR}^\bullet(A)$ extending the one on $\mathsf{R}(A)=\mathsf{DR}^0(A)$, defined above.

Let $\mathtt{Der}_{\mathcal{P}}(A,\omega)$ denote the Lie subalgebra in $\mathtt{Der}_{\mathcal{P}}A$ formed by all derivations $\theta\in\mathtt{Der}_{\mathcal{P}}A$ such that $L_\theta\omega=0$. Since $\omega=\omega_0$, this is equivalent to the requirement that the degree zero component $d\theta: A_1\to A_1$ induces an endomorphism of $E\otimes E$ that annihilates $\omega^\vee\in E\otimes E$. Using the same argument as in §§2-3, one proves the following two results

Lemma 6.4. *The assignment:* $\theta\mapsto i_\theta\omega$ *gives graded vector space isomorphisms:*

$$\mathtt{Der}^\bullet_{\mathcal{P}}(A) \xrightarrow{\sim} \mathsf{DR}^1(A^\bullet) \quad\textit{and}\quad \mathtt{Der}^\bullet_{\mathcal{P}}(A,\omega) \xrightarrow{\sim} \mathsf{DR}^1(A^\bullet)_{\text{closed}}\,. \qquad\square$$

Proposition 6.5. *There is a canonical graded Lie algebra central extension:*

$$0\longrightarrow\Bbbk\longrightarrow\mathsf{R}^{\bullet-1}(A)\longrightarrow\mathtt{Der}^\bullet_{\mathcal{P}}(A,\omega)\longrightarrow 0\,. \qquad\square$$

We call a pair $(\mathcal{S},\mathtt{tr})$, where $\mathcal{S}$ is a $\mathcal{P}$-algebra and $\mathtt{tr}$ is a symmetric nondegenerate invariant bilinear form $\mathtt{tr}:\mathcal{S}\otimes\mathcal{S}\to\Bbbk$, a *symmetric* $\mathcal{P}$-algebra. Any such bilinear form is determined, cf. [GeK], by a linear function $\mathtt{tr}:\mathsf{R}(\mathcal{S})\to\Bbbk\,,\ b\otimes b'\mapsto\mathtt{tr}(b\otimes b')=\mathtt{tr}(b,b')$.

From now on, fix a finite-dimensional symmetric $\mathcal{P}$-algebra $(\mathcal{S},\mathtt{tr})$. Let $\mathtt{Aut}(\mathcal{S},\mathtt{tr})$ denote the algebraic group of automorphisms of the $\mathcal{P}$-algebra $\mathcal{S}$ that preserve the bilinear form $\mathtt{tr}$. The corresponding Lie algebra $\mathtt{Der}_{\mathcal{P}}(\mathcal{S},\mathtt{tr})$ is formed by all the derivations $\theta\in\mathtt{Der}_{\mathcal{P}}\mathcal{S}$ such that, for any $b,b'\in\mathcal{S}$, one has: $\mathtt{tr}(\theta(b)\,,\,b')+\mathtt{tr}(b\,,\,\theta(b'))=0\,.$

Representation functor. For any finitely generated $\mathcal{P}$-algebra A, the set $\mathtt{Hom}_{\mathcal{P}\text{-alg}}(A,\mathcal{S})$ has the natural structure of a finite dimensional affine algebraic variety, acted on by the algebraic group $\mathtt{Aut}(\mathcal{S},\mathtt{tr})$. We put $\mathsf{Rep}(A,\mathcal{S}):=\Bbbk[\mathtt{Hom}_{\mathcal{P}\text{-alg}}(A,\mathcal{S})]$.

