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BOUNDARY BEHAVIOR OF SOLUTIONS TO LINEAR AND NONLINEAR ELLIPTIC EQUATIONS IN PLANE CONVEX DOMAINS

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1. Introduction

Boundedness of the first derivatives of solutions to the Dirichlet problem for the Poisson equation in any n-dimensional convex domain is a classical fact, and it is well known that the convexity of the domain implies the square summability of the second derivatives of solutions to the same problem ([Kad], [LU]). Both properties fail in the presence of reentrant corners. In the last few years, considerable progress was made in the study of other differentiability properties of solutions to the Poisson equation in arbitrary convex domains [A], [AJ], [F], [FJ], [J1], [J2]. But to our knowledge, no similar results have been obtained for systems and higher order equations. The present paper is a short communication on our recent work in this direction. It contains only statements of the results and the proofs will appear elsewhere.

We start with the Dirichlet problem for elliptic equations of order $2m$ with constant coefficients in an arbitrary bounded plane convex domain Ω. We state that the m-th order derivatives of these solutions are bounded if the coefficients of the equation are real (Theorem 2). For the case of strongly elliptic operators with complex coefficients we obtain the same result under the additional (and also necessary in general) assumption that the angles on $\partial\Omega$ are absent or sufficiently close to π (Theorem 1). In Theorem 3, we give a pointwise estimate for derivatives of Green's function.

These results rely heavily upon a precise pointwise asymptotic estimate for solutions near a boundary point, which is of independent interest (Lemma 1). This estimate has been established without the convexity assumption and under the sole requirement of smallness of the local Lipschitz constant of the boundary. It can be extended to elliptic operators on n-dimensional Lipschitz domains but we do not dwell upon this generalization in the sequel. We emphasize that this estimate is of different nature in comparison with the well known results on elliptic equations in Lipschitz domains (see the book [Ken] and more recent papers [JK] and [AP]).

Our first application to problems of mathematical physics concerns the system of two-dimensional anisotropic elasticity theory in a convex domain. We claim

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that all elements of the stress tensor are uniformly bounded if zero traction conditions are prescribed on the boundary (Theorem 4).

As the second application, we guarantee the boundedness of the gradient of the velocity vector satisfying the Navier–Stokes system and zero Dirichlet conditions on the boundary of a convex domain (Theorem 5).

Finally, we assert the boundedness of the second derivatives of a solution to the system of von Kármán equations in a convex domain whose boundary is clamped in transversal direction and free in horizontal direction (Theorem 6).

2. Linear higher order equation in a convex domain

By w we denote a unique solution to the Dirichlet problem

$$L(\partial_x)w = f \quad w \in \mathring{W}^{m,2}(\Omega)\,, \tag{1}$$

where $f \in W^{1-m,q}(\Omega)$ with $q \in (2,\infty)$. Here $\mathring{W}^{l,p}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ in the Sobolev space $W^{l,p}(\Omega)$, $1 < p < \infty$, and $W^{-l,p'}(\Omega)$ with $p' = p/(p-1)$ is the dual of $\mathring{W}^{l,p}(\Omega)$. The operator $L(\partial_x)$ is strongly elliptic and given by

$$L(\partial_x) = \sum_{0\le k\le 2m} a_k \partial_{x_1}^k \partial_{x_2}^{2m-k}\,.$$

Theorem 1. *Let Ω be a bounded convex domain in $\mathbb{R}^2$ such that the jumps of all angles between the exterior normal vector to $\partial\Omega$ and the x–axis do not exceed a sufficiently small constant depending on m, q and the coefficients of $L(\partial_x)$. Then the solution w belongs to the space $C^{m-1,1}(\overline{\Omega})$, i.e. its derivatives of order $m-1$ are Lipschitz on $\overline{\Omega}$. The estimate holds*

$$||w||_{C^{m-1,1}(\Omega)} \le C\,||f||_{W^{1-m,q}(\Omega)} \tag{2}$$

where the constant C does not depend on f.

Generally, this assertion does not hold without the above restriction on the jumps of the normal vector. More precisely, if there exists an angle vertex on $\partial\Omega$, one can construct a second order strongly elliptic operator L with complex coefficients such that the Dirichlet problem (1) with $f \in C^\infty(\overline{\Omega})$ has a solution with unbounded gradient [Koz] (see also [KMR], Sect. 8.4.3).

