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L^2 CURVATURE AND VOLUME RENORMALIZATION OF AHE METRICS ON 4-MANIFOLDS

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Abstract. This paper relates the boundary term in the Chern-Gauss-Bonnet formula on 4-manifolds M with the renormalized volume V, as defined in the AdS/CFT correspondence, for asymptotically hyperbolic Einstein metrics on M. In addition we compute and discuss the differential or variation dV of V, or equivalently the variation of the L^2 norm of the Weyl curvature, on the space of such Einstein metrics.

0. Introduction

The Chern-Gauss-Bonnet formula for a compact Riemannian 4-manifold (M, g) without boundary states that

$$(0.1)\quad \frac{1}{8\pi^2}\int_M (|R|^2 - 4|z|^2)dV = \frac{1}{8\pi^2}\int_M (|W|^2 - \frac{1}{2}|z|^2 + \frac{1}{24}s^2)dV = \chi(M),$$

where R, W, z, s are respectively the Riemann, Weyl, trace-free Ricci and scalar curvatures.

In particular, if g is an Einstein metric, then $z = 0$, and so Einstein metrics minimize the L^2 norm of the curvature over all metrics on M. Hence the L^2 norm of the full curvature of an Einstein metric on M is apriori bounded by the topology of M.

If (M, g) is a compact 4-manifold with non-empty boundary, then (0.1) no longer holds; there is a correction or defect term given by certain curvature integrals over the boundary ∂M. If (M, g) is complete and open, then the boundary integrals relate to the asymptotic geometry of (M, g).

When (M, g) is a complete non-compact, Ricci-flat 4-manifold, then the defect term in (0.1) is easily identified if the manifold asymptotically approaches that of a quotient of $\mathbb{R}^4$, i.e. M is asymptotically locally Euclidean (ALE), flat, (AF) or locally flat, (ALF), c.f. [5].

In this paper, we consider this issue when (M, g) is an Einstein 4-manifold of negative scalar curvature, which is asymptotically hyperbolic. To define this, let M be an arbitrary compact, connected and oriented 4-manifold with non-empty boundary ∂M; we do not assume that ∂M is connected. According to Penrose, c.f. [14] and also [12], a complete metric g on M is *conformally compact* if there

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is a smooth defining function ρ on $\bar{M} = M \cup \partial M$, i.e. $\rho(\partial M) = 0$, $d\rho \neq 0$ on ∂M and $\rho > 0$ on M, such that the metric

$$\bar{g} = \rho^2 \cdot g, \tag{0.2}$$

extends to a smooth metric on $\bar{M}$. We require that $\bar{g}$ is at least C^3 smooth up to ∂M, although this condition could be relaxed somewhat.

Conversely, if $\bar{g}$ is any smooth Riemannian metric on $\bar{M}$ and ρ is any C^1 defining function, then $g \equiv \rho^{-2} \cdot \bar{g}$ gives a complete conformally compact metric on the open manifold M.

The defining function ρ is not unique, since it can be multiplied by any smooth positive function on $\bar{M}$. Hence, both the metric $\bar{g}$ and its induced metric γ on ∂M are not uniquely defined by (M, g). However, the conformal class $[\gamma]$ of the metric $\gamma = \bar{g}_{T\partial M}$ is uniquely determined by the complete metric g; $(\partial M, [\gamma])$ is called the *conformal infinity* of (M, g). Conversely, any conformal class $[\gamma]$ on ∂M is the conformal infinity of a complete metric on M.

When (M, g) is a complete conformally compact Einstein metric with $\mathrm{Ric}_g = -3g$, then the sectional curvatures of g necessarily approach -1 uniformly at infinity at an exponential rate, c.f. (1.3) below or [8]. Such manifolds are called asymptotically hyperbolic (AH).

The study of complete AH Einstein (AHE) manifolds has become very active recently due to the AdS/CFT correspondence in string theory, c.f. [16] and references therein. In order to produce an effective gravitational action, one needs to renormalize the volume of such a metric, since the volume itself is obviously infinite. If (M, g) is an AH Einstein 4-manifold and ρ is any defining function on M, then one has the following asymptotic expansion for the volume of compact domains $B(r) = \{\log \rho^{-1} \leq r\}$ in M as $r \to \infty$;

$$\mathrm{vol}\, B(r) = v_o e^{3r} + v_1 e^r + V + o(1). \tag{0.3}$$

The coefficients v_o and v_1 depend on the geometry of $(\partial M, \gamma)$ as well as the defining function ρ in this generality.

Clearly, since there are numerous defining functions, the exact exponential growth rates in r, as well as the coefficients, depend on the compactification $(\bar{M}, \bar{g})$, and are not defined intrinsically w.r.t. (M, g). However, the constant term V in (0.3) is an invariant of (M, g), i.e. is independent of the choice of ρ. This is one of the elementary consequences of the AdS/CFT correspondence, c.f. [16]; a proof appears in [7].

The first purpose of this paper is to relate the renormalized volume V in (0.3) with the Chern-Gauss-Bonnet theorem in dimension 4.

Theorem 0.1. *Let (M, g) be a complete AH Einstein 4-manifold. Then, up to a constant, the boundary term at infinity in the Chern-Gauss-Bonnet formula renormalizes the volume in the sense of* (0.3). *In fact,*

$$\frac{1}{8\pi^2}\int_M |W|^2 dV = \chi(M) - \frac{3}{4\pi^2}V. \tag{0.4}$$

An analogous result holds for arbitrary AH metrics on M which are suitably asymptotic to an Einstein metric at infinity, c.f. Remark 1.2. Of course (0.4) shows directly that V is an intrinsic invariant of (M, g), independent of any compactification $(\bar{M}, \bar{g})$.

One thus has the following universal upper bound on V for any AH Einstein metric:

$$V \leq \frac{4\pi^2}{3}\chi(M), \tag{0.5}$$

with equality if and only if (M, g) is hyperbolic. Even when (M, g) is hyperbolic, i.e. $M = H^4(-1)/\Gamma$, (0.5) gives non-trivial information, since it implies that V is an integer, mod $4\pi^2/3$. The renormalized volume in this case may serve as an analogue of Thurston's theory of the volume of closed hyperbolic 3-manifolds. (After completion of the paper, the referee informed the author that the equality in (0.5) for hyperbolic manifolds has also been proved by C. Epstein in Appendix A to [13]).

A result analogous to (0.4) holds for AHE metrics on $M = M^n$ in any dimension $n \geq 2$ and relates the Chern-Gauss-Bonnet integrand (Euler density) with $\chi(M)$ and the volume renormalization V in even dimensions. In odd dimensions, it (re)-produces the formula for the conformal anomaly, c.f. [10]. This will be detailed elsewhere, and we restrict here to dimension 4.

It is interesting to compare, and combine, Theorem 0.1 with a result of Hitchin [11], where an analogous result is proved for the signature via the Atiyah-Patodi-Singer index theorem. Thus for any AH Einstein metric, (or any AH metric suitably asymptotic to an Einstein metric at infinity), one has

$$\frac{1}{12\pi^2}\int_M (|W^+|^2 - |W^-|^2)dV = \tau(M) - \eta_\gamma, \tag{0.6}$$

where $\tau(M)$ is the signature of M and η_γ is the eta-invariant of the conformal infinity $(\partial M, \gamma)$. In particular, (0.4) and (0.6) imply the following analogue of the Hitchin-Thorpe inequality for AHE metrics:

$$\chi(M) - \frac{3}{4\pi^2}V \geq \frac{3}{2}|\tau(M) - \eta_\gamma|, \tag{0.7}$$

with equality if and only if (M, g) is self-dual Einstein.

