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MONOPOLE EQUATION AND THE $\frac{11}{8}$-CONJECTURE

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1. Introduction

Let M be a 4-dimensional oriented closed spin manifold. We write $b_l(M)$ for the l-th Betti number of M and $\operatorname{sign}(M)$ for the signature of M. In [13] Y. Matsumoto conjectured the following inequality.

Conjecture.
$$b_2(M) \geq \frac{11}{8}|\operatorname{sign}(M)|.$$

This conjecture is now known as *the 11/8 conjecture.* Since the K3 surface satisfies the equality, the coefficient 11/8 cannot be replaced by a larger number. In this paper we show a weaker inequality.

Theorem 1. *If the intersection form of M is not definite, the following inequality is satisfied.*
$$b_2(M) \geq \frac{5}{4}|\operatorname{sign}(M)| + 2.$$

Note that if the intersection form of M is definite, a theorem of S.K. Donaldson implies $b_2(M) = \operatorname{sign}(M) = 0$ ([6,7]).

V. A. Rohlin's theorem implies that $k = -\operatorname{sign}(M)/16$ is an integer ([15]). Let b_+ be the dimension of a maximal positive definite subspace of $H_2(M, \mathbb{Q})$. The 11/8-conjecture is equivalent to the inequality $3k \leq b_+$. The inequality in Theorem 1 is equivalent to $2k + 1 \leq b_+$.

Donaldson's theorem mentioned above says that $k \geq 1$ implies $b_+ \geq 1$. Donaldson also proved that $k \geq 1$ implies $b_+ \geq 3$ when $H_1(M, \mathbb{Z})$ is 2-torsion free ([6,7]). To obtain these estimates he used moduli spaces of instantons on M with small instanton numbers. In 1994 P. B. Kronheimer explained, in his lecture in Cambridge, how to use N. Seiberg and E. Witten's monopole equation to get similar results ([10]). In particular he showed that $k \geq 1$ always implies $b_+ \geq 3$ without any condition on $H_1(M, \mathbb{Z})$.

In this paper we consider moduli spaces of monopoles following Kronheimer's lecture. Our key idea is to use a finite dimensional approximation of the monopole equation.

We formulate the monopole equation in Section 2 so that we can see the Pin_2-symmetry of the equation explicitly. The finite dimensional approximation of

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the equation is constructed in Section 3 and its Pin_2-symmetry is described in Section 4. We prove Theorem 1 in Section 5 by using equivariant K-theory. In Appendix we summarize some elementary properties of the Adams operations used in Section 5.

Since the preprint version of this article appeared in 1995, there have been some related works. See [13], [4] and [8].

2. Monopole equation

To show Theorem 1, we can assume (1) $\mathrm{sign}(M)$ is non-positive and (2) $b_1(M) = 0$ without loss of generality, since (1) if we reverse the orientation of M, then the sign of $\mathrm{sign}(M)$ changes and (2) if $b_1(M)$ is not zero, then we can construct, by using surgery along non-trivial loops, another spin manifold with its first Betti number zero and with the same second Betti number and signature. We assume these two conditions in the rest of this paper.

In this section we formulate the monopole equation for the spin structure of M. Let $\mathbb{H}$ be the quaternion numbers and Sp_1 be the group of the quaternions with norm 1. The monopole equation has an S^1-symmetry from definition, where S^1 is the intersection of Sp_1 with $\mathbb{C}$ in $\mathbb{H}$. Let Pin_2 be the normalizer of S^1 in Sp_1. An important aspect of the equation is the equation has actually a Pin_2-symmetry ([11,16]). This extra symmetry will play a crucial role in the proof of Theorem 1.

Recall that Spin_4 is isomorphic to the product of two copies of Sp_1. We define four $\mathrm{Spin}_4 \times \mathrm{Pin}_2$-modules ${}_-\mathbb{H}_+$, ${}_+\mathbb{H}$, ${}_-\mathbb{H}$ and ${}_+\mathbb{H}_+$ as follows. As vector spaces, they are just four copies of $\mathbb{H}$. The actions of $(q_-, q_+, q_0) \in \mathrm{Spin}_4 \times \mathrm{Pin}_2 = Sp_1 \times Sp_1 \times \mathrm{Pin}_2$ on $a \in {}_-\mathbb{H}_+$, $\phi \in {}_+\mathbb{H}$, $\psi \in {}_-\mathbb{H}$ and $\omega \in {}_+\mathbb{H}_+$ are defined by $q_- a {q_+}^{-1}$, $q_+ \phi {q_0}^{-1}$, $q_- \psi {q_0}^{-1}$ and $q_+ \omega {q_+}^{-1}$ respectively.

For a principal Spin_4-bundle P on the 4-manifold M, we have four associated vector bundles T, S^+, S^- and Λ from the $\mathrm{Spin}_4 \times \mathrm{Pin}_2$-modules ${}_-\mathbb{H}_+$, ${}_+\mathbb{H}$, ${}_-\mathbb{H}$ and ${}_+\mathbb{H}_+$. Then they are Pin_2-equivariant vector bundles. Let $\tilde{\mathbb{R}}$ be the nontrivial real 1-dimensional Pin_2-module defined by the multiplication of $\mathrm{Pin}_2 / S^1 = \{\pm 1\}$ and we write $\tilde{E} = E \otimes \tilde{R}$ for Pin_2-modules or Pin_2-equivariant vector bundles E. We shall consider $\tilde{T}$ and $\tilde{\Lambda}$ associated to ${}_-\tilde{\mathbb{H}}_+$ and ${}_+\tilde{\mathbb{H}}_+$.

