DIRAC OPERATOR ON EMBEDDED HYPERSURFACES

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Abstract. New extrinsic lower bounds are given for the classical Dirac operator on the boundary of a compact domain of a spin manifold. The main tool is to solve some boundary problems for the Dirac operator of the domain under boundary conditions of Atiyah-Patodi-Singer type. Spinorial techniques are used to give simple proofs of classical results for compact embedded hypersurfaces.

1. Introduction

The spectrum of the fundamental Dirac operator on closed manifolds have been extensively studied over the past three decades. First, the intrinsic aspect has been systematically studied by many authors (see [BFGK, BHMM] and for references therein). The striking fact in this setup is the Lichnerowicz' Theorem which gives topological obstructions to the existence of metrics with positive scalar curvature. Another important feature in this approach is the geometric characterisation of manifolds admitting solutions of some special field equations as the Killing spinor equation. Second, the extrinsic aspect has been recently studied in [An, Bm, Bä] where mainly extrinsic upper bounds are obtained. More recently in [Z, HZ, Mo], extrinsic lower bounds for the hypersurface Dirac operator are established.

In this paper, we investigate the spectral properties of the Dirac operator on a compact manifold with boundary for the Atiyah-Patodi-Singer type boundary condition corresponding to the spectral resolution of the classical Dirac operator of the boundary hypersurface. We start by recalling the Schrödinger-Lichnerowicz' integral formula (16) for a compact $(n+1)$–dimensional manifold Ω with boundary $\Sigma = \partial\Omega$ from which we deduce a spinorial Reilly type inequalities (17) and (22). Under some curvature assumptions, we show that the Dirac operator on the ambient space Ω, subject to an APS type boundary condition,

has zero kernel and we derive the analogue of the Friedrich inequality (26) with its generalization (28) involving the energy-momentum tensor.

We then use the Reilly type Inequality (17) to prove that (see Theorem 6) if the scalar curvature of Ω and the mean curvature H of Σ are nonnegative, then the lowest nonnegative eigenvalue of the intrinsic hypersurface Dirac operator is at least equal to

$$\frac{n}{2} \inf_{\Sigma} H.$$

It is shown that this estimate improves previous results. In particular, it is valid even if the scalar curvature of the boundary is negative.

Finally, we make use of the spinorial techniques to study constant mean curvature and minimal embedded hypersurfaces. A spinorial simple proof for the classical Alexandrov Theorem (see Theorem 7) is given. We then prove that (see Theorem 8) minimal compact hypersurfaces bounding a compact domain admitting a parallel spinor are totally geodesic. We end with a rigidity type result.

The conformal aspect of the Dirac operator on embedded hypersurfaces is the object of [HMZ1] while general lower bound estimates on compact manifolds with boundary are established in [HMZ2].

2. Preliminaries

Let $(M, \langle\ ,\ \rangle)$ be an $(n+1)$-dimensional Riemannian spin manifold and denote by $\overline{\nabla}$ its Levi-Civita connection. We use the same symbol to denote the corresponding lift of $\overline{\nabla}$ to the spinor bundle $\mathbb{S}M$ of M. On the spinor bundle $\mathbb{S}M$ there exist a natural Hermitian structure (also denoted by $\langle\ ,\ \rangle$), and a Clifford module structure
$\gamma : \text{Clif}(M) \to \text{End}(\mathbb{S}M)$ which are compatible with $\overline{\nabla}$. That is

$$(1) \qquad\qquad X\langle\psi, \varphi\rangle \;=\; \langle\overline{\nabla}_X\psi, \varphi\rangle + \langle\psi, \overline{\nabla}_X\varphi\rangle$$

$$(2) \qquad\qquad \langle\gamma(X)\psi, \gamma(X)\varphi\rangle \;=\; |X|^2\langle\psi, \varphi\rangle$$

$$(3) \qquad\qquad \overline{\nabla}_X\Big(\gamma(Y)\psi\Big) \;=\; \gamma(\overline{\nabla}_X Y)\psi + \gamma(Y)\overline{\nabla}_X\psi,$$

for any spinor fields $\psi, \varphi \in \Gamma(\mathbb{S}M)$ and any tangent vector fields $X, Y \in \Gamma(TM)$. The Dirac operator $\overline{D}$ on $\mathbb{S}M$ is locally given by

$$(4) \qquad\qquad \overline{D} = \sum_{i=1}^{n+1} \gamma(e_i)\overline{\nabla}_{e_i},$$

where $\{e_1, \ldots, e_{n+1}\}$ is a local orthonormal frame of TM. Consider an orientable hypersurface Σ in M. Let ∇ be the Levi-Civita connection of the Riemannian metric on Σ induced by the metric of M. The Gauss formula says that

$$(5) \qquad\qquad \nabla_X Y = \overline{\nabla}_X Y - \langle AX, Y\rangle N,$$

where X, Y are vector fields tangent to the hypersurface Σ, the vector field N is the global unit field (inner) normal to Σ and A stands for the shape operator

corresponding to N, that is,

$$(6) \qquad\qquad \overline{\nabla}_X N = -AX, \qquad \forall X \in \Gamma(T\Sigma).$$

