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STRENGTHENING THE THEOREM OF EMBEDDED DESINGULARIZATION

A. Bravo and O. Villamayor U.

1. Introduction

Resolution of singularities is one of the central areas of research in Algebraic Geometry. It is a basic prerequisite for the classification of algebraic varieties up to birational equivalence, since it allows to consider only regular varieties. Hironaka's monumental work [Hi1] gave a non-constructive, existence proof of resolution of singularities over fields of characteristic zero. Constructive versions of Hironaka's Theorem have been proposed in [BM], [V1], [V2] and [EV1], while simplified weak non-constructive versions can be found in [AJ], [AW] and [BP].

Here we announce the following stronger form of resolution of singularities:

Theorem 2.2 *Let X be a reduced subscheme embedded in a scheme W, smooth over a field $\mathbf{k}$ of characteristic zero, and let $\mathcal{I}(X)$ be the sheaf of ideals defining X. There exists a proper, birational morphism $\pi : W_r \longrightarrow W$, obtained as a composition of monoidal transformations, such that if $X_r \subset W_r$ denotes the strict transform of $X \subset W$, then:*

(i) *X_r is regular in W_r and $\mathrm{Reg}(X) \simeq \pi^{-1}(\mathrm{Reg}(X))$ via π.*
(ii) *X_r has normal crossings with $\pi^{-1}(\mathrm{Sing}(X))$, which is a union of hypersurfaces with normal crossings.*
(iii) *The* total transform *of $\mathcal{I}(X)$ at $\mathcal{O}_{W_r}$ factors as a product of an invertible sheaf of ideals $\mathcal{L}$ supported on the exceptional locus, times the sheaf of ideals defining the strict transform of X (i.e. $\mathcal{I}(X)\mathcal{O}_{W_r} = \mathcal{L} \cdot \mathcal{I}(X_r)$).*

Parts (i) and (ii) are the usual conditions of embedded desingularization (Hironaka's Theorem). Part (iii) is new and provides, in an elementary way, equations defining the embedded desingularization, from the equations defining the original singular scheme ($X \subset W$). This result answers a question formulated by A. Nobile, which was the starting point of this research. A complete proof can be found in [BV].

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In Section 3 we present an example in order to explain the difference between our results and previous algorithms that follow Hironaka's line of proof, which are based on the notion of strict transforms of ideals (cf. [BM], [V1], [V2] and [EV1]). This example shows that such algorithms will never yield in a direct fashion the result contained in Theorem 2.2 (iii). We also discuss the simplification introduced by Theorem 2.2 from an algorithmic point of view, and we point out that it provides a form of lifting Koszul complexes (cf. Corollary 3.1).

As it happens with a previous algorithm described in [EV3], the proof of Theorem 2.2 avoids the notion of strict transform of ideals: Desingularization is achieved as a byproduct of a much simpler problem, namely that of *algorithmic principalization of ideals* (see Definition 2.3). The only invariant involved is the order of an ideal; so Hilbert Samuel functions, normal flatness, and Hironaka's iterated use of strict transforms of ideals, are avoided in the proof of desingularization with this strategy. In Sections 4 and 5 we sketch the answer to the following question:

How can we desingularize a singular scheme without considering the strict transform of the corresponding ideal?

Note that even if we start with a prime ideal defining a singular subscheme, after applying monoidal transformations the total transform of this prime ideal will have, together with the strict transform, several other primary components. Now Theorem 2.2 (iii) says that by making all other primary components locally principal (principalization!), the strict transform is obtained by factoring out a locally principal ideal. In 5.5 we give a simple geometric description of how to achieve this.

The fact that algorithmic principalization of ideals implies immediately resolution of singularities appears for the first time in [EV3] (see also the section "added in proof" of [EV2], pp. 224-225 for an elementary resolution of irreducible schemes). While the algorithm from [EV3] does not yield Theorem 2.2 (iii), it does open the gates to a generation of algorithms (including ours) based on a philosophy different from Hironaka's. One of the advantages of this new generation of proofs, is that they provide algorithms of resolutions of singularities that can be implemented in computer programs. In fact the algorithm of principalization indicated in the addendum of [EV2] has been implemented in MAPLE by G. Bodnár and J. Schicho (cf. [BS1], [BS2]).

The idea that principalization implies desingularization has also been used in [ENV] to study equiresolution of families of schemes. Finally, we mention that the result from [EV3] has been extended to the non-equidimensional case in [EH].