Recall that a spin structure of M is a pair of a principal Spin_4-bundle P and an isomorphism $TM \cong T$. When we fix this isomorphism, M has a canonical orientation and a canonical Riemannian metric.

The $\mathrm{Spin}_4 \times \mathrm{Pin}_2$-equivariant map ${}_-\mathbb{H}_+ \times {}_+\mathbb{H} \to {}_-\mathbb{H}$ defined by $(a, \phi) \mapsto a\phi$ induces the Clifford multiplication $C \colon T \otimes S^+ \to S^-$. Similarly the $\mathrm{Spin}_4 \times \mathrm{Pin}_2$-equivariant map ${}_-\mathbb{H}_+ \times {}_-\tilde{\mathbb{H}}_+ \to {}_+\tilde{\mathbb{H}}_+$ defined by $(a, b) \mapsto \bar{a}b$ induces the twisted Clifford multiplication $\bar{C} \colon T \otimes \tilde{T} \to \tilde{\Lambda}$. From the construction, C and $\bar{C}$ are Pin_2-equivariant.

The Riemannian connection induces the covariant derivatives ∇_1 on $\Gamma(S^+)$ and ∇_2 on $\Gamma(\tilde{T})$. Let D_1 and D_2 be the (twisted) Dirac operators

$$D_1 = C\nabla_1 \colon \Gamma(S^+) \to \Gamma(S^-) \quad \text{and} \quad D_2 = \bar{C}\nabla_2 \colon \Gamma(\tilde{T}) \to \Gamma(\tilde{\Lambda}).$$

We write D for the direct sum of D_1 and D_2:

$$D = D_1 \oplus D_2 \colon \Gamma(S^+ \oplus \tilde{T}) \to \Gamma(S^- \oplus \tilde{\Lambda}).$$

The operators D_1, D_2 and D are Pin_2-equivariant.

We also need a Pin_2-equivariant quadratic map

$$Q \colon S^+ \oplus \tilde{T} \to S^- \oplus \tilde{\Lambda}$$

induced from the $\mathrm{Spin}_4 \times \mathrm{Pin}_2$-equivariant map

$$_+\mathbb{H} \times {_-\tilde{\mathbb{H}}_+} \to {_-\mathbb{H}} \times {_+\tilde{\mathbb{H}}_+}, \qquad (\phi, a) \mapsto (a\phi i, \phi i \bar{\phi}).$$

Then the monopole equation we shall consider is the nonlinear Pin_2-equivariant map

$$D + Q \colon V \to W,$$

where V is the L^2_4-completion of $\Gamma(S^+ \oplus \tilde{T})$ and W is the L^2_3-completion of $\Gamma(S^- \oplus \tilde{\Lambda})$. Note that if u is in L^2_4, then Du is in L^2_3 and Qu is in L^2_4.

Remark. (1) The twisted Clifford multiplication $\bar{C} \colon T \otimes \tilde{T} \to \tilde{\Lambda}$. is identified with $T^*M \otimes T^*M \to \mathbb{R} \oplus \Lambda^+$ defined by $(a, b) \mapsto \langle a, b \rangle \oplus p^+(a \wedge b)$. Here $\langle \cdot, \cdot \rangle$ is the inner product, Λ^+ is the self-dual part of $\Lambda^2 T^*M$ and $p^+ \colon \Lambda^2 T^*M \to \Lambda^+$ is the orthogonal projection. The real part and the imaginary part of $\tilde{\Lambda}$ are identified with $\mathbb{R}$ and Λ^+ respectively. Since $\phi i \bar{\phi}$ is purely imaginary, the image of Q is contained in $S^- \oplus \Lambda^+$.

(2) The twisted Dirac operator $D_2 \colon \Gamma(T) \to \Gamma(\tilde{\Lambda})$ is identified with $d^* + d^+ \colon \Omega^1 \to \Omega^0 \oplus \Omega^+$. Here Ω^k is the k-th forms on M, Ω^+ is the self-dual 2-forms, d^* is the formal adjoint of the exterior derivative $d \colon \Omega^0 \to \Omega^1$, and d^+ is the composition of $d \colon \Omega^1 \to \Omega^2$ with $p^+ \colon \Omega^2 \to \Omega^+$. In particular we have $\mathrm{Ker}\ D_2 = H^1(M, \mathbb{R}) = 0$ if $b_1(M) = 0$, and also $\mathrm{Coker}\ D_2 = H^0(M, \mathbb{R}) \oplus H^+(M, \mathbb{R})$, where $H^+(M, \mathbb{R})$ is the space of the self-dual harmonic 2-forms. Hence the index of D_2 is equal to $-(1 + b^+)$.

