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SPIN 4-MANIFOLDS WITH SIGNATURE$= -32$

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Abstract. We show that if X is a closed spin 4-manifold with $\mathrm{sign}(X) = -32$, then X satisfies $b_2(X) \geq 44 = (11/8)|\mathrm{sign}(X)|$.

1. Introduction

Let X be a closed spin 4-manifold. We denote by $b_i(X)$ the i-th Betti number of X. We write $b_2^+(X)$ (resp. b_2^-) for the maximal dimension of the positive (resp. negative) definite subspace of $H^2(X, \mathbf{R})$ with respect to the intersection form of X. The signature of X is defined as $\mathrm{sign}(X) := b_2^+(X) - b_2^-(X)$. Rohlin's theorem implies that $\mathrm{sign}(X)$ is divisible by 16.

The first author showed that if $\mathrm{sign}(X)$ is not zero, then the inequality $b_2^+(X) \geq 2(-\mathrm{sign}(X)/16) + 1$ holds. It implies that if $\mathrm{sign}(X) = -32$, then $b_2^+(X) \geq 5$. The purpose of this note is to improve this inequality.

Theorem 1. *Suppose that X is a closed spin 4-manifold with $\mathrm{sign}(X) = -32$. Then X satisfies that $b_2^+(X) \geq 6$.*

We give three proofs of the above theorem.

The first proof depends on Theorem 22 of [6]. The second proof is more fundamental and uses stable-homotopy version of Seiberg-Witten invariant. The third proof is essentially a translation of the second proof by using the language of spin cobordism.

In 1994 P. B. Kronheimer gave a lecture discussing the inequality in Theorem 1 [7]. The formulation of the third proof is motivated by his argument.

We also give some applications of the above theorem. Some generalizations along the line of this note is obtained in [5].

Alternative approaches for generalizations of Theorem 1 are discussed by N. Minami [8], and the first and the second authors [4].(See Section 3.3.)

2. The first two proofs

We will give two proofs of Theorem 1.

By using the surgery along homologically nontrivial loops, we can assume $b_1(X) = 0$ without loss of generality.

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2.1. The first proof: adjunction inequality. Suppose there exists a spin 4-manifold X which satisfies

$$b_1(X) = 0, \quad -\frac{\mathrm{sign}(X)}{16} = 2, \quad b_2^+(X) = 5.$$

Let Y be the connected sum $X \# S^2 \times S^2$. Then Y has the same rational cohomology ring as $K3 \# K3$. From Theorem 22 of [6], any embedded oriented closed surface Σ in Y satisfies the generalized adjunction inequality

$$\max(2g(\Sigma) - 2, 0) \geq [\Sigma] \cdot [\Sigma].$$

On the other hand, there exists an embedded sphere in $S^2 \times S^2$ with arbitrary large self-intersection number. Since it implies that Y has the same property, this contradicts the generalized adjunction inequality. □

2.2. The second proof: the stable-homotopy Seiberg-Witten invariant I. Let s be a spin structure on X. We put

$$k := -\frac{\mathrm{sign}(X)}{16}, \quad l := b_2^+(X).$$

We would like to show that if $k = 2$ then $l \geq 6$.

Suppose $l \leq 5$. Then the stable-homotopy Seiberg-Witten invariant $SW(X, s)$ is an element of $\{S(\mathbf{H}^2), S(\tilde{\mathbf{R}}^l)\}^{Pin_2}$. (See Section 3.3 for its definition, and [6] for details.) So it suffices to show the following proposition:

Proposition 2. *If $l \leq 5$, then $\{S(\mathbf{H}^2), S(\tilde{\mathbf{R}}^l)\}^{Pin_2}$ is empty.*

Proof. By using the inclusion $\tilde{\mathbf{R}}^l \subset \tilde{\mathbf{R}}^6$, and by restricting the group action, we have a composition of maps:

$$\{S(\mathbf{H}^2), S(\tilde{\mathbf{R}}^l)\}^{Pin_2} \to \{S(\mathbf{H}^2), S(\tilde{\mathbf{R}}^6)\}^{Pin_2} \to \{S(\mathbf{C}^4), S(\mathbf{R}^6)\}^{S^1} \cong \mathbf{Z}_2\,.$$

Proposition 16 of [6] implies that the image of the composition is $\{1\}$, if not empty.