The spin structure of M can be also induced on Σ in such a way that the restricted bundle $\mathbb{S}M_{|\Sigma}$ is isomorphic to either $\mathbb{S}\Sigma$ or $\mathbb{S}\Sigma \oplus \mathbb{S}\Sigma$ according to the dimension n of Σ is either even or odd (see [Bä], [Mo] for example). *A spinor field on M and its restriction to the hypersurface will be denoted by the same symbol.* Since the n–dimensional Clifford algebra is the even part of the $(n + 1)$–dimensional Clifford algebra, Clifford multiplication on $\mathbb{S}M_{|\Sigma}$ is given by

$$\gamma^\Sigma(X)\psi = \gamma(X)\gamma(N)\psi,$$

where $\psi \in \Gamma(\mathbb{S}M_{|\Sigma})$ and $X \in \Gamma(T\Sigma)$. It is not difficult to check that for any $X \in \Gamma(T\Sigma)$ and $\psi \in \Gamma(\mathbb{S}M_{|\Sigma})$, the Levi-Civita connection on $\mathbb{S}\Sigma$ is given by the following spinorial Gauss formula

$$(7) \qquad \nabla_X \psi = \overline{\nabla}_X \psi - \frac{1}{2}\gamma^\Sigma(AX)\psi = \overline{\nabla}_X \psi - \frac{1}{2}\gamma(AX)\gamma(N)\psi\,.$$

Hence, if D denotes the Dirac operator associated with the spin structure of the hypersurface Σ, then for any spinor field $\psi \in \Gamma(\mathbb{S}M_{|\Sigma})$

$$(8) \qquad D\psi = \sum_{j=1}^{n} \gamma^\Sigma(e_j)\nabla_{e_j}\psi = \frac{n}{2}H\psi - \gamma(N)\sum_{j=1}^{n}\gamma(e_j)\overline{\nabla}_{e_j}\psi,$$

where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame on Σ such that $\{e_1, \ldots, e_n, e_{n+1} = N\}$ is the corresponding local orthonormal frame on M and

$$H = \frac{1}{n}\mathrm{trace}\, A$$

is the mean curvature of Σ corresponding to the orientation N. From (8), if $\psi \in \Gamma(\mathbb{S}M)$ is a spinor field on the ambient manifold M, we have

$$(9) \qquad\qquad D\psi = \frac{n}{2}H\psi - \gamma(N)\overline{D}\psi - \overline{\nabla}_N\psi.$$

We end this section by showing that the sepctrum of the Dirac operator D on the hypersurface Σ is symmetric w.r.t zero. We have

Proposition 1. *For any spinor field $\psi \in \Gamma(\mathbb{S}M)$, and any tangent vector field $X \in \Gamma(T\Sigma)$, the following relations hold*

$$(10) \qquad\qquad \nabla_X\left(\gamma(N)\psi\right) = \gamma(N)\nabla_X\psi,$$

$$(11) \qquad\qquad D\left(\gamma(N)\psi\right) = -\gamma(N)D\psi.$$

Proof. By (7) and (6) it follows

$$
\begin{aligned}
\nabla_X \big(\gamma(N)\psi\big) &= \Big(\overline{\nabla}_X - \tfrac{1}{2}\gamma(AX)\gamma(N)\Big)\big(\gamma(N)\psi\big) \\
&= -\gamma(AX)\psi + \gamma(N)\overline{\nabla}_X\psi + \tfrac{1}{2}\gamma(AX)\psi \\
&= \gamma(N)\Big(\overline{\nabla}_X - \tfrac{1}{2}\gamma(AX)\gamma(N)\Big)\psi \\
&= \gamma(N)\nabla_X\psi.
\end{aligned}
$$

For the second relation it is sufficient to use (10) and the Clifford algebra relations to get

$$
\begin{aligned}
D\big(\gamma(N)\psi\big) &= \sum_{j=1}^{n}\gamma(e_j)\gamma(N)\nabla_{e_j}\big(\gamma(N)\psi\big) \\
&= \sum_{j=1}^{n}\gamma(e_j)\gamma(N)\gamma(N)\nabla_{e_j}\psi \\
&= -\gamma(N)\sum_{j=1}^{n}\gamma(e_j)\gamma(N)\nabla_{e_j}\psi \\
&= -\gamma(N)D\psi.
\end{aligned}
$$