(3) Let $\mathcal{A}$ be the set of smooth connections on a complex line bundle L on M. The monopole equation (for a spin manifold M) is usually defined as a map

$$\mathcal{F} \colon \Gamma(S^+ \otimes L) \times \mathcal{A} \to \Gamma((S^- \otimes L) \oplus \Lambda^+), \quad \mathcal{F}(\phi, A) = (D_A \phi, F_A^+ + \phi i \bar{\phi}).$$

Here D_A is the Dirac operator twisted by A and F_A^+ is the self-dual part of the curvature F_A of A. The gauge group $\mathcal{G} = \Gamma(M, S^1)$ of L naturally acts on $\Gamma(S^+ \otimes L) \times \mathcal{A}$ and $\Gamma((S^- \otimes L) \oplus \Lambda^+)$. The map $\mathcal{F}$ is $\mathcal{G}$-equivariant. Instead of dividing out by $\mathcal{G}$, we can take the slice at a base point A_0 in $\mathcal{A}$. Note that $\mathcal{A} = A_0 + \Omega^1 i$. The slice is given by $A_0 + \mathrm{Ker}(d^* \colon \Omega^1 \to \Omega^0)\, i$. In particular when L is the trivial bundle $M \times \mathbb{C}$ and A_0 is the trivial flat connection on it, $\mathcal{F}$ together with the equation of cutting slice is identified with $D + Q$. (4) When we restrict $\mathcal{F}$ on the slice, the $\mathcal{G}$-symmetry is reduced to the symmetry of the stabilizer of A_0. The stabilizer is equal to the group of the harmonic maps $\mathrm{Harm}(M, S^1)$ from M to S^1 whose group structure is induced from that of S^1. (Here we do not have to take L to be trivial.) If we assume $b_1(M) = 0$, then the harmonic maps are constant, hence we have just an S^1-symmetry.

(5) When L is trivial and A_0 is the trivial flat connection, we have an extra symmetry explained earlier. Actually before taking the slice, the map has the symmetry of $\langle \mathrm{Map}(M, S^1), j\rangle$, which is a subgroup of $\mathrm{Map}(M, Sp_1)$. Here j is the constant map with value j. Restricted on the slice, the symmetry becomes $\langle \mathrm{Harm}(M, S^1), j\rangle$.

3. Finite dimensional approximation

An important property of the monopole equation is compactness of the moduli space of solutions.

Lemma 3.1([11, 16]). *The zero set of $D + Q$ is compact.*

We introduce the norms of V and W explicitly. Let $\|\cdot\|_V$ and $\|\cdot\|_W$ be the L^2_4-norm on V and the L^2_3-norm on W defined by

$$\|v\|_V^2 = \int_M (|(D^*D)^2 v|^2 + |v|^2), \quad \text{and} \quad \|w\|_W^2 = \int_M (|(DD^*)^{\frac{3}{2}} w|^2 + |w|^2)$$

respectively. Here D^* is the formal adjoint of D. More explicitly, D^* is the sum of $D_1^*\colon \Gamma(S^-) \to \Gamma(S^+)$ and $D_2^*\colon \Gamma(\tilde{\Lambda}) \to \Gamma(\tilde{T})$, where D_1^* and D_2 are the formal adjoint of D_1 and D_2, and identified with the (twisted) Dirac operators associated to the (twisted) Clifford multiplications $T \otimes S^- \to S^+$ and $T \otimes \tilde{\Lambda} \to \tilde{T}$ which are induced from the $\mathrm{Spin}_4 \times \mathrm{Pin}_2$-equivariant maps ${}_-\mathbb{H}_+ \times {}_-\mathbb{H} \to {}_+\mathbb{H}$, $(a, \psi) \mapsto \bar{a}\psi$ and ${}_-\mathbb{H}_+ \times {}_+\tilde{\mathbb{H}}_+ \to {}_-\tilde{\mathbb{H}}_+$, $(a, \omega) \mapsto a\omega$ respectively.

These norms are preserved by the Pin_2-action. From Lemma 3.1, if R is sufficiently large, then $(D + Q)v \neq 0$ for any $v \in V$ satisfying $\|v\|_V \geq R$. For this R, we have the following estimate.

Lemma 3.2. *There is a positive real number ϵ such that $\|(D + Q)v\|_W \geq \epsilon$ for any $v \in V$ with $\|v\|_V = R$.*

The proof of the lemma will be given later.

For a nonnegative real number λ, let V_λ be the subspace of V spanned by the eigenspaces of D^*D with eigenvalues less than or equal to λ. Similarly we define W_λ by using DD^*. We write p_λ for the L^2-orthogonal projection from W to W_λ, and p^λ for $1 - p_\lambda$.

Lemma 3.3. *For sufficiently large λ, we have the estimate $\|p^\lambda Q v\|_W < \epsilon$ for any $v \in V$ satisfying $\|v\|_V = R$.*

The proof of the lemma will be given later.

From the lemmas 3.2 and 3.3, the map $D + p_\lambda Q \colon V \to W$ has no zeros on the sphere with radius R centered in 0. The image of V_λ by the map $D + p_\lambda Q$ is contained in W_λ. We write

$$D_\lambda + Q_\lambda \colon V_\lambda \to W_\lambda$$

for this restriction, where D_λ is linear and Q_λ is quadratic. We consider this restriction as a finite dimensional approximation of $D + Q \colon V \to W$. A direct consequence of this construction is:

Lemma 3.4. *The map $D_\lambda + Q_\lambda$ has no zeros on the sphere with radius R centered in* 0.

The proofs of the lemmas 3.2 and 3.3 are immediate if we use the following facts which are easily shown from the Sobolev embedding theorem, Hölder's inequality, the elliptic estimate for D and the spectral decomposition of DD^*.

(1) For any bounded sequence $v_1, v_2, \cdots$ in V, there is a subsequence $v_1', v_2', \cdots$ weakly convergent to some v_∞ such that the sequence $Qv_1', Qv_2', \cdots$ is strongly convergent to Qv_∞ in W.

(2) If the sequence $v_1, v_2, \cdots$ is weakly convergent to v_∞ in V and the sequence $Dv_1, Dv_2, \cdots$ is strongly convergent to w_∞ in W, then $v_1, v_2, \cdots$ is strongly convergent to v_∞ and we also have $Dv_\infty = w_\infty$.