The volume term V clearly depends, apriori, on the global geometry of the 'bulk' manifold (M, g). However, the η-invariant of $(\partial M, \gamma)$ depends only on the intrinsic geometry of the conformal class $[\gamma]$ on ∂M. Thus, for a self-dual Einstein metric on M, it follows that V is also an intrinsic invariant of $(\partial M, [\gamma])$, given that the topology of M is fixed.

The second purpose of this paper is to discuss to what extent this might be true for a general AH Einstein metric on M. First, recall that η is a global conformal invariant of $[\gamma]$, i.e. it is not computable from the local geometry of $[\gamma]$. However, the variation of η in the space of metrics is a local quantity; thus, if $h_{(0)}$ is an infinitesimal variation of γ on ∂M, then

$$d\eta(h_{(0)}) = -\frac{1}{24\pi^2}\int_{\partial M} < *d\,\mathrm{Ric}, h_{(0)} > d\,\mathrm{vol}_\gamma, \tag{0.8}$$

c.f. [4, Thm. 6.9] and [1, Prop. 4.19]. Here Ric is the Ricci curvature of γ, viewed as a 1-form with values in the tangent bundle, d is the exterior derivative on Λ^1 induced by the metric, and $*$ is the Hodge star operator $* : \Lambda^2 \to \Lambda^1$. Recall that $d\,\mathrm{Ric}$ is the well-known Cotton-York tensor of conformal geometry, whose vanishing characterizes conformal flatness.

Now let g be an AH Einstein metric on M and h an infinitesimal AHE variation of g, with $h_{(0)}$ the induced variation of the boundary metric γ. We then have the following expression for the variation or differential dV of V.

Theorem 0.2. *Let g be an AH Einstein metric, and h an infinitesimal AHE deformation. Then the differential of the renormalized volume V in the direction h is given by*

$$dV(h) = -\frac{1}{4}\int_{\partial M} < g_{(3)}, h_{(0)} > d\,\mathrm{vol}_\gamma, \tag{0.9}$$

where $g_{(3)}$ is the 3^{rd} order term in the Taylor expansion of the metric $\bar{g}$ at ∂M, w.r.t. the special defining function determined by γ, c.f. §1.

A formula similar to (0.9) holds in all dimensions ≥ 4. Thus the variation of V at g is determined solely by the behavior of $\bar{g}$ at the boundary ∂M. Formally speaking, we may consider $g_{(3)}$ as the gradient of the volume function V, modulo the factor $-\frac{1}{4}$. The term $g_{(3)}$ is formally undetermined, in the sense that the Einstein equations do not determine any local expression for $g_{(3)}$ at ∂M, c.f. [6], [7]. This is in contrast to the situation for the terms $g_{(j)}, j \leq 2$, which are determined locally by the geometry of $\gamma = g_{(0)}$.

In Proposition 2.6, we relate the formulas (0.8) and (0.9). Namely, let g be an AH Einstein metric on M with boundary metric γ. If γ is not conformally flat and $dV \neq 0$, then

$$dV(h) = -\frac{1}{12}\int_{\partial M} < *d\,\mathrm{Ric}, h^+_{(0)} > d\,\mathrm{vol}_\gamma + \frac{1}{12}\int_{\partial M} < *d\,\mathrm{Ric}, h^-_{(0)} > d\,\mathrm{vol}_\gamma, \tag{0.10}$$

where $h^\pm_{(0)} = \pi^\pm(h_{(0)})$ and $\pi^\pm$ are linear projection operators on the space of symmetric bilinear forms $S^2(\partial M)$. As in (0.9), the projections $\pi^\pm$ depend, apriori, on the term $g_{(3)}$.

It is a rather delicate open question whether the dependence of dV in (0.9) or (0.10) on $g_{(3)}$ can be reduced to a dependence only on the boundary metric $(\partial M, \gamma)$, as is the case for $d\eta$. We point out at the end of §2 that at a minimum, this depends on the global topology of the bulk or filling manifold M, (again in contrast to $d\eta$, which is independent of M). This is illustrated by observing that the hyperbolic metric on $H^4(-1)/\mathbb{Z} \approx \mathbb{R}^3 \times S^1$ and the Schwarzschild AdS metric on $\mathbb{R}^2 \times S^2$ have $\partial M = S^2 \times S^1$, and with suitable normalization, have the same conformally flat boundary metric γ on $S^2 \times S^1$. However, both V and dV are different for these metrics.

On the positive side, we will show elsewhere that an AH Einstein metric on a given manifold M is uniquely determined, up to diffeomorphism, by dV and the boundary metric γ, at least when the induced map $\pi_1(\partial M) \to \pi_1(M)$ is surjective. In addition, the results of this paper will be applied elsewhere to study the Dirichlet problem for AHE metrics with prescribed conformal infinity, c.f. [6], [8].

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1. Chern-Gauss-Bonnet and V

This section is concerned with the proof of Theorem 0.1. Before starting the proof, we discuss some further background material on conformally compact metrics, c.f. also [7], [8].

If g is a complete conformally compact metric on M, with defining function ρ, let

$$r = \log \rho^{-1}, \rho = e^{-r}. \tag{1.1}$$

A simple computation shows that

$$|\nabla r|_g^2 = |\nabla \rho|_{\bar{g}}^2 \equiv |\bar{\nabla}\rho|^2, \tag{1.2}$$

and that this quantity is independent of the choice of defining function ρ at ∂M. Hence when the compactification $\bar{g}$ is C^1, $|\bar{\nabla}\rho|^2$ on ∂M is an invariant of the conformal structure $(\partial M, [\gamma])$.

Now a computation for the change of curvature under conformal change in the metric shows

$$\bar{K}_{ij} = \rho^{-2}(K_{ij} + |\bar{\nabla}\rho|^2) - \rho^{-1}\{\bar{D}^2\rho(\bar{e}_i, \bar{e}_i) + \bar{D}^2\rho(\bar{e}_j, \bar{e}_j)\}, \tag{1.3}$$

and, if (M, g) is Einstein with $\mathrm{Ric} = -3g$, then

$$\bar{\mathrm{Ric}} = -2\rho^{-1}\bar{D}^2\rho + \{3\rho^{-2}(|\bar{\nabla}\rho|^2 - 1) - \rho^{-1}\bar{\Delta}\rho\}\bar{g}. \tag{1.4}$$

Here $\bar{K}_{ij}$, (resp. K_{ij}), denotes the sectional curvature of $(M, \bar{g})$, (resp. (M, g)), in the $(\bar{e}_i, \bar{e}_j)$ direction, where $\{\bar{e}_i\}$ form an orthonormal basis w.r.t. $(M, \bar{g})$. Hence if $\bar{g}$ is C^2 smooth up to ∂M, then

$$K_{ij} = -|\bar{\nabla}\rho|^2 + O(\rho^2). \tag{1.5}$$

Thus, the complete metric g is asymptotically of variable strictly negative curvature; the curvature varies between two negative constants. The metric g is called *asymptotically hyperbolic* (AH) if the invariant $|\bar{\nabla}\rho|^2$ satisfies

$$|\bar{\nabla}\rho|^2 = 1 \ \text{ on } \ \partial M. \tag{1.6}$$

Note that if (M, g) is Einstein, then (1.4) implies that (1.6) must hold, so that any conformally compact Einstein metric is automatically AH.