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2. Formulation of the Main Theorem

We briefly explain the notions of *pairs* and *transformation of pairs* suitable for the formulation of both *Strong Embedded Desingularization* and *Principalization of Ideals* (cf. Theorem 2.2 and Definition 2.3 below).

Definition 2.1. [EV2]. Let W be a pure dimensional scheme, smooth over a field $\mathbf{k}$ of characteristic zero, and let $E = \{H_1, \dots, H_r\}$ be a set of smooth hypersurfaces in W with normal crossings (i. e. $\cup_{i=1}^{r} H_i$ has normal crossings). The couple (W, E) is said to be a *pair*. A regular closed subscheme $Y \subset W$ is said to be *permissible* for the pair (W, E) if Y has normal crossings with E.

If $Y \subset W$ is permissible for a pair (W, E), we define a *transformation of pairs* in the following way: Consider the blowing-up with center Y, $W \overset{\Pi}{\longleftarrow} W_1$, and define $E_1 = \{H'_1, \dots, H'_r, H_{r+1}\}$, where H'_i denotes the strict transform of H_i, and $H_{r+1} = \Pi^{-1}(Y)$ the exceptional hypersurface in W_1. Note that W_1 is smooth and that E_1 has normal crossings. We say that $(W, E) \longleftarrow (W_1, E_1)$ is a *transformation of the pair* (W, E).

2.2. Main Theorem. (*of Strong Embedded Desingularization*) *Let* $(W_0, E_0 = \{\emptyset\})$ *be a pair and let* $X_0 \subset W_0$ *be a closed subscheme defined by* $\mathcal{I}(X_0) \subset \mathcal{O}_{W_0}$. *We assume that the open set* $\mathrm{Reg}(X)$ *of regular points is dense in* X (*e.g.* X *reduced*). *Then there is a finite sequence of transformations of pairs*

$$(W_0, E_0) \longleftarrow \cdots \longleftarrow (W_r, E_r)\,, \tag{2.2.1}$$

inducing a proper birational morphism $\Pi_r : W_r \longrightarrow W_0$, *so that setting* $E_r = \{H_1, \dots, H_r\}$ *and* $X_r \subset W_r$ *the strict transform of* X_0:

(i) X_r *is regular in* W_r, *and* $W_r - \cup_{i=1}^{r} H_i \simeq W_0 - \mathrm{Sing}(X)$. *In particular* $\mathrm{Reg}(X) \cong \Pi_r^{-1}(\mathrm{Reg}(X)) \subset X_r$ *via* Π_r.

(ii) X_r *has normal crossings with* $E_r = \cup_{i=1}^{r} H_i$ (*the exceptional locus of* Π_r).

(iii) *The total transform of the ideal* $\mathcal{I}(X_0) \subset \mathcal{O}_{W_0}$ *factors as a product of ideals in* $\mathcal{O}_{W_r}$:

$$\mathcal{I}(X)\mathcal{O}_{W_r} = \mathcal{L} \cdot \mathcal{I}(X_r),$$

where now $\mathcal{I}(X_r) \subset \mathcal{O}_{W_r}$ *denotes the sheaf of ideals defining* X_r, *and* $\mathcal{L} = \mathcal{I}(H_1)^{a_1} \cdot \ldots \cdot \mathcal{I}(H_r)^{a_r}$ *is an invertible sheaf of ideals supported on the exceptional locus of* Π_r.

The proof of Theorem 2.2 follows from the simpler problem of principalization: It is indicated in the addendum in [EV2] that desingularization follows from principalization (see below).

Definition 2.3. Let $I \subset \mathcal{O}_W$ be a sheaf of ideals. A *principalization of* I is a proper birational morphism $W_1 \longrightarrow W$ such that W_1 is regular and $I\mathcal{O}_{W_1}$ is an invertible sheaf of ideals. A *strong principalization of* I is a chain of transformations of pairs

$$(W_0, E_0 = \emptyset) = (W, E) \longleftarrow \ldots \longleftarrow (W_r, E_r)$$

such that $W \longleftarrow W_r$ defines an isomorphism over the open subset $W \setminus V(I)$, and

$$\mathcal{L} = I\mathcal{O}_{W_r} = \mathcal{I}(H_1)^{c_1} \cdot \ldots \cdot \mathcal{I}(H_s)^{c_s},$$

where $E' = \{H_1, H_2, \ldots, H_s\}$ are regular hypersurfaces with normal crossings and all $c_i \geq 1$. In case that $V(I)$ is of codimension ≥ 2, this means that $E' = E_r$ and the total transform of I is locally spanned by a monomial supported on the exceptional locus of $\Pi_r : W_r \to W$.