Let now (E,ω) be a finite dimensional symplectic vector space, and $A=T_{\mathcal{P}}E$, the free $\mathcal{P}$-algebra on E. Then we clearly have: $\mathtt{Hom}_{\mathcal{P}\text{-alg}}(A,\mathcal{S})=\mathtt{Hom}_{\Bbbk}(E,\mathcal{S})=$

$E^* \otimes_{\Bbbk} \mathcal{S}$, is a finite dimensional $\Bbbk$-vector space. The symplectic 2-form ω on E gives rise, as in §3, to the symplectic 2-form $\omega_{\mathsf{Rep}} := \omega^\vee \otimes \mathtt{tr}$ on $\mathtt{Hom}_{\Bbbk}(E, \mathcal{S}) = E^* \otimes_{\Bbbk} \mathcal{S}$. The action of the group $\mathtt{Aut}(\mathcal{S}, \mathtt{tr})$ on $\mathtt{Hom}_{\mathcal{P}\text{-}\mathsf{alg}}(A, \mathcal{S})$ preserves this symplectic form and is, moreover, Hamiltonian. In other words, the vector field on $E^* \otimes_{\Bbbk} \mathcal{S}$ arising from a derivation $\theta \in \mathtt{Der}_{\mathcal{P}}(\mathcal{S}, \mathtt{tr})$ is induced by an $\mathtt{Aut}(\mathcal{S}, \mathtt{tr})$-invariant Hamiltonian function $H_\theta \in \mathsf{Rep}(A, \mathcal{S})$. Explicitly, the function H_θ is given by the following quadratic polynomial on $E^* \otimes_{\Bbbk} \mathcal{S}$:

$$H_\theta : \ \sum\nolimits_k \ \check{x}_k \otimes s_k \ \mapsto \ \sum\nolimits_{i<j} \ \omega^\vee(\check{x}_i\,,\,\check{x}_j) \cdot \mathtt{tr}(\theta(s_i)\,,\,s_j)\ ,$$
$$x_l \in E^*\,,\ s_l \in \mathcal{S}\,,\ l = i, j, k.$$

Write $\mathsf{Rep}(A, \mathcal{S})^{\mathtt{Aut}(\mathcal{S},\mathtt{tr})}$ for the $\Bbbk$-algebra of $\mathtt{Aut}(\mathcal{S}, \mathtt{tr})$-invariant polynomial functions on the $\Bbbk$-vector space $\mathtt{Hom}_{\Bbbk}(E, \mathcal{S})$. The symplectic form $\omega_{\mathsf{Rep}} = \omega \otimes \mathtt{tr}$ makes $\mathsf{Rep}(A, \mathcal{S})^{\mathtt{Aut}(\mathcal{S},\mathtt{tr})}$ into a *Poisson algebra*. We have the standard Lie algebra central extension:

$$(6.6) \quad 0 \ \to \ \Bbbk \ \to \ \mathsf{Rep}(A, \mathcal{S})^{\mathtt{Aut}(\mathcal{S},\mathtt{tr})} \ \xrightarrow{\delta} \ \mathtt{Der}_{\omega_{\mathsf{Rep}}}\big(\mathsf{Rep}(A, \mathcal{S})^{\mathtt{Aut}(\mathcal{S},\mathtt{tr})}\big) \ \to \ 0\,,$$

where $\mathtt{Der}_{\omega_{\mathsf{Rep}}}\big(\mathsf{Rep}(A, \mathcal{S})^{\mathtt{Aut}(\mathcal{S},\mathtt{tr})}\big)$ stands for the Lie algebra of derivations of the commutative algebra $\mathsf{Rep}(A, \mathcal{S})^{\mathtt{Aut}(\mathcal{S},\mathtt{tr})}$ respecting the Poisson bracket. It is straightforward to check that the assignment: $\theta \mapsto H_\theta$ gives a Lie algebra splitting: $\mathtt{Der}_{\omega_{\mathsf{Rep}}}\big(\mathsf{Rep}(A, \mathcal{S})^{\mathtt{Aut}(\mathcal{S},\mathtt{tr})}\big) \longrightarrow \mathsf{Rep}(A, \mathcal{S})^{\mathtt{Aut}(\mathcal{S},\mathtt{tr})}$ of the surjective morphism δ in the exact sequence above.