It is also elementary to see, (c.f. [8]) that if (M, g) is AH, then there is a defining function ρ such that in a collar neighborhood U of ∂M,

$$|\nabla r| = |\bar{\nabla}\rho| \equiv 1, \tag{1.7}$$

in U. The metrics g and $\bar{g}$ thus split in U as

$$g = dr^2 + g_r \text{ and } \bar{g} = d\rho^2 + \bar{g}_\rho, \tag{1.8}$$

where $g_r = \rho^{-2} \cdot g_\rho$ is a curve of metrics on the 3-manifold ∂M. Thus, w.r.t. the metric g or $\bar{g}$, the flow lines of ∇r or $\bar{\nabla}\rho$ are geodesics. The function ρ gives the distance to ∂M w.r.t. $\bar{g}$, while the function $r = \log \rho^{-1}$ is a distance function w.r.t. g from the boundary of some compact set in M.

Defining functions satisfying (1.7) are called *special*, or alternately *geodesic*, defining functions. Special defining functions are still not unique; as an example, r may be the distance function from the boundary of any compact convex subset of $(M, g) = H^4(-1)$.

If (M, g) has a special C^k conformal compactification $(M, \bar{g})$, then one may expand $\bar{g}$, i.e. $\bar{g}_\rho$ in (1.8), in a Taylor series in powers of ρ, as

$$\bar{g}_\rho = g_{(0)} + \rho g_{(1)} + \rho^2 g_{(2)} + \rho^3 g_{(3)} + ... + \rho^k g_{(k)} + o(\rho^k), \tag{1.9}$$

where the terms $g_{(i)}$ are bilinear forms on $T(\partial M)$, i.e. are annihilated when evaluated on $\bar{\nabla}\rho$. The term $g_{(0)}$ is just the boundary metric γ, while $g_{(j)} = \dfrac{1}{j!}\mathcal{L}^{(j)}_{\bar{\nabla}\rho}\bar{g}_\rho|_{\rho=0}$, where $\mathcal{L}$ is the Lie derivative.

Now if g is Einstein, then results of Fefferman-Graham [6], c.f. also [7], imply that

$$g_{(1)} = 0, \tag{1.10}$$

so that ∂M is totally geodesic in $\bar{M}$ w.r.t. $\bar{g}$ and further that

$$trg_{(3)} = 0, \quad \delta g_{(3)} = 0, \tag{1.11}$$

where the trace and divergence are w.r.t. γ. The term $g_{(2)}$ is intrinsically and locally determined by $\gamma = g_{(0)}$, c.f. [7] or (2.18) below, but the Einstein equations do not imply any local intrinsic determination of $g_{(j)}$, for $j \geq 3$, beyond (1.11).

The expansion (1.9) gives the following expansion for $\mathrm{vol}\, \bar{S}(\rho)$:

$$\mathrm{vol}\, \bar{S}(\rho) = \mathrm{vol}\, \bar{S}(0) + \rho^2 v_{(2)} + O(\rho^4). \tag{1.12}$$

There is no ρ^3 term, by (1.11). Hence, $\mathrm{vol}\, S(r)$ has the expansion

$$\mathrm{vol}\, S(r) = v_{(0)}e^{3r} + v_{(2)}e^r + O(e^{-r}), \tag{1.13}$$

and so, as in (0.3), $\mathrm{vol}\, B(r)$ has the expansion

$$\mathrm{vol}\, B(r) = \frac{1}{3}v_{(0)}e^{3r} + v_{(2)}e^r + V + O(e^{-r}). \tag{1.14}$$

We now begin with the proof of Theorem 0.1 itself. We assume that g is an AHE metric on the 4-manifold M, and let ρ be a special defining function for (M, g). Since g is Einstein, the curvature tensor R is pure Weyl and scalar, i.e.

$$R = W - \frac{1}{2} g \wedge g,$$

where $\wedge$ denotes the Kulkarni-Nomizu product, c.f. [2, Ch.1G]. When $W = 0$, the curvature tensor $R = -\frac{1}{2} g \wedge g$ gives the curvature of hyperbolic space $H^4(-1)$, with sectional curvature -1. Thus

$$(1.15)\quad \int_M (|R|^2 - 6) d\,\mathrm{vol}_g = \int_M |R + \frac{1}{2} g \wedge g|^2 d\,\mathrm{vol}_g = \int_M |W|^2 d\,\mathrm{vol}_g = \int_M |\bar{W}|^2 d\,\mathrm{vol}_{\bar{g}},$$

where the second equality uses the conformal invariance of the L^2 norm of W on 4-manifolds. Since, by assumption, g has a C^2 conformal compactification, this integral is finite. The norm here is the usual L^2 norm of R or W, as a symmetric map $\Lambda^2(TM) \to \Lambda^2(TM)$, so that $|R|^2 = 6$ on $H^4(-1)$; this is $\frac{1}{4}|R|^2$, when R is viewed as a (4,0) tensor.

Let D be a compact domain in M, with smooth boundary $\partial D \subset M$. Since g is Einstein, the Chern-Gauss-Bonnet formula for manifolds with boundary states

$$(1.16)\quad \frac{1}{8\pi^2} \int_D |R|^2 = \chi(D) - \frac{1}{2\pi^2} \int_{\partial D} \prod_1^3 \lambda_i - \frac{1}{8\pi^2} \int_{\partial D} \sum_{\sigma \in S_3} K_{\sigma_1 \sigma_2} \lambda_{\sigma_3},$$

c.f. [3]. Here λ_i are the eigenvalues of the 2^{nd} fundamental form A and the indices σ_i run over an orthonormal basis of the tangent spaces to ∂D. The sign on A is chosen so that $\lambda_i > 0$ for convex domains; K denotes sectional curvature.

Let $\partial D = S(r)$ be the r-level set of the function r in (1.1) and let $D = B(r)$ be the corresponding sublevel set. The 2^{nd} fundamental form of $S(r)$ is then given by $A = D^2(r) = -D^2 \log \rho$. For r sufficiently large, i.e. ρ sufficiently small, D is diffeomorphic to M. Hence (1.16) may be rewritten as

$$(1.17)\quad \frac{1}{8\pi^2} \int_{B(r)} |W|^2 = \chi(M) - \frac{3}{4\pi^2}\big(\mathrm{vol}\, B(r) + \frac{2}{3} \int_{S(r)} \prod_1^3 \lambda_i + \frac{1}{6} \int_{S(r)} \sum_{\sigma \in S_3} K_{\sigma_1 \sigma_2} \lambda_{\sigma_3}\big).$$

All three terms in the parenthesis diverge to $\pm\infty$ as $r \to \infty$, and so we need to understand their cancellation properties. From the expansion (1.9) and (1.14), using (1.3) and (1.18) below, one may prove purely formally that these terms must converge to V as $r \to \infty$. However, it is worthwhile to calculate this explicitly to see just how the Einstein condition is being used. We begin by analysing the boundary integrals over $S(r)$. Following this, we analyse the bulk integral over $B(r)$.