(3) For each w in W, $\|p^\lambda w\|_W$ is decreasing and convergent to 0 as $\lambda \to \infty$.

Proof of Lemma 3.2. Suppose there is a sequence $v_1, v_2, \cdots$ in V satisfying $\|v_d\|_V = R$ for every d and $\|(D+Q)v_d\|_W \to 0$ as $d \to \infty$. From (1) above, if we replace the sequence by a subsequence, we can assume the sequence is weakly convergent to some v_∞ and the sequence $Qv_1, Qv_2, \cdots$ is strongly convergent to Qv_∞ in W. Then $Dv_1, Dv_2, \cdots$ is strongly convergent to $-Qv_\infty$. Hence from (2) above, the sequence $v_1, v_2, \cdots$ is strongly convergent to v_∞. This strong convergence implies $\|v_\infty\|_V = R$ and $(D+Q)v_\infty = 0$, which is a contradiction. □

Proof of Lemma 3.3. Suppose there are sequences $v_1, v_2, \cdots$ in V and $\lambda_1, \lambda_2, \cdots$ satisfying $\|v_d\|_V = R$ and $\|p^{\lambda_d} Q v_d\|_W \geq \epsilon$ for every d and $\lambda_d \to \infty$ as $d \to \infty$. From (1) above, if we replace the sequence by a subsequence, we can assume the sequence is weakly convergent to some v_∞ and the sequence $Qv_1, Qv_2, \cdots$ is strongly convergent to Qv_∞ in W. From (3) above, we have $\|p^{\lambda_{d_0}} Qv_\infty\|_W < \epsilon/2$ for some large d_0. Since Qv_d is strongly convergent to Qv_∞, we have $\|p^{\lambda_{d_0}}(Qv_{d_1} - Qv_\infty)\|_W < \epsilon/2$ for some $d_1 \geq d_0$. Then, by using (3) again, we obtain

$$\begin{aligned}\|p^{\lambda_{d_1}} Qv_{d_1}\|_W &\leq \|p^{\lambda_{d_0}} Qv_{d_1}\|_W \leq \|p^{\lambda_{d_0}}(Qv_{d_1} - Qv_\infty)\|_W + \|p^{\lambda_{d_0}} Qv_\infty\|_W \\ &< \epsilon/2 + \epsilon/2 = \epsilon,\end{aligned}$$

which is a contradiction. □

Remark. An alternative way to construct a finite dimensional approximation is to use an extended version of the Kuranishi map. Let V^λ and W^λ be the completions of the spaces spanned by the eigenspaces of D^*D and DD^* with eigenvalues larger then λ respectively. Then the restriction of D on V^λ has the inverse $D^{-1}\colon W^\lambda \to V^\lambda$. Let $\Phi\colon V \to V$ be the map defined by $\Phi = Id_V + D^{-1}p^\lambda Q$. Suppose $u = \Phi v$. Then v is a zero of $D+Q$ if and only if u is in V_λ and $p_\lambda(D+Q)v = 0$. It is not hard to show that Φ is close to the identity on the disk centered in 0 with a fixed radius R, if λ is sufficiently large. So the map $p_\lambda(D+Q)\Phi^{-1}\colon V_\lambda \to W_\lambda$, which is defined only on a disk centered in 0, is a finite dimensional approximation in the sense that Φ gives a bijection between its zeros and the zeros of $D+Q\colon V \to W$, if we restrict these maps on the disks. When $\lambda = 0$, this construction is called the Kuranishi construction, and the approximation describes a neighborhood of 0 of the zero set of $D+Q$. If we take a larger λ, then the radius R for which the approximation is valid becomes larger. Since the zero set is bounded, we eventually have an approximation to describe the whole zero set as λ becomes large enough.

4. Pin_2-module structures

Since D is Pin_2-equivariant, V_λ and W_λ are Pin_2-modules and the approximations D_λ and Q_λ are still Pin_2-equivariant. We think of $\mathbb{H}$ as a Pin_2-module by using the right Pin_2-multiplication. Recall that we assumed $k = -\operatorname{sign}(M)/16 \geq 0$.

Lemma 4.1. *There are nonnegative integers m and n such that as Pin_2-modules*

$$V_\lambda = \mathbb{H}^{k+m} \oplus \tilde{\mathbb{R}}^n \quad \textit{and} \quad W_\lambda = \mathbb{H}^m \oplus \tilde{\mathbb{R}}^{b_+ + n + 1}.$$

Proof. Recall $D\colon V \to W$ is the direct sum of $D_1\colon L^4_2(S^+) \to L^3_2(S^-)$ and $D_2\colon L^4_2(T) \to L^3_2(\tilde{\Lambda})$. Let V^1_λ be the subspace of $L^4_2(S^+)$ spanned by the eigenspaces of $D_1^*D_1$ with eigenvalues less than or equal to λ. We define W^1_λ, V^2_λ and W^2_λ similarly, then we have the decompositions $V_\lambda = V^1_\lambda \oplus V^2_\lambda$ and $W_\lambda = W^1_\lambda \oplus W^2_\lambda$ as Pin_2-modules. There are finitely many points $p_1, p_2, \cdots, p_l$ on M such that the restriction on the fibers over these points is an injection from V^1_λ to $\oplus_{d=1}^l (S^+)_{p_d}$ which is isomorphic to $\mathbb{H}^l$ as a Pin_2-module. Since the Pin_2-module $\mathbb{H}$ is irreducible, V^1_λ itself is isomorphic to a Pin_2-module of the form $\mathbb{H}^{m'}$ for some m'. Similarly W^1_λ is of the form $\mathbb{H}^m$ for some m. The index of D_1 is equal to $\dim V^1_\lambda - \dim W^1_\lambda = 4m' - 4m$, which is on the other hand calculated from the Atiyah-Singer index theorem as follows.