On the other hand, the image of the composition

$$\{S(\mathbf{H}^2), S(\tilde{\mathbf{R}}^l)\}^{Pin_2} \to \{S(\mathbf{C}^4), S(\mathbf{R}^l)\}^{S^1} \to \{S(\mathbf{C}^4), S(\mathbf{R}^6)\}^{S^1} \cong \mathbf{Z}_2$$

is $\{0\}$, if not empty, since $S(\mathbf{R}^l)$ is contractible in $S(\mathbf{R}^6)$.

Since the above two compositions are the same map, we have a contradiction if $\{S(\mathbf{H}^2), S(\tilde{\mathbf{R}}^l)\}^{Pin_2}$ is non-empty. □

3. The third proof

We will give a direct proof of Theorem 1.

3.1. Spin structures on surface with free involution.

Lemma 3. *Let M be an oriented surface, possibly with boundary. Let s_M be a spin structure on M. Suppose M has a free $\mathbf{Z}_2$-action preserving the orientation. Let $\iota : M \to M$ be the action of the non-trivial element of $\mathbf{Z}_2$. We assume that ι lifts to an automorphism $\tilde{\iota}$ of s_M which satisfies $\tilde{\iota}^2 = -\mathrm{id}_{s_M}$. We write $\bar{M}$ for the quotient space M/ι and $p : M \to \bar{M}$ for the quotient map.*

Let $u \in H^1(\bar{M}, \mathbf{Z}_2)$ be the element classifying the double covering p. Then there is a bijection between the set of spin structures on $\bar{M}$ and the set of $\mathbf{Z}_4$-lifts of u:

$$\{w \in H^1(\bar{M}, \mathbf{Z}_4) \mid u = w \bmod 2\}.$$

Let $s^w_{\bar{M}}$ be the spin structure on $\bar{M}$ corresponding to w. Then we have the relation

$$p^* s^w_{\bar{M}} - s_M = \frac{p^* w}{2} \bmod 2 \in H^1(M, \mathbf{Z}_2).$$

Remark 4.

1. We used the following notation: the Bockstein exact sequence implies that the natural map
$$H^1(A, \mathbf{Z}_2) \to \operatorname{Ker}(H^1(A, \mathbf{Z}_4) \to H^1(A, \mathbf{Z}_2))$$
is an isomorphism for any space A. We write $\bullet/2 \bmod 2$ for the inverse map of this isomorphism.
2. The bijection is well-defined up to sign. More precisely, if we fix one of the two lifts of ι to s_M, then the bijection is well-defined.

Proof of Lemma 3. Let $\tilde{\mathbf{R}}$ be the real line bundle on $\bar{M}$ such that $w_1(\tilde{\mathbf{R}}) = u$. We write $\tilde{\mathbf{C}}$ for $\tilde{\mathbf{R}} \otimes \mathbf{C}$. We first show that $T\bar{M} \oplus \tilde{\mathbf{C}}$ has a canonical spin structure, which implies that the spin structures on $\bar{M}$ is in one-to-one correspondence with the spin structures on $\tilde{\mathbf{C}}$.

Let $\mathbf{C}_1$ and $\mathbf{C}_2$ be two copies of the product bundle $M \times \mathbf{C}$. We regard the circle bundle $S(\mathbf{C}_2)$ as a trivial spin structure of $\mathbf{C}_1$ by using the square map $S(\mathbf{C}_2) \to S(\mathbf{C}_1)$, $w \mapsto w^2$. Let $\tilde{\iota}_1$ and $\tilde{\iota}_2$ be the lifts of ι to $\mathbf{C}_1$ and $S(\mathbf{C}_2)$ defined by

$$\begin{aligned} \tilde{\iota}_1 : \mathbf{C}_1 \to \mathbf{C}_1, &\qquad (x, z) \mapsto (\iota x, -z) \\ \tilde{\iota}_2 : S(\mathbf{C}_2) \to S(\mathbf{C}_2), &\qquad (x, w) \mapsto (\iota x, \sqrt{-1} w). \end{aligned}$$