The eigenvalues λ_i of $D^2 r$ are related to the eigenvalues $\bar{\lambda}_i$ of $\bar{D}^2\rho$ by

$$\lambda_i = 1 - \rho\bar{\lambda}_i, \tag{1.18}$$

c.f. [2, Ch. 1J] for instance for formulas on conformal changes of the metric. Hence

$$\prod \lambda_i = 1 - \bar{H}\rho + \bar{\sigma}\rho^2 - \bar{\pi}\rho^3,$$

where $\bar{H} = \mathrm{tr}\ \bar{D}^2\rho$ is the mean curvature of $S(r) = \bar{S}(\rho), \bar{\sigma} = \prod_{i<j} \bar{\lambda}_i\bar{\lambda}_j$ and $\bar{\pi} = \bar{\lambda}_1\bar{\lambda}_2\bar{\lambda}_3$. Here and in the following, the indices 1,2,3 refer to directions tangent to $S(r)$, while the index 4 refers to the normal direction.

Next, for the boundary curvature term in (1.17), using (1.3) and (1.18) we have

$$\sum_{\sigma\in S_3} K_{\sigma_1\sigma_2}\lambda_{\sigma_3} = -6 + 6\bar{H}\rho + 2\bar{\tau}\rho^2 + O(\bar{\lambda}^2)\bar{\rho}^2 + O(\bar{K}\bar{\lambda})\rho^3,$$

where $\bar{\tau} = \bar{K}_{12} + \bar{K}_{13} + \bar{K}_{23}$, and $O(\bar{\lambda}^2)$ and $O(\bar{K}\bar{\lambda})$ denote terms quadratic in $\bar{\lambda}$ or products of $\bar{K}$ and $\bar{\lambda}$. For the last two terms in (1.17), we thus have

$$\int_{S(r)} \{(\frac{2}{3} - 1) + (-\frac{2}{3} + 1)\bar{H}\rho + \frac{1}{3}\bar{\tau}\rho^2 + O(\bar{\lambda}^2)\rho^2 + O(\bar{\pi}, \bar{K}\bar{\lambda})\rho^3\}. \tag{1.19}$$

Now $\mathrm{vol}_g S(r) = \rho^{-3}\,\mathrm{vol}_{\bar{g}} S(\rho) \sim \rho^{-3}$. On the other hand, by (1.10), we have

$$\bar{D}^2\rho = \bar{A} = \frac{1}{2}\mathcal{L}_{\bar{\nabla}\rho}\bar{g} = O(\rho), \tag{1.20}$$

where $\bar{A}$ is the 2nd fundamental form of $\bar{S}(\rho)$ in $(M, \bar{g})$. Hence, the last two terms in (1.19) are $O(\rho^4)$. This shows that (1.19) may be rewritten in the form

$$-\frac{1}{3}\,\mathrm{vol}\, S(r) + \frac{1}{3}\int_{S(r)} (\bar{H}\rho + \bar{\tau}\rho^2) + O(\rho).$$

We rewrite the second term as follows. From (1.4), one computes

$$\bar{\mathrm{R}}\mathrm{ic}(4,4) = \frac{1}{6}\bar{s} = -\frac{\bar{\Delta}\rho}{\rho}, \tag{1.21}$$

where $\bar{\mathrm{R}}\mathrm{ic}$ denotes Ricci curvature w.r.t. $\bar{g}$. The first equality gives $5\,\bar{\mathrm{R}}\mathrm{ic}(4,4) = \sum_{i<4}\bar{\mathrm{R}}\mathrm{ic}(i,i) = \bar{\mathrm{R}}\mathrm{ic}(4,4) + 2\bar{\tau}$, and so $2\,\bar{\mathrm{R}}\mathrm{ic}(4,4) = \bar{\tau} = -2\frac{\bar{H}}{\rho}$. Hence

$$\frac{1}{3}\int_{S(r)} (\bar{H}\rho + \bar{\tau}\rho^2) = -\frac{1}{3}\int_{S(r)} \bar{H}\rho.$$

Finally, the integral curves of $\bar{\nabla}\rho$ are geodesics, and so the Ricatti equation

$$\frac{d\bar{H}}{d\rho} + |\bar{A}|^2 + \bar{\mathrm{R}}\mathrm{ic}(\bar{\nabla}\rho, \bar{\nabla}\rho) = 0,$$

holds. Since $|\bar{A}|^2 = O(\rho^2)$ by (1.20), and $\bar{H}/\rho = -\,\bar{\mathrm{R}}\mathrm{ic}(\bar{\nabla}\rho, \bar{\nabla}\rho)$, we obtain

$$\int_{S(r)} \bar{H}\rho = \int_{S(r)} \bar{H}'\rho^2 + O(\rho),$$

where $\bar{H}' = d\bar{H}/d\rho$. In summary, we thus have the last two terms in (1.17) equal to

$$-\frac{1}{3}\operatorname{vol} S(r) - \frac{1}{3}\rho^2 \int_{S(r)} \bar{H}' + O(\rho). \tag{1.22}$$

Now we claim that the two terms in (1.22) are exactly the first two terms in the ρ-expansion of $\operatorname{vol} B(r)$.

Lemma 1.1. *As $r \to \infty$, we have the expansion*

$$\operatorname{vol} B(r) = \frac{1}{3}\operatorname{vol} S(r) + \frac{1}{3}\rho^2 \int_{S(r)} \bar{H}' + V + o(1). \tag{1.23}$$

Proof. Let $\bar{S}(\rho)$ be the ρ-level set of ρ in $(M, \bar{g})$. Then for ρ small,

$$\operatorname{vol} \bar{S}(\rho) =$$
$$\operatorname{vol} \bar{S}(0) + \rho \int_{\bar{S}(0)} \bar{H} + \frac{1}{2}\rho^2 \int_{\bar{S}(0)} (\bar{H}' + \bar{H}^2) + \frac{1}{6}\rho^3 \int_{\bar{S}(0)} (\bar{H}'' + 3\bar{H}\bar{H}' + \bar{H}^3) + O(\rho^4).$$

Recall that $\operatorname{vol} S(r) = \rho^{-3} \cdot \operatorname{vol} \bar{S}(\rho)$ and $\bar{H} = \bar{H}'' = 0$ at $\bar{S}(0) = \partial M$ by (1.10) and (1.11). Thus

$$\operatorname{vol} S(r) = \rho^{-3} \operatorname{vol} \bar{S}(0) + \frac{1}{2}\rho^{-1} \int_{\bar{S}(0)} \bar{H}' + O(\rho). \tag{1.24}$$

Integrating this from 0 to r gives

$$\operatorname{vol} B(r) = \int_0^r \operatorname{vol} S(r) dr = \int_\rho^1 \rho^{-1} \operatorname{vol} \bar{S}(\rho) d\rho =$$
$$\operatorname{vol} \bar{S}(0) \int_\rho^1 \rho^{-4} d\rho + (\frac{1}{2} \int_{\bar{S}(0)} \bar{H}') \int_\rho^1 \rho^{-2} d\rho + O(1),$$

which implies

$$\operatorname{vol} B(r) = \frac{1}{3}\rho^{-3} \operatorname{vol} \bar{S}(0) + \frac{1}{2}\rho^{-1} \int_{\bar{S}(0)} \bar{H}' + V + o(1).$$

Substituting in (1.24) shows that

$$\operatorname{vol} B(r) = \frac{1}{3}\operatorname{vol} S(r) + \frac{1}{3}\rho^{-1} \int_{\bar{S}(0)} \bar{H}' + V + o(1). \tag{1.25}$$

Finally, we have

$$\rho^{-1} \int_{\bar{S}(0)} \bar{H}' = \rho^{-1} \int_{\bar{S}(\rho)} \bar{H}'(\rho) + \rho^{-1} (\int_{\bar{S}(0)} \bar{H}' - \int_{\bar{S}(\rho)} \bar{H}'(\rho)).$$

But $\bar{H}'(\rho) = \bar{H}'(0) + \rho \bar{H}''(0) + o(\rho^2) = \bar{H}'(\rho) + o(\rho^2)$. Hence

$$\rho^{-1} \int_{\bar{S}(0)} \bar{H}' = \rho^{-1} \int_{\bar{S}(\rho)} \bar{H}'(\rho) + o(1),$$

and the result follows. $\square$

Combining (1.17), (1.22) and (1.23) and letting $r \to \infty$ then completes the proof of Theorem 0.1.