$\square$

3. Bounding domains hypersurfaces

Suppose now that the hypersurface Σ is the boundary of a compact domain Ω in the manifold M (which could be the manifold itself). Let $\psi \in \Gamma(\mathbb{S}\Omega)$ be a spinor field on the domain Ω. The Schrödinger–Lichnerowicz formula says that

$$
(12) \qquad\qquad \overline{D}^2\psi = \overline{\nabla}^*\overline{\nabla}\psi + \tfrac{1}{4}\overline{R}\psi,
$$

where $\overline{R}$ is the scalar curvature of M. Then

$$
\langle \overline{D}^2\psi, \psi\rangle = \langle \overline{\nabla}^*\overline{\nabla}\psi, \psi\rangle + \tfrac{1}{4}\overline{R}|\psi|^2.
$$

Consider the 1-forms α and β on Ω defined by

$$
\alpha(X) = \langle \gamma(X)\overline{D}\psi, \psi\rangle, \qquad \beta(X) = \langle \overline{\nabla}_X\psi, \psi\rangle,
$$

for any $X \in \Gamma(T\Omega)$. It is clear that

$$
\delta\alpha = \langle \overline{D}^2\psi, \psi\rangle - |\overline{D}\psi|^2,
$$

because γ acts by skew–symmetric endomorphisms for $\langle\ ,\ \rangle$. Also

$$
\delta\beta = -\langle \overline{\nabla}^*\overline{\nabla}\psi, \psi\rangle + |\overline{\nabla}\psi|^2.
$$

Hence we obtain

$$
\delta\alpha + \delta\beta = |\overline{\nabla}\psi|^2 - |\overline{D}\psi|^2 + \tfrac{1}{4}\overline{R}|\psi|^2.
$$

Integrating on Ω and applying the divergence theorem

$$-\int_\Sigma \langle \gamma(N)\overline{D}\psi + \overline{\nabla}_N\psi, \psi \rangle \, d\Sigma = \int_\Omega \left(|\overline{\nabla}\psi|^2 - |\overline{D}\psi|^2 + \frac{1}{4}\overline{R}|\psi|^2 \right) d\Omega,$$

where N is the inner unit normal field along Σ. Using (9), this equation could be written as

$$(13) \quad \int_\Sigma \left(\langle D\psi, \psi \rangle - \frac{nH}{2}|\psi|^2 \right) d\Sigma = \int_\Omega \left(|\overline{\nabla}\psi|^2 - |\overline{D}\psi|^2 + \frac{1}{4}\overline{R}|\psi|^2 \right) d\Omega,$$

for any spinor field $\psi \in \Gamma(\mathbb{S}\Omega)$.

On the other hand, for any spinor field ψ on M, we have the following decomposition :

$$(14) \quad |\overline{\nabla}\psi|^2 = |\overline{P}\psi|^2 + \frac{1}{n+1}|\overline{D}\psi|^2,$$

where $\overline{P}$ is the Twistor operator of M defined by

$$(15) \quad \overline{P}_X\psi := \overline{\nabla}_X\psi + \frac{1}{n+1}\gamma(X)\overline{D}\psi, \qquad \forall X \in \Gamma(TM).$$

A non-trivial spinor field ψ such that $\overline{P}\psi \equiv 0$ is *called* a twistor-spinor. Combining the identities (13) and (14), it follows

$$(16) \quad \int_\Sigma \left(\langle D\psi, \psi \rangle - \frac{nH}{2}|\psi|^2 \right) d\Sigma =$$

$$\frac{1}{4}\int_\Omega \overline{R}|\psi|^2 \, d\Omega - \frac{n}{n+1}\int_\Omega |\overline{D}\psi|^2 \, d\Omega + \int_\Omega |\overline{P}\psi|^2 \, d\Omega,$$

for all $\psi \in \Gamma(\mathbb{S}\Omega)$.

Remark 1. Since $|\overline{P}\psi|^2 \geq 0$, identity (16) immediately translates to

$$(17) \quad \int_\Sigma \left(\langle D\psi, \psi \rangle - \frac{nH}{2}|\psi|^2 \right) d\Sigma \geq \frac{1}{4}\int_\Omega \overline{R}|\psi|^2 \, d\Omega - \frac{n}{n+1}\int_\Omega |\overline{D}\psi|^2 \, d\Omega,$$

which is the analogue of the Reilly Inequality [Re] for the gradient of a function. Moreover, equality occurs if and only if ψ is a twistor–spinor.