Then $\tilde{\iota}_2$ is a lift of $\tilde{\iota}_1$. The square of $\tilde{\iota}_1$ is the identity and the quotient of $\mathbf{C}_1$ by the involution $\tilde{\iota}_1$ is identified with $\tilde{\mathbf{C}}$. Let S_M be the $Spin(2)$-bundle over M that corresponds to the spin structure s_M. A spin structure of $TM \oplus \mathbf{C}_1$ is given by $(S_M \times S(\mathbf{C}_2))/\pm 1$ via the inclusion $(Spin(2) \times Spin(2))/\pm 1 \subset Spin(4)$, where we write ± 1 for $\{(1,1), (-1,-1)\}$. We denote by s'_M this spin structure on $TM \oplus \mathbf{C}_1$, and by $\tilde{\iota}'$ the lift of ι to s'_M induced from the product $\tilde{\iota} \times \tilde{\iota}_2$. The square of $\tilde{\iota}'$ is the identity. It implies that s'_M descends to a spin structure on the quotient $T\bar{M} \oplus \tilde{\mathbf{C}}$.

Let w be a $\mathbf{Z}_4$-lift of u. We will construct a spin structure on $\tilde{\mathbf{C}}$ by using w.

Let α be the standard generator of $H^1(B\mathbf{Z}_4, \mathbf{Z}_4) \cong \mathbf{Z}_4$. Since $B\mathbf{Z}_4 = K(\mathbf{Z}_4, 1)$, the $\mathbf{Z}_4$-lift w of u corresponds to the homotopy class of a continuous map

$f : \bar{M} \to B\mathbf{Z}_4$ by the relation $w = f^*\alpha$. Let $q : E\mathbf{Z}_4/\mathbf{Z}_2 \to B\mathbf{Z}_4$ be the double covering corresponding to α mod 2. Then $p : M \to \bar{M}$ is the pullback of q by f, and hence $\tilde{\mathbf{C}}$ is canonically isomorphic to the pullback of the complex line bundle $(E\mathbf{Z}_4/\mathbf{Z}_2 \times \mathbf{C})/\mathbf{Z}_2$ over $B\mathbf{Z}_4$, where $\mathbf{Z}_2$ acts on $\mathbf{C}$ nontrivially. A spin structure of this complex line bundle over $B\mathbf{Z}_4$ is given by the $Spin(2)$-bundle $(E\mathbf{Z}_4 \times Spin(2))/\mathbf{Z}_4$ over $B\mathbf{Z}_4$, where the $\mathbf{Z}_4$-action on $Spin(2)$ is given by using the embedding $\mathbf{Z}_4 \subset Spin(2)$. We define $s^w_{\bar{M}}(\tilde{\mathbf{C}})$ to be the pullback of this spin structure by f.

The pullback of this spin structure by q is given by the $Spin(2)$-bundle $(E\mathbf{Z}_2 \times Spin(2))/\mathbf{Z}_2$ over $B\mathbf{Z}_2 = E\mathbf{Z}_4/\mathbf{Z}_2$, which is not isomorphic to the trivial spin structure on $B\mathbf{Z}_2 \times \mathbf{C}$. The difference of the spin structures of the former and the latter is given by the nontrivial element of $H^1(B\mathbf{Z}_2, \mathbf{Z}_2) \cong \mathbf{Z}_2$. Note that this nontrivial element is equal to

$$\frac{q^*\alpha}{2} \bmod 2.$$

Pulling back the equality to M, we obtain

$$p^* s^w_{\bar{M}}(\tilde{\mathbf{C}}) - s_M(\mathbf{C}) = \frac{p^* w}{2} \bmod 2,$$

where $s_M(\mathbf{C})$ is the trivial spin structure on the trivial bundle $M \times \mathbf{C}$.

If w and w' are two $\mathbf{Z}_4$-lifts of u, then the difference of the corresponding spin structures $s^w_{\bar{M}}$ and $s^{w'}_{\bar{M}}$ is given by the element $(w - w')/2$ mod 2 in $H^1(\bar{M}, \mathbf{Z}_2)$. This implies that all the spin structures on $\tilde{\mathbf{C}}$ appears by this construction. □

The following lemma is a variant of the above one.