Remark 1.2. The same proof as above evaluates the right side of (1.17) whenever (M, g) is any AH metric which is Einstein to 3^{rd} order, i.e. for which the expansion (1.9) agrees with the expansion of an Einstein metric to order 3. Hence, for such metrics, we obtain

$$\frac{1}{8\pi^2}\int_M (|W|^2 - \frac{1}{2}|z|^2 + \frac{1}{24}s^2 - 6)dV = \chi(M) - \frac{3}{4\pi^2}V. \tag{1.26}$$

It follows for instance that an Einstein metric minimizes V in its conformal class, among AH metrics. Note that V itself is, of course, not a conformal invariant among such AH metrics.

2. Boundary determination of dV

This section is concerned with the question of to what extent the renormalized volume V, or the L^2 norm of the Weyl curvature, is determined by the conformal infinity γ of an AHE metric. To do this, we study the variation dV of V in the space of AHE metrics on M.

Thus, let g be an AHE metric on M and let h be an infinitesimal variation of g, so that the curve of metrics $g_t = g + th$ is AHE, to first order in t. From Theorem 0.1, we have

$$dV(h) = \frac{dV_t}{dt}|_{t=0} = -\frac{1}{6}\frac{d}{dt}(\int_M |W|^2 d\,\mathrm{vol}_t)_{t=0} \equiv -\frac{1}{6}d\mathcal{W}(h). \tag{2.1}$$

To analyse dV recall that, by definition, Einstein metrics are critical points of the scale-invariant Einstein-Hilbert action

$$\mathcal{S} = \mathrm{vol}^{-1/2}\int s d\,\mathrm{vol}, \tag{2.2}$$

in dimension 4. Hence the variation $d\mathcal{S}$ of $\mathcal{S}$ is determined by the behavior of the variation of the metric at the boundary. We first make this precise in the Lemma below, and then relate it to the variation of V. The following result has recently also been proved in [15]; the proof below however is simpler and more transparent.

Lemma 2.1. *Let g be an Einstein metric on a smooth compact domain D in M^4, with scalar curvature s, and let h be an infinitesimal deformation of g, so that $g_t = g + th$ is Einstein, with scalar curvature s, to first order in t. Then*

$$(\mathrm{vol}\, D)' = \frac{d}{dt}\mathrm{vol}_{g_t} D|_{t=0} = -\frac{2}{s}\int_{\partial D}(2H' + < A, h >), \tag{2.3}$$

where A is the 2^{nd} fundamental form of $\partial D, H = trA$, and $H' = \frac{dH}{dt}|_{t=0}$.

Proof. Take the derivative of (2.2) w.r.t. t and use the fact that s is constant. A brief computation shows that at $t = 0$,

$$s \cdot (\mathrm{vol}^{1/2} D)' = \mathrm{vol}^{-1/2} \int_D (L(h) + \frac{s}{4} < g, h >) d\,\mathrm{vol},$$

where $L(h) = s'(h)$ is the linearization of the scalar curvature, given by

$$L(h) = -\Delta trh + \delta\delta h - < \mathrm{Ric}, h >,$$

c.f. [2, 1.174]. Since $z = 0$, this gives

$$\frac{1}{2} s(\mathrm{vol}\, D)' = \int_D (-\Delta trh + \delta\delta h) d\,\mathrm{vol},$$

and hence by the divergence theorem

$$\frac{1}{2} s(\mathrm{vol}\, D)' = -\int_{\partial D} < dtrh, N > - \int_{\partial D} \delta h(N), \tag{2.4}$$

where N is the unit outward normal.

Now choose local normal exponential (Fermi) coordinates for a neighborhood of ∂D. Thus, N is the field tangent to geodesics, and normal to equidistant hypersurfaces $S(r)$, with $S(0) = \partial D$. Let $\{e_i\}$ be a local orthonormal basis for $T(S(r))$, so that $\{e_i, N\}$ are a local orthonormal basis for $T(D)$ near ∂D. We then have

$$\delta h(N) = -Nh(N, N) - < \nabla_{e_i} h(e_i), N >, \tag{2.5}$$

and

$$< dtrh, N > = Nh(N, N) + N(< g^T, h^T >). \tag{2.6}$$

where T denotes tangential part. When combined, the first terms in (2.5) and (2.6) cancel. For the second term in (2.5), we have $< \nabla_{e_i} h(e_i), N >= \mathrm{div}_S(h(N)) - < h, A >$, where div_S is the divergence on the hypersurfaces S. This integrates to 0 on $S(0) = \partial D$. Hence, (2.4) becomes

$$\frac{1}{2} s(\mathrm{vol}\, D)' = -\int_{\partial D} N < g^T, h^T > - \int_{\partial D} < A, h > . \tag{2.7}$$

To evaluate the first term, for each metric g_t we have the hypersurfaces $S_t(r)$ constructed above, with the induced metric $g_{ij}(t, r)$. In a fixed local coordinate system, the volume form $dV_S(t, r)$ of $S_t(r)$ is given by

$$dV_S(t, r) = (\det g_{ij}(t, r))^{1/2} dx_1 \wedge dx_2 \wedge dx_3.$$

Then $\frac{1}{2} < g^T, h^T > dV_S = \frac{\partial}{\partial t}[\det g_{ij}(t, r)^{1/2}] dx_1 \wedge dx_2 \wedge dx_3$, and

$$\tfrac{1}{2} N < g^T, h^T > dV_S = \frac{\partial}{\partial r}\frac{\partial}{\partial t}[\det g_{ij}(t, r)^{1/2}] dx_1 \wedge dx_2 \wedge dx_3.$$

The coefficients g_{ij} are smooth functions of the parameters r and t, and so

$$\frac{\partial}{\partial r}(\frac{\partial}{\partial t}(\det(g_{ij})^{1/2})) = \frac{\partial}{\partial t}(\frac{\partial}{\partial r}(\det(g_{ij})^{1/2})) = H' = \frac{dH}{dt}.$$

It follows that (2.7) becomes

$$\frac{1}{2}s(\operatorname{vol} D)' = -\int_{\partial D}(2H' + < A, h >)d\operatorname{vol},$$

which gives (2.3). □

This result, with the same proof, holds in all dimensions, with the coefficient $\frac{1}{2}$ replaced by $2/n, n = \dim M$.