We now make use of the energy-momentum tensor to derive another useful expression of the r.h.s of identity (13) (see [Hi]). Recall that the energy-momentum tensor $\overline{Q}_\psi$ associated with a spinor field $\psi \in \Gamma(\mathbb{S}\Omega)$ is the symmetric 2-tensor, defined on the complement set of zeros of ψ and for any tangent vector fields $X, Y \in \Gamma(T\Omega)$ by

$$(18) \quad \overline{Q}_\psi(X, Y) = \frac{1}{2}\Re \left\langle \gamma(X)\overline{\nabla}_Y\psi + \gamma(Y)\overline{\nabla}_X\psi, \frac{\psi}{|\psi|^2} \right\rangle.$$

If the associated symmetric endomorphism of the tangent bundle $T\Omega$ is denoted by the same symbol, then one can easily check that the modified connection $\overline{\nabla}^{\overline{Q}_\psi}$ defined by

$$(19) \quad \overline{\nabla}_X^{\overline{Q}_\psi}\psi := \overline{\nabla}_X\psi + \gamma(\overline{Q}_\psi(X))\psi$$

satisfies, for any spinor field ψ the relation

$$(20) \qquad |\overline{\nabla}\psi|^2 = |\overline{\nabla}^{\overline{\mathcal{Q}}_\psi}\psi|^2 + |\overline{\mathcal{Q}}_\psi|^2|\psi|^2.$$

Hence, this identity combined with (13) imply

$$\int_\Sigma \left(\langle D\psi, \psi \rangle - \frac{nH}{2}|\psi|^2 \right) d\Sigma \;\;=$$

$$(21) \qquad \int_\Omega \left((\frac{1}{4}\overline{R} + |\overline{\mathcal{Q}}_\psi|^2)|\psi|^2 - |\overline{D}\psi|^2 + |\overline{\nabla}^{\overline{\mathcal{Q}}_\psi}\psi|^2 \right) d\Omega.$$

Remark 2. As before, since $|\overline{\nabla}^{\overline{\mathcal{Q}}_\psi}\psi|^2 \geq 0$, identity (21) translates to

$$\int_\Sigma \left(\langle D\psi, \psi \rangle - \frac{nH}{2}|\psi|^2 \right) d\Sigma \;\;\geq$$

$$(22) \qquad \int_\Omega \left((\frac{1}{4}\overline{R} + |\overline{\mathcal{Q}}_\psi|^2)|\psi|^2 - |\overline{D}\psi|^2 \right) d\Omega.$$

Moreover, if equality occurs in (22) for a spinor field ψ, then the function $|\psi|^2$ is constant.

4. Boundary problems for the Dirac operator

Since the hypersurface $\Sigma = \partial\Omega$ is compact, the Dirac operator D has a discrete spectrum

$$\cdots \leq \lambda_{-k} \leq \cdots \leq \lambda_{-1} \leq 0 \leq \lambda_1 \leq \cdots \leq \lambda_k \leq \cdots .$$

But from (11), we have $\lambda_{-k} = -\lambda_k$ for all $k \in \mathbb{Z}$. That is, the spectrum of D is

$$\cdots \leq -\lambda_k \leq \cdots \leq -\lambda_1 \leq 0 \leq \lambda_1 \leq \cdots \leq \lambda_k \leq \cdots \nearrow .$$

Denote by $\pi_+ : \Gamma(\mathbb{S}\Sigma) \to \Gamma(\mathbb{S}\Sigma)$ the projection onto the subspace of $\Gamma(\mathbb{S}\Sigma)$ spanned by the eigenspinors corresponding to the nonnegative eigenvalues of D. It is clear that

$$(23) \qquad D\pi_+ = \pi_+ D \qquad \text{and} \qquad \int_\Sigma \langle D\psi, \psi \rangle \, d\Sigma \leq \int_\Sigma \langle D\pi_+\psi, \pi_+\psi \rangle \, d\Sigma,$$

for any spinor field ψ on Σ and the equality holds if and only if $\pi_+\psi = \psi$. This projection π_+ provides an Atiyah-Patodi-Singer type boundary condition for the Dirac operator $\overline{D}$ of the domain Ω. We have proved in [HMZ2] that this is a global self-adjoint elliptic condition. This can be done either by using standard facts on pseudo-differential operators [BW, GLP, S] or by obtaining basic elliptic estimates and standard results from functional analysis. The second approach was discovered in [FS] and closely followed in [HMZ2]. Following either one of these approaches, we prove the following result:

Theorem 2. *Let Ω be a compact Riemannian spin manifold with boundary $\partial\Omega = \Sigma$. The inhomogeneous boundary problem for the Dirac operator*

$$(24) \qquad \begin{cases} \overline{D}\psi & = \Psi \qquad\qquad \text{on } \Omega \\ \pi_+\psi & = \pi_+\varphi \qquad\;\; \text{on } \Sigma, \end{cases}$$

has a smooth solution for each $\Psi \in \Gamma(\mathbb{S}\Omega)$ *and* $\varphi \in \Gamma(\mathbb{S}\Omega_{|\Sigma})$ *satisfying the following integrability condition*

$$\int_\Omega \langle \Psi, \Phi \rangle \, d\Omega + \int_\Sigma \langle \gamma(N)\varphi, \Phi \rangle \, d\Sigma = 0$$

for any harmonic spinor Φ *on* Ω *such that* $\pi_+\Phi = 0$. *This solution is unique up to an arbitrary harmonic spinor field* Φ *of this type.*

Under some curvature assumptions, we now show the following particular case of Theorem 2:

Proposition 3. *Let* Ω *be a compact Riemannian spin manifold with nonnegative scalar curvature* $\overline{R}$, *whose boundary* $\partial\Omega = \Sigma$ *has nonnegative mean curvature* H *(with respect to the inner normal). Then the following inhomogeneous problem for the Dirac operator* $\overline{D}$ *of* Ω *with the Atiyah-Patodi-Singer boundary condition*

$$(25) \qquad \begin{cases} \overline{D}\psi & = \Psi & \text{on } \Omega \\ \pi_+\psi & = \pi_+\varphi & \text{on } \Sigma, \end{cases}$$

has a unique smooth solution for any $\Psi \in \Gamma(\mathbb{S}\Omega)$ *and* $\varphi \in \Gamma(\mathbb{S}\Omega_{|\Sigma})$.

Proof. Since $\overline{R}$ and H are both nonnegative, Inequality (17) could be written as

$$\int_\Sigma \langle D\psi, \psi \rangle \, d\Sigma \geq -\frac{n}{n+1} \int_\Omega |\overline{D}\psi|^2 \, d\Omega,$$

for any spinor field $\psi \in \Gamma(\mathbb{S}\Omega)$. Assume that $\overline{D}\psi = 0$ and $\pi_+\psi = 0$. Then,

$$0 = \int_\Sigma \langle D\pi_+\psi, \pi_+\psi \rangle \, d\Sigma \geq \int_\Sigma \langle D\psi, \psi \rangle \, d\Sigma \geq -\frac{n}{n+1} \int_\Omega |\overline{D}\psi|^2 \, d\Omega = 0.$$

Hence, since we have equality in (17), ψ is a harmonic twistor–spinor, that is a parallel spinor. In particular the function $|\psi|^2$ is constant. On the other hand, since the equality on the right side of (23) is achieved, we have $\psi = \pi_+\psi = 0$ on Σ. Therefore ψ is identically zero on Ω and the integrability condition of Theorem 2 is satisfied. $\square$

5. Extrinsic lower bounds for the hypersurface Dirac operator

We start this section by proving a Friedrich type inequality [Fr] for compact spin manifolds with non-empty boundary.

Theorem 4. *Let* Ω *be a compact* $(n+1)$–*dimensional Riemannian spin manifold of nonnegative scalar curvature* $\overline{R}$ *with boundary* $\partial\Omega$ *of nonnegative mean curvature. Then the first eigenvalue* $\overline{\lambda}_1$ *of the Dirac operator on* Ω, *with the Atiyah-Patodi-Singer condition, satisfies*

$$(26) \qquad \overline{\lambda}_1^2 > \frac{n+1}{4n} \inf_\Omega \overline{R}.$$

Proof. Consider the Dirac operator $\overline{D}$ acting on the spinor bundle $\Gamma(\mathbb{S}\Omega)$ satisfying the Atiyah-Patodi-Singer boundary condition

$$\pi_+\psi = 0 \qquad \text{on } \partial\Omega = \Sigma.$$

The spectrum of $\overline{D}$ consists of entirely isolated real eigenvalues with finite multiplicity and smooth eigenspinors (see [BW, FS] or 1.5.8 in [GLP]). Let $\overline{\lambda}_1$ be the eigenvalue of $\overline{D}$ with the lowest absolute value and take a corresponding eigenspinor ψ. Then from (23), we have

$$\overline{D}\psi = \overline{\lambda}_1\psi, \qquad \int_\Sigma \langle D\psi, \psi\rangle\, d\Sigma \le 0.$$

Inequality (17) applied to this particular spinor field ψ could be written as

$$0 \ge -\frac{n}{2}\int_\Sigma H|\psi|^2\, d\Sigma \ge \frac{1}{4}\int_\Omega \overline{R}|\psi|^2\, d\Omega - \frac{n}{n+1}\overline{\lambda}_1^2 \int_\Omega |\psi|^2\, d\Omega.$$

Hence

$$\tag{27} \overline{\lambda}_1^2 \ge \frac{n+1}{4n}\inf_\Omega \overline{R}.$$

If equality in (27) is achieved, then ψ is a Killing spinor since it is simultaneously a twistor–spinor and an eigenspinor. Moreover, $\psi = \pi_+\psi$ and $H = 0$. Since a (real) Killing spinor has constant length and $\psi = \pi_+\psi = 0$ on the boundary Σ, the spinor field $\psi \equiv 0$ on Ω. This contradicts the fact that ψ is a non-trivial eigenspinor of $\overline{D}$. Therefore, equality in (27) could not hold. $\qquad\square$

The next result is an immediate generalization of (26) whose proof follows the same arguments as Theorem 4, when inequality (22) is used instead of (17). More precisely, we have

Theorem 5. *Let Ω be a compact $(n+1)$–dimensional Riemannian spin manifold with scalar curvature $\overline{R}$ satisfying $\frac{1}{4}\overline{R} + |\overline{\mathcal{Q}}_\psi|^2 \ge 0$ whose boundary $\partial\Omega$ is of nonnegative mean curvature. Then the first eigenvalue $\overline{\lambda}_1$ of the Dirac operator on Ω, with the Atiyah-Patodi-Singer condition, satisfies*

$$\tag{28} \overline{\lambda}_1^2 > \inf_{\Omega_\psi}\left(\frac{1}{4}\overline{R} + |\overline{\mathcal{Q}}_\psi|^2\right),$$

where Ω_ψ is the complement of the set of zeros of the associated eigenspinor field ψ.