3. Total transform versus strict transform

Let $W_0 = A^3_{\mathbb{Q}} = \mathrm{Spec}(\mathbb{Q}[x_1, x_2, x_3])$ and consider the curve C defined by

$$\mathcal{I}(C) = < x_1, x_2x_3 + x_2^3 + x_3^3 > .$$

Set $W_0 \overset{\Pi}{\longleftarrow} W_1$ the quadratic transformation at the origin, $H \subset W_1$ the exceptional divisor, and C_1 the strict transform of C. This defines an embedded desingularization of C, in the usual sense, since both (i) and (ii) of Theorem 2.2 hold.

A) (*On condition* 2.2 (*iii*)). Since the ideal $\mathcal{I}(C)$ has order 1 at the center of the quadratic transformation, the *total transform* of $\mathcal{I}(C)$, namely $\mathcal{I}(C)\mathcal{O}_{W_1}$, can be factored as a product, $\mathcal{I}(C)\mathcal{O}_{W_1} = \mathcal{I}(H)^1\overline{J}_1$ for some coherent ideal $\overline{J}_1 \subset \mathcal{O}_{W_1}$ which *does not* vanish along H.

Note that $\mathcal{I}(C_1)$ is a primary component of $\overline{J}_1$. However, $\overline{J}_1 \subsetneqq \mathcal{I}(C_1)$, and hence Theorem 2.2(iii) does not hold. To see why, it is convenient to express both ideals in terms of conductors: By definition,

$$\overline{J}_1 = (\mathcal{I}(C)\mathcal{O}_{W_1} : \mathcal{I}(H)^1)\,.$$

On the other hand, the ideal of the strict transform is

$$\mathcal{I}(C_1) = \cup_{k \geq 0}(\mathcal{I}(C)\mathcal{O}_{W_1} : \mathcal{I}(H)^k)\,,$$

or, in other words, $\mathcal{I}(C_1) = (\mathcal{I}(C)\mathcal{O}_{W_1} : \mathcal{I}(H)^N)$ for N large enough, since $(\mathcal{I}(C)\mathcal{O}_{W_1} : \mathcal{I}(H)^k) \subset (\mathcal{I}(C)\mathcal{O}_{W_1} : \mathcal{I}(H)^{k+1})$.

In this example $H \simeq P^2_{\mathbb{Q}}$ and C_1 cuts $P^2_{\mathbb{Q}}$ transversally at two different points. Let $L \subset P^2_R$ be the line defined by these two points, and let $\mathcal{I}(L) \subset \mathcal{O}_{W_1}$ be the ideal of $L(\subset W_1)$. Here

$$\overline{J}_1 = (\mathcal{I}(C)\mathcal{O}_{W_1} : \mathcal{I}(H)) \subsetneqq \mathcal{I}(C_1) = (\mathcal{I}(C)\mathcal{O}_{W_1} : \mathcal{I}(H)^2)\,.$$

In fact, looking at a suitable affine chart it follows that $\mathcal{I}(L)$ is a primary component of $\overline{J}_1$, and (of course), not of $\mathcal{I}(C_1)$. Therefore (iii) of Theorem 2.2 does not hold for the embedded desingularization defined by Π.

In Hironaka's line of proof the centers of monoidal transformations are always *included in the strict transform* of the scheme. In the case of our singular curve, the first monoidal transformation must be the one we have defined above, and any other center will have dimension zero. Now $\mathcal{I}(L)$ is a primary component of $\overline{J}_1$ supported on L, which has dimension 1; so we will never eliminate $\mathcal{I}(L)$ by blowing up closed points; hence (iii) will never hold for any desingularization of this curve defined as in Hironaka's proof.

In order to achieve (iii) one must blow up L (or some strict transform of L). Using the new algorithm that we propose, we first consider the quadratic transformation $\Pi : W_1 \longrightarrow W_0$, and then we blow-up at L. Since $L \subset H$, the first isomorphism in (i) is preserved after such monoidal transformation.