We now apply Lemma 2.1 to the domains $B(r)$ in an AH Einstein manifold (M, g), and let $r \to \infty$. This leads to the proof of Theorem 0.2, which we restate as:

Theorem 2.2. *Let $h = dg_t/dt$ be an infinitesimal AHE deformation of an AHE metric g on M and let $h_{(0)} = d\gamma_t/dt$ be the induced variation of γ on ∂M, where $\bar{g}$ is determined by a special defining function ρ, as in* (1.7)*. Then*

$$dV(h) = -\frac{1}{4}\int_{\partial M} < g_{(3)}, h_{(0)} >, \tag{2.8}$$

for $g_{(3)}$ as in (1.9)*. The inner product and volume form in* (2.8) *are w.r.t. γ.*

Proof. By Lemma 2.1, we have with $r = \log\rho^{-1} >> 1$,

(2.9)

$$(\operatorname{vol} B(r))' = \frac{1}{6}\int_{S(r)}(2H' + < A, h >)d\operatorname{vol} = \frac{1}{6}\rho^{-3}\int_{\bar{S}(\rho)}(2H' + < A, h >)d\,\overline{\operatorname{vol}}\,.$$

As in the proof of Theorem 0.1, we analyse the terms on the right from the expansion of $\bar{g}_t$, given by

$$\bar{g}_t = d\rho_t \otimes d\rho_t + (g_{(0),t} + \rho_t^2 g_{(2),t} + \rho_t^3 g_{(3),t}). \tag{2.10}$$

Taking the derivative of (2.10) w.r.t. t gives

$$\bar{h} = 2d\rho' \otimes d\rho + (h_{(0)} + \rho^2 h_{(2)} + 2\rho\rho' g_{(2)} + O(\rho^3)). \tag{2.11}$$

Here $\rho' = d\rho_t/dt$, and we have used the fact that $\rho' = O(\rho)$, since $\rho'(\partial M) = 0$. In fact, since ρ_t are special defining functions w.r.t. g_t, a simple computation gives

$$\rho' = \phi_{(1)}\rho + \phi_{(3)}\rho^3 + o(\rho^3), \tag{2.12}$$

c.f. also [7, Lemma 2.2]. Note also that $\bar{h}$ has no tangential terms of order ρ.

Next, from (1.18), we have $H = 3 - \bar{H}\rho$, and, by (1.10)-(1.11), $\bar{H} = h_1\rho + h_3\rho^3$, so that

$$H = 3 - h_1\rho^2 - h_3\rho^4. \tag{2.13}$$

Similarly, $A = g^T - \rho\bar{A}$, and $\bar{A} = A_1\rho + A_2\rho^2$, so that

$$A = g^T - A_1\rho^2 - A_2\rho^3 + O(\rho^4). \tag{2.14}$$

Further, by (2.13), $H' = h_1'\rho^2 + 2\rho\rho' h_1 + O(\rho^4)$ which with (2.12) gives

$$H' = \xi_1\rho^2 + O(\rho^4). \tag{2.15}$$

Now we substitute these computations in (2.9). The estimate (2.15) shows that the first term in (2.9) contains only an $O(\rho^{-1})$ term, and hence gives no contribution to V', where V is the renormalized volume. Hence we may ignore this term. For the next term, we have

$$< A, h > = < h, g^T > - < A_1, h > \rho^2 - < A_2, h > \rho^3 + O(\rho^4). \tag{2.16}$$

Now for any (1,1) tensors A, B, $< A, B >_g = < A, B >_{\bar{g}}$, and so the g-inner products in (2.16) may be replaced by $\bar{g}$-inner products. We have $\frac{1}{2} < h, g^T >_{\bar{g}}$ $d\,\mathrm{vol}_{\bar{S}(\rho)} = \frac{d}{dt}(d\,\mathrm{vol}_{\bar{g}_t}(\bar{S}_t(\rho)))$, which vanishes at order $O(\rho^3)$ by (1.11). Similarly, $< A_1, h >_{\bar{g}} \rho^2$ has no terms of order $O(\rho^3)$ by (2.11). Hence the only term in (2.16) of order $O(\rho^3)$ is

$$- < A_2, h > \rho^3 = - < A_2, h >_{\bar{g}} \rho^3 = - < A_2, h_{(0)} >_{\bar{g}} \rho^3 + o(\rho^3).$$

Taking the limit $\rho \to 0$ then implies that

$$V' = -\frac{1}{6}\int_{\partial M} < A_2, h_{(0)} > .$$

To complete the proof, we have $\mathcal{L}_{\bar{\nabla}\rho}\bar{g} = 2\bar{A}$, while $A_2 = \frac{1}{2}\mathcal{L}^2_{\bar{\nabla}\rho}\bar{A} = \frac{1}{4}\mathcal{L}^3_{\bar{\nabla}\rho}\bar{g} = \frac{3}{2}g_{(3)}$, which then gives (2.8). $\square$

The formula (2.8) shows that although apriori the renormalized volume V depends on the global geometry of the bulk manifold (M, g), its variation dV depends only on the (3^{rd} order) behavior of the compactification $\bar{g}$ at ∂M. We note that one may prove, via Lemma 2.1 again, that there is a similar formula in higher dimensions.

Remark 2.3. Using the fact that $g_{(2)} = \mathcal{L}_{\bar{\nabla}\rho}\bar{A}$ together with (1.4), a brief computation shows that

$$g_{(2)} = -\frac{1}{2}(\bar{\mathrm{Ric}} - \frac{\bar{s}}{6}g_{(0)}), \tag{2.17}$$

where the curvatures are w.r.t. the metric $\bar{g}$ on M. In fact one may compute that

$$-\frac{1}{2}(\bar{\mathrm{Ric}} - \frac{\bar{s}}{6}g_{(0)}) = -(\mathrm{Ric}_\gamma - \frac{s_\gamma}{4}\gamma), \tag{2.18}$$

at ∂M, where the curvatures on the right of (2.18) are *intrinsic* w.r.t. the boundary metric $g_{(0)} = \gamma$, c.f. also [7, (2.10)] for example. We note that (1.21) and (1.11) imply that $\bar{\nabla}\rho(\bar{s}) = 0$ at ∂M and further computation shows that $(\bar{\nabla}_X \bar{\mathrm{Ric}})(\bar{\nabla}\rho) = 0$ at ∂M, for X tangent to ∂M. Hence, (2.17) and the relation $g_{(3)} = \frac{1}{3}\mathcal{L}_{\bar{\nabla}\rho}g_{(2)}$ imply

$$g_{(3)} = \frac{1}{6}\nabla_N \bar{\mathrm{Ric}} = \frac{1}{6}d\,\bar{\mathrm{Ric}}(N), \tag{2.19}$$

where $N = -\bar{\nabla}\rho$ and $d = d^{\bar{\nabla}}$ is the exterior derivative w.r.t. the ambient metric $\bar{g}$, c.f. [2, 4.69].

Remark 2.4. We verify briefly that (2.8) also gives, up to a constant, the variation $d\mathcal{W}$, where $\mathcal{W} = \int |W|^2$; of course, this must be the case by (2.1).