Observe next that the 'infinite dimensional' group $\mathtt{Aut}(A)$ acts naturally on $\mathtt{Hom}_{\mathcal{P}\text{-}\mathsf{alg}}(A, \mathcal{S})$. This action commutes with that of the group $\mathtt{Aut}(\mathcal{S}, \mathtt{tr})$, preserves the symplectic form ω_{Rep}, but it is *not* Hamiltonian, in general. That means that the induced Lie algebra morphism $\xi : \mathtt{Der}_{\mathcal{P}} A \longrightarrow \mathtt{Der}_{\omega_{\mathsf{Rep}}}\big(\mathsf{Rep}(A, \mathcal{S})^{\mathtt{Aut}(\mathcal{S},\mathtt{tr})}\big)$ cannot be lifted, in general, to a Lie algebra morphism:

$$\mathtt{Der}_{\mathcal{P}} A \ \longrightarrow \ \mathsf{Rep}(A, \mathcal{S})^{\mathtt{Aut}(\mathcal{S},\mathtt{tr})}\,, \quad \text{see (6.6).}$$

The following result shows that the $\mathtt{Aut}(A)$-action becomes Hamiltonian after a 1-dimensional central extension. The result below agrees also with the philosophy advocated in [KR], saying that, for any finite-dimensional (symmetric) $\mathcal{P}$-algebra $\mathcal{S}$, 'functions' on the non-commutative space corresponding to a $\mathcal{P}$-algebra A should go into genuine regular functions on the affine algebraic variety $\mathtt{Hom}_{\mathcal{P}\text{-}\mathsf{alg}}(A, \mathcal{S})$.

Theorem 6.7. *There is a natural Lie algebra homomorphism* $\widehat{\mathsf{tr}} : \mathsf{R}(A) \to \mathsf{Rep}(A, \mathcal{S})^{\mathtt{Aut}(\mathcal{S},\mathtt{tr})}$ *making the following diagram commute:*

$$\begin{tikzcd}
0 \arrow[r] & \Bbbk \arrow[r] \arrow[d, equal] & \mathsf{R}(A) \arrow[r, "6.5"] \arrow[d, "\hat{\mathsf{tr}}"'] & \mathtt{Der}_{B}(A, \omega) \arrow[r] \arrow[d, "\xi"'] & 0 \\
0 \arrow[r] & \Bbbk \arrow[r] & \mathsf{Rep}(A, \mathcal{S})^{\mathtt{Aut}(\mathcal{S},\mathtt{tr})} \arrow[r, "(6.6)"] & \mathtt{Der}_{\omega_{\mathsf{Rep}}}\big(\mathsf{Rep}(A, \mathcal{S})^{\mathtt{Aut}(\mathcal{S},\mathtt{tr})}\big) \arrow[r] & 0
\end{tikzcd}$$

Proof. Very similar to the proof of Proposition 3.4. ☐

Given an action of an algebraic group on a smooth affine algebraic variety X, let $\mathcal{DR}^{\bullet}_{G}(X)$ denote the G-equivariant algebraic De Rham complex of X, computing the equivariant cohomology of X.

Problem. Construct a natural morphism of complexes:
$\mathsf{DR}^{\bullet}A \to \mathcal{DR}^{\bullet}_{\mathtt{Aut}(\mathcal{S},\mathtt{tr})}\big(\mathtt{Hom}_{\mathcal{P}\text{-}\mathsf{alg}}(A, \mathcal{S})\big)$ that gives an equivariant lifting of the morphism: $\mathsf{DR}^{\bullet}A \longrightarrow \mathcal{DR}^{\bullet}\big(\mathtt{Hom}_{\mathcal{P}\text{-}\mathsf{alg}}(A, \mathcal{S})\big)$, constructed in the Remark at the end of §2.

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