Lemma 5. *Let B be a manifold, possibly with boundary. Let ι be a free involution on B and E an ι-equivariant real vector bundle over B. We write $\bar{B}$ (resp. $\bar{E}$) for the quotient of B (resp. E) by ι. Let u be the element of $H^1(\bar{B}, \mathbf{Z}_2)$ classifying the double covering $p : B \to \bar{B}$. Suppose the following conditions are satisfied.*

$$\dim B - \mathrm{rank} E = 2,\ w_1(\bar{E}) - w_1(\bar{B}) = 0,\ w_2(\bar{E}) - w_2(\bar{B}) = u^2.$$

Then we can take an "orientation" and a "spin structure" on the virtual bundle $TM - E$ so that the following properties are satisfied: Let s be an ι-invariant section of E transverse to the zero section. We write M for the zero set $s^{-1}(0)$ and $\bar{M}$ for the quotient M/ι.

1. *There exists a canonical orientation o_M and a canonical spin structure s_M on M.*
2. *The action of ι preserves o_M.*
3. *The involution ι lifts to an automorphism $\tilde{\iota}$ of s_M which satisfies $\tilde{\iota}^2 = -\mathrm{id}_{s_M}$.*
4. *There exists a canonical orientation $o_{\bar{M}}$ on $\bar{M}$ such that $p^* o_{\bar{M}} \cong o_M$.*

5. *There is a bijection between the set of spin structures on* $\bar{M}$ *and the set*
$$\{w \in H^1(\bar{M}, \mathbf{Z}_4) \mid u|\bar{M} = w \bmod 2\}.$$
such that if $s_{\bar{M}}^w$ *is the spin structure corresponding to* w*, then we have the relation*
$$p^* s_{\bar{M}}^w - s_M = \frac{p^* w}{2} \bmod 2 \in H^1(M, \mathbf{Z}_2). \tag{1}$$

Remark 6. 1. B or E could be non-orientable.

2. Let $\tilde{\mathbf{C}}$ be the complex line bundle over $\bar{B}$ defined to be the quotient of $B \times \mathbf{C}$ where ι acts as -1 on $\mathbf{C}$. Then the assumption of the above lemma is equivalent to the following conditions for the element $\kappa := [TB - E - \tilde{\mathbf{C}}]$ in $KO(\bar{B})$:
$$\mathrm{rank}(\kappa) = 0, \quad w_1(\kappa) = 0, \quad w_2(\kappa) = 0.$$

Proof of Lemma 5. Let $\tilde{\mathbf{C}}$ be the complex line bundle over $\bar{B}$ defined in Remark 6. Take and fix a real vector bundle $\bar{F}$ over $\bar{B}$ satisfying
$$w_1(\bar{F}) = w_1(\bar{E}), \quad w_2(\bar{F}) = w_1(\bar{E})^2 + w_2(\bar{E}).$$
For instance we can take $\bar{F} = \bar{E} \oplus \bar{E} \oplus \bar{E}$. Let $F = p^*\bar{F}$ be the pullback of $\bar{F}$ on M.

1. Since $w_1(\bar{E} \oplus \bar{F})$ and $w_2(\bar{E} \oplus \bar{F})$ vanish, $\bar{E} \oplus \bar{F}$ is orientable and spin. Take and fix an orientation and a spin structure on $\bar{E} \oplus \bar{F}$.
2. Since we have $w_1(\tilde{\mathbf{C}}) = 0$ and $w_2(\tilde{\mathbf{C}}) = u^2$, $w_1(T\bar{B} \oplus \bar{F} \oplus \tilde{\mathbf{C}})$ and $w_2(T\bar{B} \oplus \bar{F} \oplus \tilde{\mathbf{C}})$ vanish, and hence $T\bar{B} \oplus \bar{F} \oplus \tilde{\mathbf{C}}$ is orientable and spin. Take and fix an orientation and a spin structure on $T\bar{B} \oplus \bar{F} \oplus \tilde{\mathbf{C}}$.