Now, we use the Reilly type inequality (17) to get a lower bound for the first eigenvalue of the Dirac operator D on the boundary hypersurface Σ of the compact manifold Ω. More precisely, we have

Theorem 6. *Let M be a Riemannian spin manifold of nonnegative scalar curvature $\overline{R}$ and Σ a compact hypersurface. Suppose that Σ has nonnegative mean curvature H with respect to its inner unit normal field N and that it bounds a*

*compact domain Ω in M. Then, the lowest nonnegative eigenvalue λ_1 of the
Dirac operator on Σ satisfies*

$$(29) \qquad \lambda_1 \geq \frac{n}{2} \inf_{\Sigma} H.$$

*Moreover, if the equality holds, then Ω is a Ricci flat manifold, Σ has constant
mean curvature and the eigenspace corresponding to λ_1 consists of the restric-
tions to Σ of parallel spinors on the domain Ω.*

Proof. From Theorem 4 above or directly from Proposition 3, we have $\overline{\lambda}_1 > 0$
and so the following inhomogeneous boundary problem has a unique solution:

$$(30) \qquad \begin{cases} \overline{D}\psi &= 0 & \text{on } \Omega \\ \pi_+\psi &= \pi_+\varphi = \varphi & \text{on } \partial\Omega = \Sigma, \end{cases}$$

where $\varphi \in \Gamma(\mathbb{S}\Sigma)$ is an eigenspinor on Σ corresponding to the first eigenvalue
$\lambda_1 \geq 0$ of D. That is

$$D\varphi = \lambda_1\varphi \qquad \text{and so} \quad \pi_+\varphi = \varphi.$$

From the Reilly inequality (17), we get

$$\int_{\Sigma} \left(\lambda_1 - \frac{n}{2}H\right) |\psi|^2 \, d\Sigma \geq \frac{1}{4} \int_{\Omega} \overline{R}|\psi|^2 \, d\Omega,$$

which implies (29). For the equality case in (29) is satisfied for a spinor field
ψ, then ψ is harmonic spinor and a twistor–spinor, hence parallel. As $\pi_+\psi = \varphi$
along the boundary Σ, ψ is a non-trivial parallel spinor and also $\lambda_1 = nH/2$.
The existence of such a spinor field implies that Ω is a Ricci flat Riemannian
manifold (see, for example, [Ht] or [W]). On the other hand, since ψ is parallel,
one deduces from (9) that $D\psi = (nH/2)\psi$. Hence, as the equality $\lambda_1 = nH/2$
implies that H is constant, we have

$$\varphi = \pi_+\psi = \psi.$$

Conversely, the fact that the restriction to Σ of a parallel spinor on Ω is an
eigenspinor with eigenvalue $nH/2$ is a direct consequence of (9). $\qquad\square$

Remark 3. If R denotes the scalar curvature of the induced metric on the
embedded hypersurface Σ, we have the Friedrich Inequality [Fr]

$$(31) \qquad \lambda_1^2 \geq \frac{n}{4(n-1)} \inf_{\Sigma} R.$$

A consequence from the Gauss formula for the embedding $\Sigma \subset \Omega$ is that

$$R = \overline{R} - 2\overline{\mathrm{Ric}}\,(N, N) + n^2 H^2 - |\sigma|^2,$$

where $\overline{\mathrm{Ric}}$ is the Ricci tensor of Ω and σ is the second fundamental form of
the embedding. From this equation it is clear that, in general, we cannot hope
to obtain a relation between R and H allowing us to compare the Friedrich
inequality (31) and (29). However, if the Einstein tensor $\overline{\mathrm{Ric}} - (\overline{R}/2)\langle\,,\,\rangle$ of the
ambient manifold is positive semidefinite, then

$$R \leq n^2 H^2 - |\sigma|^2 \leq n(n-1)H^2,$$

where the last inequality is true because of the Schwarz inequality. Moreover, if the last inequality is in fact an equality, then the embedding is totally umbilical. Hence, we can state that

> *When the Einstein tensor of the ambient manifold Ω is positive semi-definite, then the extrinsic lower bound (29) for the first eigenvalue of the Dirac operator of an embedded hypersurface is sharper than the corresponding intrinsic Friedrich inequality (31).*

This is the situation, for example, when the ambient space M is Euclidean. In this case, if the scalar curvature R of the hypersurface Σ is positive, then H should be positive too (see Lemma 1 in [MR]), but it is possible for an embedded hypersurface to have positive mean curvature (always with respect to the inner normal) and negative scalar curvature (for example, consider the revolution tori in $\mathbb{R}^3$). So there are situations in which only inequality (29) will be significant.