$$\text{index } D_1 = 2\langle \hat{A}(M), [M]\rangle = -\frac{p_1(M)}{12} = -\frac{\langle L(M), [M]\rangle}{4} = -\frac{\operatorname{sign}(M)}{4} = 4k.$$

Here $\hat{A}(M)$ and $L(M)$ are the $\hat{A}$-genus and the L-genus of M. Hence we have $m' = k+m$ and $V^1_\lambda = \mathbb{H}^{k+m}$. Similarly V^2_λ and W^2_λ are of the form $\tilde{\mathbb{R}}^n$ and $\tilde{\mathbb{R}}^{n'}$

for some n and n' as Pin_2-modules. From Remark (2) in Section 2, the index of D_2 is $-1-b_+$, which is on the other hand equal to $\dim V_\lambda^2 - \dim W_\lambda^2 = n - n'$. Hence we have $n' = b_+ + n + 1$ and $W_\lambda^2 = \tilde{\mathbb{R}}^{b_+ + n + 1}$. □

Remark. The above argument also gives a proof of Rohlin's theorem: the kernel and the cokernel of the Dirac operator D_1 are the sums of copies of $\mathbb{H}$, so the index of D_1 is divisible by 4, while the index is equal to $-\operatorname{sign}(M)/4$ from the index theorem. Our argument to use the monopole equation would be regarded as a nonlinear version of this proof of Rohlin's theorem.

We next show that the image of $D+Q$ is actually contained in a subspace of W of codimension 1. Let s_0 be the parallel section of $\tilde{\Lambda} \subset S^- \oplus \tilde{\Lambda}$ corresponding to the Spin_4-invariant element 1 in ${}_+\tilde{\mathbb{H}}_+$. Parallel sections are contained in the kernel of the twisted Dirac operator D^*. Since the image of D is L^2-orthogonal to the kernel of D^*, the image of D is contained in the L^2-orthogonal complement $s_0^\perp$ of s_0 in W. The image of Q is also contained in $s_0^\perp$ from Remark (1) in Section 2. From the construction of the finite dimensional approximation, the image of $D_\lambda + Q_\lambda$ is still contained in the subspace $\bar{W}_\lambda = W_\lambda \cap s_0^\perp$ of codimension 1. Note that $\mathbb{R}s_0 = \tilde{\mathbb{R}}$ as a Pin_2-module and hence $\bar{W}_\lambda = \mathbb{H}^m \oplus \tilde{\mathbb{R}}^{b_+ + n}$.

Remark. The origin of the codimension 1 of $\bar{W}_\lambda$ is the dimension 1 of $H^0(M, \mathbb{R})$, which is also identified with the dimension of the symmetry group Pin_2.

We summarize the results of Section 3 and Section 4:

Theorem 4.2. *Let M be a closed spin 4-manifold with $b_1(M) = 0$ and* $\operatorname{sign}(M) \le 0$. *Then there are finite dimensional real* Pin_2*-modules V_λ and $\bar{W}_\lambda$ and a* Pin_2*-equivariant linear map D_λ and a* Pin_2*-equivariant quadratic map Q_λ from V_λ to $\bar{W}_\lambda$ which satisfy the following properties.*

(1) *There are* Pin_2*-module isomorphisms $V_\lambda = \mathbb{H}^{k+m} \oplus \tilde{\mathbb{R}}^n$ and $\bar{W}_\lambda = \mathbb{H}^m \oplus \tilde{\mathbb{R}}^{b_+ + n}$ for some m and n, where $k = -\operatorname{sign}(M)/16$.*

(2) *There are no zeros of $D_\lambda + Q_\lambda$ on the sphere centered in 0 with some radius R defined by using some* Pin_2*-invariant metric on V_λ.*

Remark. For λ_1 larger than λ, we have another finite dimensional approximation $D_{\lambda_1} + Q_{\lambda_1} \colon V_{\lambda_1} \to \bar{W}_{\lambda_1}$, where $V_{\lambda_1} = \mathbb{H}^{k+m_1} \oplus \tilde{\mathbb{R}}^{n_1}$ and $\bar{W}_{\lambda_1} = \mathbb{H}^{m_1} \oplus \tilde{\mathbb{R}}^{b_+ + n_1}$ for some $m_1 \ge m$ and $n_1 \ge n$. Restricted on the sphere of radius R, $D_{\lambda_1} + Q_{\lambda_1}$ is homotopic to the join of $D_\lambda + Q_\lambda$ and the identity on the sphere $S(\mathbb{H}^{m_1 - m} \oplus \tilde{\mathbb{R}}^{n_1 - n})$ through Pin_2-equivariant maps. Hence we have an inductive system of Pin_2-equivariant maps between spheres, and can define the stable class

$$\lim_{\lambda} [D_\lambda + Q_\lambda] \in \lim_{m,n} [S(\mathbb{H}^{k+m} \oplus \tilde{\mathbb{R}}^n), S(\mathbb{H}^m \oplus \tilde{\mathbb{R}}^{b_+ + n})],$$

where $[\cdot,\cdot]$ denotes the set of Pin_2-equivariant homotopy classes. Taking cones, we also have a stable class in the inductive limit of the set of Pin_2-equivariant homotopy classes between certain compact pairs of disks and spheres. We can think of this stable class as a model of the *proper homotopy class* of the map $D + Q \colon V \to \bar{W}$, where $\bar{W} = s_0^\perp$.