B) (*On a question of complexity*). We think of a subscheme X of a smooth scheme W, at least locally, as a finite number of *equations* defining the ideal $\mathcal{I}(X)$. Fix $X \subset W$. An algorithm of desingularization should provide us with:

1. A sequence of monoidal transformations over the smooth scheme W, say $W_n \longrightarrow W_{n-1} \longrightarrow \dots \longrightarrow W_1 \longrightarrow W_0 = W$ so that conditions (i) and (ii) Theorem 2.2 hold for the strict transform of X at W_n.
2. A pattern of manipulation of equations defining X, so as to obtain, at least locally at an open covering of W_n, equations defining the strict transform of X, X_n, at W_n.

So (2) indicates how the original equations defining X have to be treated at an affine open subset of W_n in order to obtain local equations defining X_n. While this is very complicated in Hironaka's line of proof, here it is a direct consequence of (iii). In fact, for algorithms that follow Hironaka's proof, for both (1) and (2) one must consider the *strict transform* of the ideal of the subscheme at each monoidal transformation. In that setting one has to choose a *standard basis* of the ideal, which is a system of generators of the ideal of the subscheme, but such choice of generators must be changed if the maximum Hilbert Samuel invariant drops in the sequence of monoidal transformations. All of these complications are avoided in our new proof, which simplifies both (1) and (2). The simplifications attained in (2) are illustrated by the following corollary of Theorem 2.2:

Corollary 3.1. *Under the assumptions and with the notation of Theorem 2.2, if X is a complete intersection then, the resolution of $\mathcal{O}_X$ in terms of free $\mathcal{O}_W$-modules,*

$$\dots \longrightarrow \wedge^k \mathcal{O}_W \longrightarrow \wedge^{k-1}\mathcal{O}_W \longrightarrow \dots ,$$

induces the resolution of $\mathcal{O}_{W_r}$,

$$\dots \longrightarrow \mathcal{L}^{-k} \wedge^k \mathcal{O}_{W_r} \longrightarrow \mathcal{L}^{-k+1} \wedge^{k-1} \mathcal{O}_{W_r} \longrightarrow \dots ,$$

in terms of locally free $\mathcal{O}_{W_r}-$modules.

Sketch of the proof. Assume that W is affine, and that $X \subset W$ is defined by the complete intersection ideal $I(X) =< f_1, f_2, ..., f_r >\subset \mathcal{O}_W$. A resolution of $\mathcal{O}_X$ by free $\mathcal{O}_W-$modules can be defined in terms of a Koszul complex. This complex is defined by taking the tensor product of

$$C_i \ := \ 0 \longrightarrow \mathcal{O}_W.e_i \longrightarrow \mathcal{O}_W \longrightarrow 0\,,$$

where each such complex C_i is defined by $e_i \longrightarrow f_i$, for $i = 1, ..., r$.

In the setting of 2.2 we have that $\mathcal{O}_W \subset \mathcal{O}_{W_r}$ and that $f_i \in \mathcal{L} \subset \mathcal{O}_{W_r}$. In particular each C_i induces a complex

$$\underline{C}_i \ := \ 0 \longrightarrow \mathcal{L}^{-1}.e_i \longrightarrow \mathcal{O}_{W_r} \longrightarrow 0\,.$$

Finally note that Theorem 2.2 (iii) says that the tensor product of these exact sequences defines a resolution of $\mathcal{O}_{X_r}$ in terms of locally free $\mathcal{O}_{W_r}$-modules. □

The curve of our example $C \subset W = \mathbb{A}^3_{\mathbb{Q}}$ is a complete intersection; note that the result in this corollary *will never* hold for a desingularization of this curve given within Hironaka's line of proof, since condition (iii) of our Theorem will never hold as seen in **A)**.

4. Basic objects

To achieve our results we use the notions of *basic objects* and *resolution of basic objects* (cf. [EV2]) (Definitions 4.1 and 4.2). Both Embedded Desingularization and Strong Principalization of Ideals can be obtained from a resolution of suitably defined basic objects.