Note that we have the following relations.
$$\begin{aligned} (TB \oplus F) \oplus (B \times \mathbf{C}) &= p^*(T\bar{B} \oplus \bar{F} \oplus \tilde{\mathbf{C}}) \\ E \oplus F &= p^*(\bar{E} \oplus \bar{F}), \end{aligned}$$
By pulling back the spin structures on $T\bar{B} \oplus \bar{F} \oplus \tilde{C}$ and $\bar{E} \oplus \bar{F}$, we obtain the pullback spin structures on the left-hand sides. Using the trivial orientation and the trivial spin structure on $B \times \mathbf{C}$, we obtain the spin structure on $TB \oplus F$. By stabilizing the isomorphism $T\bar{M} \oplus \bar{E}|\bar{M} \cong T\bar{B}|\bar{M}$, we have the isomorphism
$$(T\bar{M} \oplus \tilde{\mathbf{C}}|\bar{M}) \oplus (\bar{E} \oplus \bar{F})|\bar{M} \cong (T\bar{B} \oplus \bar{F} \oplus \tilde{\mathbf{C}})|\bar{M}.$$
It implies that $T\bar{M} \oplus \tilde{\mathbf{C}}|\bar{M}$ has an induced orientation and an induced spin structure. By pulling back this orientation and the spin structure, we have an orientation and a spin structure on $TM \oplus (M \times \mathbf{C})$. By using the trivial orientation and the trivial spin structure on $M \times \mathbf{C}$, we have an orientation o_M and a spin structure s_M on M.

The claims 1, 2 and 4 immediately follow from the above construction. The induced $\mathbf{Z}_2$-action on $M \times \mathbf{C}$ cannot lift to the trivial spin structure on $M \times \mathbf{C}$. On the other hand the $\mathbf{Z}_2$-action lifts to the spin structure on $TM \oplus (M \times \mathbf{C})$. Thus we have the claim 3. The claim 5 is a corollary of Lemma 3. □

Corollary 7. *We use the notations of Lemma 5. Suppose M is a compact surface with boundary. We assume the following two conditions.*

1. *Each boundary component C is not preserved by ι as a set: $\iota C \cap C = \emptyset$.*
2. *The restriction of s_M on each boundary component is not the null-cobordant spin structure.*

Then the number of the boundary components of M is divisible by 4.

Proof. We first show that we can take a $\mathbf{Z}_4$-lift w of u so that the restriction of w on the boundary of $\bar{M}$ is trivial.

Collapse each boundary component of $\bar{M}$ to get an oriented closed surface $\hat{M}$. Since the restriction of u on each boundary component of $\bar{M}$ is trivial, we can extend u to an element $\hat{u}$ of $H^1(\hat{M}, \mathbf{Z}_2)$. Since $\hat{M}$ is torsion free, we can take a $\mathbf{Z}_4$-lift $\hat{w}$ of $\hat{u}$. Then we can take w as the pullback of $\hat{w}$. From this construction the restriction of w on the boundary of $\bar{M}$ is trivial.

The relation (1) implies that the pullback $p^* s^w_{\bar{M}}$ is isomorphic to s_M on the neighborhood of the boundary of M. Hence the restriction of $s^w_{\bar{M}}$ on each boundary component $\bar{C}$ is not the null-cobordant spin structure. It implies that the number of the boundary components of $\bar{M}$ is even. ☐

3.2. Equivariant map. We use the following notations.

1. $Sp_1 = \{q \in \mathbf{H} \mid |q| = 1\}$
2. $Pin_2 = \{\cos\theta + i \sin\theta\}_{0 \le \theta < 2\pi} \cup \{j \cos\phi + k \sin\phi\}_{0 \le \phi < 2\pi} \subset Sp_1$
3. We regard $\mathbf{H}$ as a right Pin_2 module by the right multiplication.
4. We regard $\operatorname{Im}\mathbf{H}$ as a Pin_2 module by the conjugation.
5. Let $\tilde{\mathbf{R}}$ be the non-trivial 1-dimensional real representation of $Pin_2/S^1 = \{\pm 1\}$.