5. Equivariant maps

Let $V_{\lambda,\mathbb{C}}$ and $\bar{W}_{\lambda,\mathbb{C}}$ be the complexifications of V_λ and $\bar{W}_\lambda$, which we think of as complex vector bundles over a point. In general, for a complex vector bundle E over a compact space X we write BE for the disk bundle associated to E and SE for the sphere bundle which is the boundary of BE. If X has an action of a compact Lie group G and E is a G-equivariant bundle, then we take BE and SE to be G-invariant ones.

Theorem 1 follows from:

Proposition 5.1. *Suppose there is a continuous* Pin_2*-equivariant map* $\tilde{f}\colon BV_{\lambda,\ C} \to B\bar{W}_{\lambda,\mathbb{C}}$ *preserving boundaries. If* $k > 0$*, then we have the inequality* $2k + 1 \leq b_+$.

Proof of Theorem 1 assuming Proposition 5.1. Since M is not negative-definite, we have $b_+ \geq 1$ and the required inequality $2k+1 \leq b_+$ is satisfied obviously when $k = 0$. Suppose $k > 0$. We use the notations in Theorem 4.2. Let $f\colon V_{\lambda,\mathbb{C}} \to \bar{W}_{\lambda,\mathbb{C}}$ be the *complexification* of $D_\lambda + Q_\lambda$ defined by

$$f(u \otimes 1 + v \otimes i) = (D_\lambda + Q_\lambda)u \otimes 1 + (D_\lambda + Q_\lambda)v \otimes i.$$

Let $BV_{\lambda,\mathbb{C}}$ be the Pin_2-invariant disk $\{u \otimes 1 + v \otimes i \in V_{\lambda,\mathbb{C}} \bigm| \|u\|_V, \|v\|_V \leq R\}$ and $SV_{\lambda,\mathbb{C}}$ be its boundary. The image of $BV_{\lambda,\mathbb{C}}$ by f does not contain 0. We write $S\bar{W}_{\lambda,\mathbb{C}}$ for the quotient $\bar{W}_{\lambda,\mathbb{C}}\backslash\{0\}/\mathbb{R}_+$ and $p\colon \bar{W}_{\lambda,\mathbb{C}}\backslash\{0\} \to S\bar{W}_{\lambda,\mathbb{C}}$ for the projection. Then the composition $\bar{f}$ of $f|SV_{\lambda,\mathbb{C}}\colon SV_{\lambda,\mathbb{C}} \to \bar{W}_{\lambda,\mathbb{C}}\backslash\{0\}$ with $p\colon \bar{W}_{\lambda,\mathbb{C}}\backslash\{0\} \to S\bar{W}_{\lambda,\mathbb{C}}$ is a Pin_2-equivariant map. Let $\tilde{f}\colon BV_{\lambda,\mathbb{C}} \to B\bar{W}_{\lambda,\mathbb{C}}$ be the cone of $\bar{f}$. The restriction of $\tilde{f}$ on the boundary $S\bar{W}_{\lambda,\mathbb{C}}$ is $\bar{f}\colon SV_{\lambda,\mathbb{C}} \to S\bar{W}_{\lambda,\mathbb{C}}$. Then we can use Proposition 5.1 to obtain the required inequality. □

We will use equivariant K-theory to prove Proposition 5.1. Suppose, in general, G is a compact Lie group, X is a compact G-space, E and F are G-equivariant complex vector bundles over X and $\tilde{f}\colon BE \to BF$ is a G-equivariant bundle map preserving boundaries. The Thom isomorphism theorem for equivariant K-theory implies that $K_G(BE, SE)$ and $K_G(BF, SF)$ are free $K_G(X)$-modules generated by the Thom classes τ_E and τ_F respectively. Let $\tilde{f}^*\colon K_G(BF, SF) \to K_G(BE, SE)$ be the pullback map for $\tilde{f}$.

Definition. The *degree* of $\tilde{f}$ in K-theory is the unique element α_0 of $K_G(X)$ which satisfies the relation:

$$\tilde{f}^* \tau_F = \alpha_0 \tau_E.$$

The degree α_0 satisfies the following equation in $K_G(X)$.

Lemma 5.2. $$\Sigma(-1)^d[\Lambda^d F] = \alpha_0 \Sigma(-1)^d [\Lambda^d E].$$

Proof. The restrictions of τ_E and τ_F on the zero sections are the Euler classes of E and F, which are equal to $\Sigma(-1)^d[\Lambda^d E]$ and $\Sigma(-1)[\Lambda^d F]$ respectively (see

(A.4) in Appendix). Hence the required relation immediately follows from the restriction of the defining equation of α_0. $\square$

We also need other equations for α_0. Let l be an integer larger than 1 and ψ^l the Adams operation. (See Appendix for the definition of ψ^l.) The *K-theory characteristic class* $\rho^l(E)$ is defined to be the unique element of $K_G(X)$ satisfying $\psi^l\tau_E = \rho^l(E)\tau_E$. (See (A.6) in Appendix, or [1].) The other equations for α_0 is:

Lemma 5.3. $$\rho^l(F)\alpha_0 = (\psi^l\alpha_0)\rho^l(E).$$

Proof. Apply ψ^l on the equation $\tilde{f}^*\tau_F = \alpha_0\tau_E$ to get $\tilde{f}^*(\rho^l(F)\tau_F) = (\psi^l\alpha_0)\rho^l(E)\tau_E$, where we used the multiplicative property of ψ^l (see (A.9) in Appendix). Use the definition of α_0 again and compare the coefficients of the generator τ_E to get the equation. $\square$

In our case we have

$$\begin{aligned} &X = \{\text{a point}\}, \qquad G = \mathrm{Pin}_2, \\ &E = V_{\lambda,\mathbb{C}} = (\mathbb{H}^{k+m} \oplus \tilde{\mathbb{R}}^n) \otimes \mathbb{C} \quad \text{and} \\ &F = \bar{W}_{\lambda,\mathbb{C}} = (\mathbb{H}^m \oplus \tilde{\mathbb{R}}^{b_+ + n}) \otimes \mathbb{C}. \end{aligned}$$

Recall $K_G(\text{point})$ is the character ring $R(G)$ of G.

Lemma 5.4. *(1) If $k > 0$, then*

$$\{\alpha \in R(\mathrm{Pin}_2) \mid \rho^l(F)\alpha = (\psi^l\alpha)\rho^l(E)\} \subset \mathrm{Ker}(R(\mathrm{Pin}_2) \to R(S^1)).$$

(2) There is an element α of $\mathrm{Ker}(R(\mathrm{Pin}_2) \to R(S^1))$ satisfying

$$\Sigma(-1)^d[\Lambda^d F] = \alpha\Sigma(-1)^d[\Lambda^d E]$$

if and only if $2k + 1 \leq b_+$.

Proof of Proposition 5.1. It is an immediate consequence of the lemmas 5.2, 5.3 and 5.4. $\square$

In the rest of this section we will show Lemma 5.4.

Proof of Lemma 5.4 (1). Let $\mathbb{C}$ be the standard 1-dimensional complex representation of S^1 and t the class of $\mathbb{C}$ in $R(S^1)$. Let 1 be the trivial 1-dimensional complex representation of S^1 and we use the same notation 1 for its class in $R(S^1)$. Then $R(S^1)$ is the space of Laurent polynomials in t with integer coefficients. When we regard E and F as S^1-modules, we write E' and F' for these representation spaces. More explicitly, $E' = 2(k+m)(\mathbb{C} \oplus \mathbb{C}^*) \oplus n$ and $F' = 2m(\mathbb{C} \oplus \mathbb{C}^*) + b_+ + n$, where we used the additive notation for direct sum. The multiplicative property of ρ^l implies (see (A.9) in Appendix)

$$\rho^l(E') = \{\rho^l(\mathbb{C})\rho^l(\mathbb{C}^*)\}^{2(k+m)}\rho^l(1)^n.$$

For a line bundle L, in general, we have $\rho^l(L) = 1 + [L] + [L^2] + \cdots + [L^{l-1}]$ (see (A.8) in Appendix). From this formula, we have

$$\rho^l(E') = \{(1 + t + \cdots + t^{l-1})(1 + t^{-1} + \cdots + t^{-(l-1)})\}^{2(k+m)} l^n.$$

Similarly $\rho^l(F') = \{(1 + t + \cdots + t^{l-1})(1 + t^{-1} + \cdots + t^{-(l-1)})\}^{2m} l^{b_+ + n}$.
Let α be an element of $R(\mathrm{Pin}_2)$ satisfying $\rho^l(F)\alpha = (\psi^l \alpha)\rho^l(E)$, then the image α' of α by the restriction map $R(\mathrm{Pin}_2) \to R(S^1)$ is expressed by a Laurent polynomial $h(t)$. Suppose α' is not zero. Since the action of j in Pin_2 induces the relation $h(t) = h(t^{-1})$, the degree d in t of the highest nonzero term in $h(t)$ is nonnegative. Then the degree of the highest nonzero term of $\rho^l(F')\alpha'$ is $2(l-1)m + d$. On the other hand, from $\psi^l h(t) = h(t^l)$ (see (A.1) and (A.2) in Appendix), the degree of the highest nonzero term of $(\psi^l \alpha')\rho^l(E')$ is $ld + 2(l-1)(k+m)$. If $k > 0$, since the degrees are not the same, $\rho^l(F')\alpha'$ cannot be equal to $(\psi^l \alpha')\rho^l(E')$. This is a contradiction.

Proof of Lemma 5.4 (2). The irreducible complex representations of Pin_2 are classified as follows. First we have the trivial complex 1-dimensional representation 1 and the nontrivial complex 1-dimensional representation $\tilde{1} = \tilde{\mathbb{R}} \otimes 1$. Here recall that $\tilde{\mathbb{R}}$ is the nontrivial real 1-dimensional representation. The other ones are complex 2-dimensional and are parameterized by positive integers d, and the representation corresponding to d is characterized by the property that its class in $R(S^1)$ is $t^d + t^{-d}$. This classification implies:

$$\mathrm{Ker}(R(\mathrm{Pin}_2) \to R(S^1)) = \{c(1 - \tilde{1}) \mid c \in \mathbb{Z}\}.$$

Suppose $\alpha = c(1 - \tilde{1})$ satisfies the relation $\Sigma(-1)^d \Lambda^d F = \alpha \Sigma(-1)\Lambda^d E$. Take the traces of j for the both hand sides, and we get

$$2^{2m+b_+ + n} = 2\, c\, 2^{2k+2m+n},$$

which implies $b_+ \geq 2k + 1$. On the contrary, when $b_+ \geq 2k + 1$, the relation is satisfied by $\alpha = 2^{b_+ - 2k - 1}(1 - \tilde{1})$. □

Remark. When $b_+ \geq 2k + 1$, the above proof implies that Lemma 5.1 and Lemma 5.2 provide enough equations to determine that the degree α_0 of $\tilde{f}$ is equal to $2^{b_+ - 2k - 1}(1 - \tilde{1})$.