The gradient $\nabla\mathcal{W}$, i.e the Euler-Lagrange operator for $\mathcal{W}$, is given by the Bach tensor $\nabla\mathcal{W}^2 = \frac{1}{2}(\delta\delta W + W \circ \mathrm{Ric})$, c.f. [2, 4.77]; (the factor of $\frac{1}{2}$ comes from the definition of $|W|^2$ as in §1). Einstein metrics are also critical points of $\mathcal{W}$, so that $\nabla\mathcal{W}^2 = 0$. Hence, as in the proof of Lemma 2.1, on any compact domain $D \subset M$ and at $t = 0$, we have

$$\frac{d}{dt}\int_D |W|^2 d\,\mathrm{vol}_t = \int_D \frac{d}{dt}(|W|^2 d\,\mathrm{vol}_t) = \int_D < \nabla\mathcal{W}^2, h > + \int_{\partial D} < B_W, h >,$$

i.e.

$$d\mathcal{W}(h) = \int_{\partial D} < B_W, h >, \tag{2.20}$$

where B_W is a boundary term. To determine B_W, integrate by parts as follows:

$$\int_D < \delta\delta W, h >= -\int_{\partial D} < \delta W(N), h > + \int_D < \delta W, Dh >,$$

and

$$\int_D < \delta W, Dh >= -\int_{\partial D} < W(N), Dh > + \int_D < W, DDh >,$$

where N is the unit outward normal. Hence

$$\int_{\partial D} < B_W, h >= -\frac{1}{2}\int_{\partial D} < \delta W(N), h > -\frac{1}{2}\int_{\partial D} < Dh, W(N) > . \tag{2.21}$$

Since $\mathcal{W}$ is conformally invariant, we may compute (2.21) w.r.t. the compactification $\bar{g}$ and let $D = M$. A computation as in that giving (2.19) shows that the second integral in (2.21) vanishes, and so

$$\begin{aligned} \int_{\partial M} < B_W, h_{(0)} >= \\ -\frac{1}{2}\int_{\partial M} < \delta W(N), h_{(0)} >= \frac{1}{4}\int_{\partial M} < d(\bar{\mathrm{R}}\mathrm{ic} - \frac{\bar{s}}{6}\bar{g})(N), h_{(0)} >, \end{aligned} \tag{2.22}$$

where the second equality uses the Bianchi identity, c.f. [2, 16.3]. Via (2.8) and (2.19), this confirms (2.1).

Combining (2.8) and (2.19), we have

$$dV(h) = -\frac{1}{24}\int_{\partial M} < d\,\bar{\mathrm{R}}\mathrm{ic}(N), h_{(0)} > . \tag{2.23}$$

This formula resembles, at least formally, the formula for the variation of η in (0.8), i.e.

$$d\eta(h) = -\frac{1}{24\pi^2}\int_{\partial M} < *d\,\mathrm{Ric}, h_{(0)} > . \tag{2.24}$$

Of course by (0.7) these formulas must agree if (M, g) is self-dual Einstein, and h is an infinitesimal variation of such metrics. However, in (2.24), the Ricci curvature Ric and exterior derivative d are *intrinsic*, i.e. computed on the 3-manifold ∂M w.r.t. the boundary metric γ. On the other hand, in (2.23), the Ricci curvature $\bar{\mathrm{R}}\mathrm{ic}$ and d are *extrinsic*, computed w.r.t the ambient metric $\bar{g}$ at ∂M.

As in (2.18), the term $g_{(2)}$ in the expansion (1.9) of $\bar{g}$ at ∂M is local and intrinsic; recall that $g_{(1)} = 0$. However, as pointed out in [6], the Einstein equations only imply the relations (1.11) on the third term $g_{(3)}$ in the expansion; the remaining parts of $g_{(3)}$ are formally undetermined. The term $g_{(3)}$ hence (may) depend on the bulk metric $\bar{g}$ at ∂M, and not only on γ.

In general, this issue is related to the unique solvability of the Dirichlet problem for AH Einstein metrics with prescribed conformal infinity. Namely, if, given a boundary metric γ, there is a unique AHE metric g with conformal infinity γ, then the expansion terms $g_{(k)}$ are all necessarily determined, in some manner, by the intrinsic geometry of $\gamma = g_{(0)}$. On the other hand, if this is not the case, then some $g_{(k)}$ may not be determined from γ.

Before proceeding further, we make several remarks.

Remark 2.5. (i). We observe that

$$dV \neq c \cdot d\eta \tag{2.25}$$

for any constant c, in general. This essentially follows from Theorem 0.1 and a remark of Hitchin in [11]. Thus let γ_o be the canonical round metric on S^3 and let γ be any metric sufficiently close to γ_o which is invariant under an orientation reversing reflection of S^3. Since η changes sign under orientation reversal, $\eta(\gamma) = 0$. The Graham-Lee theorem [8] shows that any such γ may be filled in with an AH Einstein metric g, with γ as conformal infinity. If we now take a curve of such metrics g_t with boundary values γ_t, then $d\eta(\gamma_t)/dt = d\eta(h) = 0$, for all t. However, such curves g_t will satisfy, for $t > 0$, $V(g_t) \neq V(g_o) = \frac{4\pi^2}{3}$ unless the curve is a constant curve, c.f. Theorem 0.1. Hence, for some $t \neq 0$ small, $dV(h) \neq 0$, which gives (2.25).

We recall that $d\,\mathrm{Ric}$ is the *only* local conformal invariant constructed from the metric in dimension 3 and hence dV cannot be a locally defined intrinsic invariant of γ in general.

(ii). In [6], Fefferman and Graham consider formally the class of AH Einstein metrics for which $\bar{g}$ has an *even* expansion (1.9), i.e. $g_{(odd)} = 0$. Of course, for such metrics Theorem 2.2 gives

$$dV = -\frac{1}{6}d\mathcal{W} = 0.$$

Thus, such points are critical points of V or $\mathcal{W}$ and so one would expect that there are very few such metrics, in the space of all AHE metrics.

Any hyperbolic metric is even in this sense, since the metrics $\bar{g}_\rho$ in (1.8) are given by $\bar{g}_\rho = (1-\rho^2)^2 \cdot g_{(0)}$. But Theorem 0.1 implies that any hyperbolic metric gives the maximal value for V on M and so of course this metric must be a critical point of V.

Next we derive the formula (0.10). For a given AH Einstein metric (M,g), let W^+ and W^- be the self-dual and anti-self-dual parts of the Weyl curvature, so that $W = W^+ + W^-$; (recall that M is an oriented 4-manifold). As in Remark 2.4, we have $\nabla\mathcal{W} = \frac{1}{2}(\delta\delta W + W \circ \mathrm{Ric}) = 0$, and hence

$$\nabla\mathcal{W}^+ = \frac{1}{2}(\delta\delta W^+ + W^+ \circ \mathrm{Ric}) = 0, \text{ and } \nabla\mathcal{W}^- = \frac{1}{2}(\delta\delta W^- + W^- \circ \mathrm{Ric}) = 0,$$

where $\mathcal{W}^{\pm}$ are the functionals $\int |W^{\pm}|^2$.

The same reasoning as in (2.20) then shows that, for an infinitesimal AHE deformation h,

$$d\mathcal{W}^{\pm}(h) = \frac{d}{dt}\Big(\int_M |W^{\pm}|^2\Big)_{t=0} = \int_{\partial M} < B^{\pm}, h_{(0)} >, \tag{2.26}$$

where $B^{\pm}$ are boundary terms. These terms may be computed in the same way as B_W in (2.22). Thus, informally, we may think of $\nabla\mathcal{W}^{\pm} = B^{\pm}$.