Note that $\operatorname{Im}\mathbf{H}$ is isomorphic to $\tilde{\mathbf{R}}^3$ as Pin_2-module. Let V_0, V_1, W_0 and W_1 be four finite dimensional right Pin_2 modules which satisfy the following conditions.

1. Any irreducible submodule of V_0 or V_1 is isomorphic to $\mathbf{H}$. In other words they are quaternionic vector spaces.
2. $\dim_{\mathbf{H}} V_0 - \dim_{\mathbf{H}} V_1 = 2$.
3. Any irreducible submodule of W_0 or W_1 is isomorphic to $\tilde{\mathbf{R}}$.
4. $\dim_{\mathbf{R}} W_0 - \dim_{\mathbf{R}} W_1 = -5$.

Proposition 8. *There exists no Pin_2-equivariant map from $S(V_0 \oplus W_0)$ to $S(V_1 \oplus W_1)$.*

Proof. Suppose there is a Pin_2-equivariant map

$$f_1 : S(V_0 \oplus W_0) \to S(V_1 \oplus W_1).$$

Fix isomorphisms

$$V_0 \cong V_1 \oplus \mathbf{H}^2, \quad (j\mathbf{R} + k\mathbf{R}) \oplus (j\mathbf{R} + k\mathbf{R}) \oplus i\mathbf{R} \oplus W_0 \cong W_1$$

as Pin_2-modules. Let f_0 be the Pin_2-equivariant map

$$f_0 : S(V_0 \oplus W_0) \to V_1 \oplus W_1$$

defined to be

$$f_0((v,q_0,q_1)\oplus w) = v\oplus((q_0 i\bar{q_0})_{jk},(q_1 i\bar{q_1})_{jk},(q_0 i\bar{q_0})_i-(q_1 i\bar{q_1})_i),w),$$

where $(q)_{jk}$ (resp. $(q)_i$) is the jk-component (resp. the i-component) of the quaternion q.

Connect f_0 and f_1 by a generic Pin_2-equivariant one-parameter path

$$f_t : S(V_0\oplus W_0)\to V_1\oplus W_1.$$

Since $\dim S(W_0)+1 < \dim W_1$, the zero set $M = f_t^{-1}(0)$ does not intersect $S(0\oplus W_0)$.

$$\begin{array}{rcl} B & := & (S(V_0\oplus W_0)\setminus S(0\oplus W_0))/U(1)\times[0,1] \\ \bar{B} & := & (S(V_0\oplus W_0)\setminus S(0\oplus W_0))/Pin_2\times[0,1] \\ E & := & (S(V_0\oplus W_0)\setminus S(0\oplus W_0))\times_{U(1)}(V_1\oplus W_1)\times[0,1] \\ \bar{E} & := & (S(V_0\oplus W_0)\setminus S(0\oplus W_0))\times_{Pin_2}(V_1\oplus W_1)\times[0,1] \\ s & := & \coprod_t f_t \end{array}$$

By using the Leray-Hirsh theorem for the fiber bundle

$$\mathbf{RP}^2\to S(V_0)/Pin_2\to\mathbf{HP}(V_0),$$

it is easy to calculate the Stiefel-Whitney classes of $\bar{B}$ and $\bar{E}$. In particular we have

$$w(\bar{B}) \quad = \quad w(\bar{E})(1+u)^{\dim W_0-\dim W_1+3} = w(\bar{E})(1+u^2) \quad \text{up to degree 2.}$$

So we can apply Lemma 5.

The boundary $\partial M = f_0^{-1}(0)$ of M consists of two components

$$\{(z_0,z_1,0)\mid z_0,z_1\in U(1)\}/U(1),\quad \{(jz_0,jz_1,0)\mid z_0,z_1\in U(1)\}/U(1).$$

The action of j exchanges these two components. Since B is simply connected, the "spin structure" of the virtual bundle $TB-E$ is unique. Let us introduce another U(1)-action on B as follows:

$$z: B\to B,\quad [(v,q_0,q_1)\oplus w]\mapsto[(v,q_0,q_1 z)\oplus w].$$

Then it is easy to see that this U(1)-action lifts to the spin structure of ∂M. It implies that the spin structure on each boundary component is the Lie group spin structure, which is not null-cobordant. Now we can apply Corollary 7 to this situation. Since the number of the boundary components is two, this is a contradiction. □

3.3. The third proof: the stable-homotopy Seiberg-Witten invariant II. Now we give the third proof of Theorem 1. Suppose there exists a 4-manifold X which satisfies all the conditions below.