Appendix

We collect some properties of the Adams operations which are used in Section 5. Let G be a compact Lie group and X a compact G-space. the *Adams operation* $\psi^l : K_G(X) \to K_G(X)$ is defined as follows. We fix a positive integer l in Appendix. For each positive integer r, let $p_r(\sigma_1, \sigma_2, \cdots, \sigma_r)$ be t he polynomial expressing $x_1^l + x_2^l + \cdots + x_r^l$ with respect to the elementary symmetric polynomials, where σ_d is the d-th elementary symmetric polynomial. When E

and F are two G-equivariant complex vector bundles over X with ranks r and s, one can check the relation

$$p_{r+s}([\Lambda^1(E\oplus F)],\cdots,[\Lambda^{r+s}(E\oplus F)]) = p_r([\Lambda^1 E],\cdots,[\Lambda^r E]) + p_s([\Lambda^1 F],\cdots,[\Lambda^s F])$$

in $K_G(X)$. Then there is a unique additive homomorphism ψ^l from $K_G(X)$ to itself which satisfies $\psi^l([E]) = p_r([\Lambda^1 E],\cdots,[\Lambda^r E])$. It is straightforward to check that ψ^l is also a multiplicative homomorphism. When Y is a G-invariant compact subset of X, the relative K_G-group $K_G(X,Y)$ for the compact pair (X,Y) is defined to be the kernel of the natural map from $K_G(X/Y)$ to $K_G(\text{point}) = R(G)$. Since ψ^l is natural for continuous maps between compact G-spaces, we can define ψ^l on $K_G(X,Y)$ so that it is natural for continuous maps between compact pairs.
(A.1) The operation $\psi^l\colon K_G(X,Y)\to K_G(X,Y)$ is a ring homomorphism.
In particular we have the formula for line bundles from the definition:
(A.2) $\psi^l[L] = [L^l]$ for a line bundle L.

Let E be an G-equivariant complex vector bundle over X, BE be its G-invariant disk bundle and SE be the boundary of BE. The *Thom class* τ_E of E is an element of $K_G(BE,SE)$ which has the following properties:
(A.3) $K_G(BE,SE)$ is a free $K_G(X)$-module generated by τ_E ([2]).
(A.4) The restriction of τ_E on the zero section is equal to $e(E) = \Sigma(-1)^d[\Lambda^d E]$.
The former property is the *Thom isomorphism theorem* in equivariant K-theory and it is the only non-elementary theorem we need in Section 5. We call the restriction $e(X)$ by the *Euler class* of E. The Thom classes have the multiplicative property:
(A.5) $\tau_{E\oplus F} = \tau_E\tau_F$.

The definition of the *K-theory characteristic class* $\rho^l(E)\in K_G(X)$ is given by:
(A.6) $\psi^l\tau_E = \rho^l(E)\tau_E$.
An explicit formula for ρ^l is:

$$\rho^l(E) = \left.\frac{\Sigma(-1)^d t^{ld}\psi^l[\Lambda^d E]}{\Sigma(-1)^d t^d[\Lambda^d E]}\right|_{t=1}. \tag{A.7}$$

¿From (A.7), (A.1) and (A.2) we have:
(A.8) $\rho^l(L) = 1 + [L] + [L^2] + \cdots + [L^{l-1}]$ for a line bundle L, and
(A.9) $\rho^l(E\oplus F) = \rho^l(E)\rho^l(F)$.
Instead of using (A.7), we can use the multiplicative properties (A.1) and (A.5) of the Thom classes and the Adams operation to show (A.9) directly from the definition of ρ^l. In the rest of Appendix we give a short proof of (A.7). (See also [3].)

Proof of (*A.7*). Since E is an G-equivariant complex vector bundle, we have the S^1-action on E defined by the multiplication of $S^1\subset\mathbb{C}$, which commutes with the G-action on E. Hence we can think of E as a $G \times S^1$-equivariant bundle, where the S^1-action on X is trivial. We write $\tilde{E}$ for this $G \times S^1$-equivariant bundle. Let $\tau_{\tilde{E}}$ and $e(\tilde{E})$ be the Thom class and the Euler class of $\tilde{E}$. Then $e(\tilde{E})$ is an element of $K_{G\times S^1}(X) = K_G(X) \otimes R(S^1)$. If we write t for the class of the standard 1-dimensional representation of S^1, then $R(S^1)$ is the ring of the Laurent polynomials in t and $e(\tilde{E}) = \Sigma(-1)^d t^d[\Lambda^d E]$ is not a zero-divisor. Restrict the relation $\psi^l \tau_{\tilde{E}} = \rho^l(\tilde{E})\tau_{\tilde{E}}$ on the zero section to get $\psi^l e(\tilde{E}) = \rho^l(\tilde{E})e(\tilde{E})$. Then, since $e(\tilde{E})$ is not a zero-divisor, we obtain $\rho^l(\tilde{E}) = (\psi^l e(\tilde{E}))/e(\tilde{E})$, from which we can deduce (A.7). □

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