Now consider the moduli space $\mathcal{M}$ of AHE metrics on M which admit a C^3 conformal compactification $\bar{g}$. The boundary values γ of such metrics give a space $\mathcal{B}$ of metrics on ∂M. The structure of $\mathcal{M}$ and $\mathcal{B}$ is not of concern here. Instead, we consider only the (formal) tangent spaces $T_g\mathcal{M}$, i.e. the vector space of solutions to the linearized AHE equations at a given $\gamma \in \mathcal{M}$, c.f. [2, Ch. 12] for background on linearized Einstein equations. Any infinitesimal AHE deformation $h \in T_g\mathcal{M}$ induces an infinitesimal deformation $h_{(0)} \in T_\gamma\mathcal{B}$.

Let $T^{\pm}$ be the subspace of tangent vectors which leave $\mathcal{W}^{\mp}$ unchanged, to first order; thus

$$T^+ = \ker d\mathcal{W}^-, \quad \text{and} \quad T^- = \ker d\mathcal{W}^+. \tag{2.27}$$

Via (2.26), $T^{\pm}$ may also be viewed as subspaces of $T_\gamma\mathcal{B}$ orthogonal to $B^{\mp}$. They are codimension 1 hyperplanes of $T_\gamma\mathcal{B}$, except when $d\mathcal{W}^+ = 0$ or $d\mathcal{W}^- = 0$. Of course $T^+ \cap T^-$ consists of the variations which change neither $\mathcal{W}^+$ or $\mathcal{W}^-$ to first order. Observe that if $d\mathcal{W}^+ = \lambda d\mathcal{W}^-$ for some $\lambda \neq 0$, then since a change in the orientation interchanges $d\mathcal{W}^+$ and $d\mathcal{W}^-$, we must have $\lambda^2 = 1$, and so either $d\mathcal{W} = 0$, (when $\lambda = 1$), or $d\eta = 0$, (when $\lambda = 1$, by (0.6) and (0.8)).

Now suppose that $d\mathcal{W}^+$ and $d\mathcal{W}^-$ are linearly independent. Then any tangent vector $h_{(0)} \in T_\gamma\mathcal{B}$ may be decomposed uniquely as

$$h_{(0)} = h^+_{(0)} + h^0_{(0)} + h^-_{(0)}, \tag{2.28}$$

where $h^{\pm}_{(0)} \in T^{\pm}$ and $h^0_{(0)} \in T^+ \cap T^-$. If $d\mathcal{W}^- = 0$, we set $h^- = 0$ in (2.28) and similarly $h^+ = 0$ if $d\mathcal{W}^+ = 0$. The decomposition (2.28) defines the projection operators $\pi^{\pm}$ in (0.10). A similar decomposition holds for $h \in T_g\mathcal{M}$.

The following result relates $d\mathcal{W}$ and $d\eta$, and gives (0.10) via (2.1).

Proposition 2.6. *Let g be an AH Einstein metric on M, with boundary metric γ. Suppose $d\mathcal{W} \neq 0$ and $d\eta \neq 0$, i.e. g is not a critical point of $\mathcal{W}$ or η. Then,* *in the notation above, we have*

$$dW(h) = \frac{1}{2}\int_{\partial M} < *d\,\mathrm{Ric}, h^+_{(0)} > -\frac{1}{2}\int_{\partial M} < *d\,\mathrm{Ric}, h^-_{(0)} >, \tag{2.29}$$

where $*d\,\mathrm{Ric}$ *is intrinsically defined w.r.t.* γ.

Proof. Since $\mathcal{W} = \mathcal{W}^+ + \mathcal{W}^-$, we have

$$d\mathcal{W}(h) = d\mathcal{W}^+(h) + d\mathcal{W}^-(h).$$

By the construction in (2.27)-(2.28), this gives

$$d\mathcal{W}(h) = d\mathcal{W}^+(h^+) + d\mathcal{W}^-(h^-),$$

where

$$d\mathcal{W}^\pm(h^\pm) = \int_{\partial M} < B^\pm, h^\pm_{(0)} > .$$

However since h^+ leaves $\mathcal{W}^-$ unchanged to first order, it follows from (0.6) and (0.8) that

$$d\mathcal{W}^+(h^+) = d(\mathcal{W}^+(h^+) - \mathcal{W}^-(h^+)) = \frac{1}{2}\int_{\partial M} < *d\,\mathrm{Ric}, h^+_{(0)} > . \tag{2.30}$$

Thus, when viewed as a linear functional restricted to $T^+ \subset T_\gamma\mathcal{B}, B^+ = *d\,\mathrm{Ric}$ is intrinsically and locally determined by γ. For the same reasons, we also obtain

$$d\mathcal{W}^-(h^-) = -\frac{1}{2}\int_{\partial M} < *d\,\mathrm{Ric}, h^-_{(0)} > . \tag{2.31}$$

Combining (2.30) and (2.31) gives the result. $\square$

While $*d\,\mathrm{Ric}$ is intrinsically determined by the boundary metric γ, it is not clear to what extent the subspaces $T^\pm$ are determined by γ. In this regard, we discuss the following examples, which show that $d\mathcal{W}$ or dV cannot be *solely* determined by the boundary metric in general; these examples are also discussed in [9] and [16]. Thus, consider first hyperbolic 4-space $\mathbb{H}^4(-1)$. For any geodesic $\sigma \subset \mathbb{H}^4(-1)$, translation by a fixed length L along σ extends to an isometry of $\mathbb{H}^4(-1)$. Let $\mathbb{H}^4(-1)/\mathbb{Z} \approx \mathbb{R}^3 \times S^1$ be the quotient, with $\mathbb{Z}$ the group generated by the translation. The metric g_{-1} on $\mathbb{H}^4(-1)/\mathbb{Z}$ may be written as

$$g_{-1} = dr^2 + \sinh^2 r g_{S^2(1)} + \cosh^2 d\theta^2,$$

where θ parametrizes a circle of length L. We have $\partial M = S^2 \times S^1$ and the conformal infinity $[\gamma]$ is the conformal class of the product metric $S^2(1) \times S^1(L)$. For instance by (0.4) and Remark 2.5(ii), we have $V = 0$ and $dV = g_{(3)} = 0$ for g_{-1}, and of course $\eta = d\eta = 0$.

On the other hand, the AdS Schwarzschild metric, c.f. [9], [16, §3.2], or [2,9.118(d)] is an AH Einstein metric on $\mathbb{R}^2 \times S^2$ given by

$$g_{AS} = (1 + r^2 - \tfrac{2m}{r})^{-1}dr^2 + r^2 g_{S^2(1)} + (1 + r^2 - \tfrac{2m}{r})d\theta^2.$$

Here $m > 0$ is the mass parameter, $r \geq r_+$, where r_+ is the largest root of the equation $1 + r^2 - \frac{2m}{r} = 0$, and θ parametrizes a circle of length $L = L(m) =$

$\frac{4\pi r_+}{1+3r_+^2}$. This metric has the same conformal infinity $S^2(1) \times S^1(L)$ as before and so $\eta = d\eta = 0$. However, by [9, (2.9)],

$$V(g_{AS}) = \frac{\pi r_+^2(1-r_+^2)}{1+3r_+^2},$$

and a straightforward computation using (0.9) gives

$$dV_{g_{AS}} = \frac{m}{2} d\theta^2.$$

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