1. X is an oriented spin closed 4-manifold.
2. $b_1(X) = 0, \quad \mathrm{sign}(X) = -32, \quad b_2^+(X) = 5.$

The stable-homotopy Seiberg-Witten invariant of X is an element of

$$\{S(\mathbf{H}^2), S(\tilde{\mathbf{R}}^5)\}^{Pin_2}$$

which is defined to be the inductive limit of the set of the homotopy classes of Pin_2-equivariant maps from $S(V_0 \oplus W_0)$ to $S(V_1 \oplus W_1)$ satisfying the conditions of the previous subsection. This contradicts Proposition 8. □

Remark 9. Let $\mathbf{Z}_4$ be the subgroup of Pin_2 generated by j. Let n and m be any natural numbers. In [9] S. Stolz has determined the necessary and sufficient condition for some $\mathbf{Z}_4$-equivariant map from $S(\mathbf{H}^n)$ to $S(\tilde{\mathbf{R}}^m)$ to exist. For example he showed that there exists no $\mathbf{Z}_4$-equivariant map from $S(\mathbf{H}^2)$ to $S(\tilde{\mathbf{R}}^5)$. If one can extend Stolz's result also for maps stabilized by direct sums of some copies of $\tilde{\mathbf{R}}$ and $\mathbf{H}$, then Proposition 8 would immediately follow. However it does not seem straightforward to extend and apply Stolz's method directly to the stable case. It is still possible to prove the stable version of the result of Stolz at least in different two ways: One method is given by N. Minami [8]. He showed more general result which enables us to reduce the stable version to the absolute case. The other method is given by the first and the second authors [4]. This method uses an extension of Adams' e-invariant and gives an alternative proof of the non-existence part of the result of Stolz[1].

4. Applications

Theorem 10. *Suppose X is a spin 4-manifold which has the same rational cohomology ring as $K3\#K3$. If X is of the form of the connected sum $X_0\#X_1$, then one of the following three cases occur.*

1. *X_0 is a rational homology 4-sphere.*
2. *X_1 is a rational homology 4-sphere.*
3. *X_0 and X_1 have the same rational cohomology ring as $K3$.*

Proof. X_i satisfies

$$b_1(X_i) = 0, \quad 16|\mathrm{sign}(X_i), \quad 44 \geq b_2(X_i) \geq \frac{11}{8}|\mathrm{sign}(X_i)|.$$

The last inequality comes from the 11/8 estimate for the spin manifolds with $|\mathrm{sign}| \leq 32$. These properties imply one of the above three possibilities. □

[1]Added in proof: After the submission of this paper for publication, Minami pointed out to the authors that M. C. Crabb's method in *Periodicity in $\mathbf{Z}/4$-equivariant stable homotopy theory*, Contemp. Math. **96** (1989), 109–124, would give a proof of the stabilized version.

Theorem 11. *Suppose X is an oriented closed spin 4-manifold which has the same rational cohomology ring as $K3\#K3$. Let α be a non-torsion class of $H_2(X, \mathbf{Z})$ with $\alpha^2 = 0$. Then α cannot be realized as the fundamental class of any embedded sphere.*

Proof. Suppose $\alpha^2 = 0$ and α is realized by an embedded sphere Σ. Then the neighborhood of Σ is diffeomorphic to $S^2 \times D^2$. Using the surgery along Σ, we obtain a 4-manifold Y. Since both of the two spin structures of $\partial(S^2 \times D^2)$ can be extended to $D^3 \times S^1$, we have a spin structure on Y. Since α is not a torsion class, Y satisfies that $b_2^+(Y) = 5$, $\mathrm{sign}(Y) = -32$. This contradicts Theorem 1. □

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