Theorem 2. *Let the coefficients of $L(\partial_x)$ be real and let Ω be an arbitrary bounded convex domain in $\mathbb{R}^2$. Then the conclusion of Theorem 1 holds.*

Theorem 3. *Let $G_L(x,y)$ denote Green's function of problem (1). Also let Ω be an arbitrary bounded convex domain if the coefficients of L are real and let, additionally, the jumps of all angles between the exterior normal vector to $\partial\Omega$ and the x–axis be smaller than a constant depending on m and the coefficients of $L(\partial_x)$ in the complex coefficient case. Then for all x, y in Ω*

$$\sum_{|\alpha|=|\beta|=m} |\partial_x^\alpha \partial_y^\beta G_L(x,y)| \le C\,|x-y|^{-2}\,,$$

where C is a positive constant depending on m, the coefficients of L and Ω.

3. Auxiliary local estimate near Lipschitz boundary

In the next lemma and only there, we refute the assumption of convexity of Ω and estimate the solution w of problem (1) in a neighborhood $B_\delta = \{x \in \mathbb{R}^2 : |x| < \delta\}$ of the point $\mathcal{O} \in \partial\Omega$. We assume that $\Omega \cap B_{2\delta}$ is described by the inequalities $x_2 > \varphi(x_1)$, $|x| < 2\delta$, where φ is a Lipschitz function on $[-2\delta, 2\delta]$ and $\varphi(0) = 0$.

Lemma 1. *Suppose $f(x) = 0$ for $|x| < 2\delta$ and the Lipschitz norm of φ on $[-2\delta, 2\delta]$ does not exceed a certain constant depending only on m and the coefficients of L. Then for all $x \in \Omega \cap B_\delta$ and $k = 1\dots, m-1$*

$$(3) \quad |\nabla_k w(x)| \le A|x|^{m-k} \exp\Big(-a\int_{|x|}^{\delta} \frac{\varphi(\rho)+\varphi(-\rho)}{\rho^2} d\rho + b\int_{|x|}^{\delta} \max_{|t|<\rho} |\varphi'(t)|^2 \frac{d\rho}{\rho}\Big) .$$

Here we use the notation

$$a = \frac{1}{2\pi} \Im \sum_{1\le k\le m} (\zeta_k^+ - \zeta_k^-) ,$$

where $\zeta_1^+, \dots, \zeta_m^+$ and $\zeta_1^-, \dots, \zeta_m^-$ are roots of the polynomial $L(1,\zeta)$ with positive and negative imaginary parts respectively. This value of a is best possible. By b we denote a positive constant depending only on m and the coefficients of L, and we put $A = c\,\delta^{-1-m}||w||_{L^2(\Omega\cap B_{2\delta})}$, where c is a constant depending on m and the coefficients of L.

Note that for the polyharmonic operator Δ^m we have $\zeta_k^\pm = \pm i$ which implies $a = m/\pi$.

In short, the proof of this key lemma is as follows. By using a particular coordinate transformation we reduce the original Dirichlet problem to a first order evolution system

$$(4) \qquad \partial_t \mathbf{U} - M\mathbf{U} - N(t)\mathbf{U} = \mathbf{F}$$

in the strip $\mathbb{R} \times (0,\pi)$. Here, M and N are matrix ordinary differential operators on $(0,\pi)$ with M independent of t, and $N(t)$ playing the role of a small perturbation. In order to obtain sharp information on the global behavior of $\mathbf{U}(t,\cdot)$ we use a spectral splitting of (4) into a one-dimensional and infinite-dimensional parts, modifying the technique developed in [KM].

4. Plane anisotropic elasticity

We consider the equations

$$(5) \qquad \partial_{x_1}\sigma_{i1} + \partial_{x_2}\sigma_{i2} = f_i \quad \text{in } \Omega \text{ for } i = 1, 2,$$

which describe the plane deformation of a homogeneous anisotropic body. The stress tensor $\sigma = \{\sigma_{ij}\}_{i,j=1,2}$ is connected with the strain tensor $\varepsilon = \{\varepsilon_{ij}\}_{i,j=1,2}$

by the Hooke law

$$\varepsilon_{ij} = \sum_{k,l=1}^{2} a_{ij,kl}\sigma_{kl}\,,$$

where $a_{ij,kl}$ are real numbers such that $a_{ij,kl} = a_{kl,ij} = a_{ji,kl}$ and

$$\sum_{i,j,k,l=1}^{2} a_{ij,kl}\sigma_{ij}\sigma_{kl} \geq c_0 \sum_{i,j=1}^{2} \sigma_{ij}^2\,, \quad c_0 > 0\,.$$