It is well known that lower bounds for the first eigenvalue of the Dirac operator on a compact Riemannian spin manifold (see [Li]) have an important topological consequence: such a manifold with positive scalar curvature must have zero Â-genus. This occurs because this topological invariant can be expressed in terms of the index of the Dirac operator and inequality (31) imply that there are no non-trivial harmonic spinors. The same argument could be used to provide a topological obstruction for a compact hypersurface of a Riemannian spin manifold, with positive mean curvature and nonnegative scalar curvature, to bound a domain.

6. Constant mean curvature and minimal embedded hypersurfaces

We shall assume now that there exists a non-trivial parallel spinor field ψ_0 on the Riemannian spin ambient manifold M. As we have pointed out before, this implies that M is Ricci flat and reduces the possibilities for the (restricted) holonomy group of the manifold. In fact, when the manifold M is complete, simply connected and irreducible, the existence of such a spinor field is equivalent (see [W]) to one of the following facts: M is either flat, a Calabi-Yau Kähler manifold (that is, its holonomy group is $SU(m)$), a hyper-Kähler manifold (that is, its holomomy group is $Sp(m)$), $\dim M = 8$ and M has holonomy group $\mathrm{Spin}(7)$ or $\dim M = 7$ and the holonomy group of M is G_2. It is clear from (9) that one has

$$D\psi_0 = \frac{nH}{2}\psi_0$$

on the hypersurface Σ. Suppose also that the mean curvature H of Σ is constant and that this constant is nonnegative when computed with respect to the inner normal vector. Then we have $\lambda_1 \leq nH/2$ (this has been shown by Bär in [Bä] when the ambient space is Euclidean). But, in the present situation, we can apply Theorem 6 and so we obtain that, in fact, $\lambda_1 = nH/2$. Moreover each eigenspinor on Σ associated with this first eigenvalue is the restriction to Σ of a parallel spinor on the enclosed domain Ω. From this, we can deduce the two

results. First, we give a spinorial proof, in the spirit of the proof discovered by Reilly in [Re], for the celebrated Alexandrov Theorem (see [A] and [MR, O] for comments and references).

Theorem 7 (The Alexandrov Theorem). *The only compact embedded hypersurfaces in the Euclidean space with constant mean curvature are the round spheres.*

Proof. Let Σ be such a hypersurface in $\mathbb{R}^{n+1}$ and Ω the enclosed domain. By studying the maxima on Σ of the distance function to a point in Ω, one can see that the mean curvature H of Σ with respect to the inner normal must be positive (see [MR] for more details). Then we are in the situation quoted before the statement of the theorem. That is, the first eigenvalue of the Dirac operator D of the induced spin structure on Σ is $nH/2$ and each corresponding eigenspinor is the restriction to Σ of a parallel spinor on Ω. If $\xi(p)$ *denotes* the position vector at $p \in \mathbb{R}^{n+1}$, then the spinor field ψ defined, for all $p \in \Sigma$ by

$$\psi(p) = \gamma\Big(H\xi(p) + N(p)\Big)\psi_0(p)$$

is also an eigenspinor on the hypersurface Σ associated with the first eigenvalue. For this, it is sufficient to consider (3), identity (8) and the fact that

$$\sum_{i=1}^{n} \gamma(e_i)\overline{\nabla}_{e_i}\psi = \sum_{i=1}^{n} \gamma(e_i)\gamma(He_i - Ae_i)\psi_0 = -n \ \text{trace}\,(HI - A) = 0$$

when H is a constant. Hence this spinor field ψ is the restriction to Σ of a parallel spinor on Ω. Then, for any vector field $X \in \Gamma(T\Sigma)$, we have

$$0 = \overline{\nabla}_X\psi = \gamma(HX - AX)\psi_0,$$

where A is the shape operator corresponding to the normal field N. As ψ_0 has constant non-trivial length, we deduce that for any $X \in \Gamma(T\Sigma)$, $HX - AX = 0$. Hence, the hypersurface is umbilical. As it is also compact, it is the round sphere. □

An analogous reasoning works only for minimal hypersurfaces, if we replace the Euclidean ambient space by a Riemannian spin manifold with a non-trivial parallel spinor.

Theorem 8. *Minimal compact hypersurfaces bounding a compact domain, in a Riemannian spin manifold admitting a non-trivial parallel spinor, are necessarily totally geodesic. This is the case for minimal compact hypersurfaces embedded in simply connected Calabi-Yau manifolds, hyper-Kähler manifolds, or manifolds with holonomy groups* Spin(7) *and* G_2.