We suppose that

$$\sigma_{i1}\nu_1 + \sigma_{i2}\nu_2 = 0 \quad \text{on } \partial\Omega \;\; \text{for } i = 1, 2, \tag{6}$$

where ν_1 and ν_2 are components of the unit normal vector to $\partial\Omega$. Evaluating ε by the displacement vector $\mathbf{u} = (u_1, u_2)$ one arrives at a second order elliptic system with respect to u_1 and u_2. It is standard that this system complemented by (6) is solvable in $(W^{1,2}(\Omega))^2$ if the right-hand side $\mathbf{f} = (f_1, f_2)$ belongs to $(L^2(\Omega))^2$ and is orthogonal to the rigid body displacements.

Theorem 4. *Let Ω be a bounded convex domain and let $\mathbf{f} \in \big(L^q(\Omega)\big)^2$ for some $q > 2$. Then all elements of the stress tensor are uniformly bounded in Ω.*

5. Navier-Stokes system

We consider the Dirichlet problem for the Navier-Stokes system

$$\begin{cases} -\nu\Delta\mathbf{v} + \nabla p + \sum_{k=1}^{2} v_k \partial_{x_k}\mathbf{v} = \mathbf{f} & \text{in } \Omega \\ \nabla\cdot\mathbf{v} = 0 & \text{in } \Omega \\ \mathbf{v} = 0 & \text{on } \partial\Omega \end{cases} \tag{7}$$

where $\mathbf{f} \in \big(W^{-1,2}(\Omega)\big)^2$ and $(\mathbf{v}, p) \in \big(\mathring{W}^{1,2}(\Omega)\big)^2 \times L^2(\Omega)$.

Theorem 5. *Let Ω be a bounded convex domain and let $\mathbf{f} \in \big(L^q(\Omega)\big)^2$ for some $q > 2$. Then $\mathbf{v} \in \big(C^{0,1}(\overline{\Omega})\big)^2$.*

It is worth mentioning that Lemma 1 leads to the following pointwise estimate for the velocity vector which is valid without the convexity requirement on Ω:

$$|\mathbf{v}(x)| \leq C|x| \exp\Big(-\frac{2}{\pi}\int_{|x|}^{\delta} \frac{\varphi(\rho)+\varphi(-\rho)}{\rho^2}\,d\rho + b\int_{|x|}^{\delta} \max_{|t|<\rho} |\varphi'(t)|^2 \frac{d\rho}{\rho}\Big) \tag{8}$$

for all $x \in \Omega \cap B_\delta$. Here b is a positive constant depending on ν and C is a positive constant depending ν, $||\mathbf{f}||_{(L^q(\Omega\cap B_{2\delta}))^2}$ and $||\mathbf{v}||_{(L^2(\Omega\cap B_{2\delta}))^2}$. The value $2/\pi$ in (8) is precise.

6. Von Kármán equations

Now we deal with the Dirichlet problem for the system

$$
\left\{
\begin{array}{ll}
\Delta^2 u_1 = [u_1, u_2] + f_1 & \text{in } \Omega \\
\Delta^2 u_2 = [u_1, u_1] + f_2 & \text{in } \Omega \\
\mathbf{u} := (u_1, u_2) \in \left(\mathring{W}^{2,2}(\Omega)\right)^2, & \mathbf{f} := (f_1, f_2) \in \left(W^{-2,2}(\Omega)\right)^2,
\end{array}
\right. \tag{9}
$$

where

$$[u, v] = \partial^2_{x_1} u \cdot \partial^2_{x_2} v + \partial^2_{x_2} u \cdot \partial^2_{x_1} v - 2\partial_{x_1}\partial_{x_2} u \cdot \partial_{x_1}\partial_{x_2} v\,.$$

Theorem 6. *Let Ω be a bounded convex domain and let $\mathbf{f} \in \left(W^{-1,q}(\Omega)\right)^2$ for some $q > 2$. Then $\mathbf{u} \in \left(C^{1,1}(\overline{\Omega})\right)^2$.*

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