Proof. Let ψ_0 be a non-trivial parallel spinor on the ambient manifold. As in the proof above, under these hypotheses, we know that the first nonnegative eigenvalue of the Dirac operator D on the hypersurface Σ is $\lambda_1 = 0$, because $H = 0$, and that the restriction of ψ_0 on Σ is a corresponding eigenspinor. But

(11) implies that also $D\big(\gamma(N)\psi_0\big) = 0$, where N is a unit normal field. Thus the spinor field $\gamma(N)\psi_0$ is the restriction to Σ of a parallel spinor on the enclosed domain Ω. Then, if $X \in \Gamma(T\Sigma)$, we have

$$0 = \overline{\nabla}_X\big(\gamma(N)\psi_0\big) = -\gamma(AX)\psi_0.$$

Hence the shape operator A is identically zero on Σ, that is, Σ is totally geodesic. $\qquad\square$

Finally, we prove a rigidity result for compact hypersurfaces if the ambient Riemannian spin manifold admits a non-trivial parallel spinor. For this, we employ the same integral inequalities as above in a way inspired from the paper [Ro] by A. Ros.

Theorem 9. *Let M be a Riemannian spin manifold with a non-trivial parallel spinor and $\Sigma \subset M$, a compact hypersurface with nonnegative mean curvature, bounding a compact domain in M. Suppose that $\iota : \Sigma \to M$ is another isometric immersion which induces on Σ the same spin structure and whose mean curvature H_ι is constant and satisfies $|H_\iota| \le H$. Then the second fundamental form of the immersion ι coincides with that of the embedding $\Sigma \subset M$. In particular, $|H_\iota| = H$.*

Proof. Under these assumptions, we can identify the spinor bundles induced on Σ from the corresponding one on M, and the connections and Dirac operators constructed by means of the embedding $\Sigma \subset M$ and the immersion ι. From a non-trivial parallel spinor on M, by using equation (9) for the immersion ι, we get a spinor field φ on Σ such that

$$D\varphi = \frac{n}{2}H_\iota\varphi \qquad \text{and } |\varphi|^2 = 1 \text{ on } \Sigma,$$

where the mean curvature $H_\iota \ge 0$ is computed with respect to a suitable unit normal field N_ι for ι (note that Σ is orientable because it bounds a domain). In particular, since H_ι is a nonnegative constant, we have $\pi_+\varphi = \varphi$. On the other hand, taking into account that M is Ricci flat and that the mean curvature H of the embedding $\Sigma \subset M$ is nonnegative, we can apply Proposition 3 to obtain a smooth solution of the following problem

$$(32) \qquad \begin{cases} \overline{D}\psi & = 0 & \text{on } \Omega \\ \pi_+\psi & = \pi_+\varphi = \varphi & \text{on } \partial\Omega = \Sigma. \end{cases}$$

Then, from inequalities (17) and (23), we deduce that

$$0 \le \int_\Sigma \left(\langle D\pi_+\psi, \pi_+\psi\rangle - \frac{nH}{2}|\psi|^2\right) d\Sigma$$

and the equality holds if and only if ψ is parallel on Ω and $\psi = \pi_+\psi$. But we know that

$$D\pi_+\psi = D\varphi = \frac{n}{2}H_\iota\varphi = \frac{n}{2}H_\iota\pi_+\psi.$$

Hence

$$0 \leq \int_\Sigma \Big(H_\iota |\pi_+ \psi|^2 - H|\psi|^2 \Big) \, d\Sigma \leq 0$$

because $0 \leq H_\iota \leq H$ and $|\pi_+ \psi| \leq |\psi|$. As the equality is achieved and we know that $|\pi_+ \psi| = |\varphi| = 1$, we have $\psi = \pi_+ \psi = \varphi$, and $H_\iota = H$, hence the embedding $\Sigma \subset M$ has constant mean curvature H. Now, from (7) and the fact that ψ is parallel on Ω, we deduce

$$\gamma(AX)\gamma(N)\psi = \gamma(A_\iota X)\gamma(N_\iota)\psi$$

for any $X \in \Gamma(T\Sigma)$, where A_ι is the shape operator corresponding to the orientation N_ι for the immersion ι. Now, if $H = 0$, then Theorem 8 above implies that A is identically zero and hence $A_\iota \equiv 0$ too. Thus, assume $H > 0$. Multiplying both sides in the last equality by $\gamma(X)$ and contracting with X, since $H = H_\iota$, we have

$$\gamma(N)\psi = \gamma(N_\iota)\psi.$$

As a consequence, if X is tangent to the hypersurface Σ, we have

$$\gamma(AX)\psi = -\overline{\nabla}_X \Big(\gamma(N)\psi \Big) = -\overline{\nabla}_X \Big(\gamma(N_\iota)\psi \Big) = \gamma(A_\iota X)\psi.$$

This implies that the two shape operators coincide, as claimed. $\qquad\square$

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