Definition 4.1. A *basic object* is a triple that consists of a pair (W, E), an ideal $J \subset \mathcal{O}_W$ such that $(J)_\xi \neq 0$ for any $\xi \in W$, and a positive integer b. It is denoted by $(W, (J, b), E)$. The *singular locus* of the basic object is the closed set:

$$\mathrm{Sing}(J, b) = \{\xi \in W \mid \nu_J(\xi) \geq b\} \subset W.$$

If the dimension of W is d, then $(W, (J, b), E)$ is said to be a *$d-$dimensional basic object*. A regular closed subscheme $Y \subset W$ is *permissible* for $(W, (J, b), E)$ if Y is permissible for the pair (W, E) and $Y \subset \mathrm{Sing}(J, b)$. We define a *transformation of basic objects* in the following way: Consider the blowing-up , $W \longleftarrow W_1$, with center Y (having normal crossings with $E = \{H_1, \ldots , H_r\}$), and denote by $H_1 \subset W_1$ the exceptional hypersurface. This blowing-up induces a transformation of pairs $(W, E) \longleftarrow (W_1, E_1)$ as in 2.1. If Y is irreducible and c_1 is the order of J at the generic point of Y (i.e. $\nu_J(Y) = c_1 \geq b$), then there is an ideal $\overline{J}_1 \subset \mathcal{O}_{W_1}$ such that

$$J\mathcal{O}_{W_1} = I(H_1)^{c_1}\overline{J}_1. \qquad (4.1.1)$$

We define the ideal

$$J_1 = I(H_1)^{c_1-b}\overline{J}_1\,, \qquad (4.1.2)$$

and set $(W,(J,b),E) \longleftarrow (W_1,(J_1,b),E_1)$ as the *transformation of the basic object* $(W,(J,b),E)$.

Note here that $\overline{J}_1$ does not vanish along H_1 (i.e. $H_1 \subsetneq V(\overline{J}_1)$). In general, given a sequence of transformations

$$(W_0,(J_0,b),E_0) \longleftarrow \cdots \longleftarrow (W_k,(J_k,b),E_k)\,, \tag{4.1.3}$$

we obtain, for each index i, $0 \leq i \leq k$, expressions

$$J_i = I(H_{r+1})^{a_1} \cdots I(H_{r+i})^{a_i}\overline{J}_i \tag{4.1.4}$$

and

$$J_0\mathcal{O}_{W_i} = I(H_{r+1})^{c_1} \cdots I(H_{r+i})^{c_i}\overline{J}_i\,, \tag{4.1.5}$$

with $c_j > a_j \geq 0$, in other words,

$$J_0\mathcal{O}_{W_i} = L_iJ_i = \overline{L}_i\overline{J}_i \tag{4.1.6}$$

for some invertible ideals L_i, $\overline{L}_i$.

Definition 4.2. Sequence 4.1.3 is a *resolution of* $(W_0,(J_0,b),E_0)$ if $\mathrm{Sing}(J_k,b)=\emptyset$.

So if (4.1.3) is a resolution, then $W_k \longrightarrow W_0$ defines an isomorphism over $W_0 \setminus V(J_0)$, and $J_0\mathcal{O}_{W_k} = L_kJ_k$, where L_k is an invertible sheaf of ideals and J_k has no points of order $\geq b$ (in W_k). In particular Strong Principalization follows by taking a resolution of a basic object with $b=1$ and $J_0 = I$, (where I is as in Definition 2.3) .

A resolution of basic objects is usually approached by means of an *algorithm of resolution of basic objects*. To prove Theorem 2.2 we modify the algorithm of resolution of basic objects which appears in [EV1].

5. On the proof of Theorem 2.2

5.1. Let $(W_0, E_0 = \emptyset)$ and X be as in Theorem 2.2. Assume that, for some index $k \leq r$ we have defined a sequence

$$(W_0, E_0 = \emptyset) \longleftarrow \ldots \longleftarrow (W_k, E_k) \tag{5.1.1}$$

by setting $W_i \longleftarrow W_{i+1}$ a monoidal transformation with center Y_i. For each such index i, let $[\mathrm{Sing}(X)]_i$ $(\subset W_i)$, be the strict transform of $\mathrm{Sing}(X) \subset W_0$. If we assume that each center $Y_i \subset [\mathrm{Sing}(X)]_i \cup_{H_j \in E_i} H_j$, it will then follow that there is an isomorphism of open sets:

$$V_k := W_k \setminus \big([\mathrm{Sing}(X)]_k \cup \big(\cup_{H_j\in E_k} H_j\big)\big) \simeq V_0 = W_0 \setminus \mathrm{Sing}(X)\,.$$

We will also obtain an expression for the total transform

$$\mathcal{I}(X)\mathcal{O}_{W_k} = \overline{L}_k\overline{J}_k\,,$$

where $\overline{L}_k = \mathcal{I}(H_1)^{a_1}\cdot\ldots\cdot\mathcal{I}(H_k)^{a_k}$, with $a_i > 0$. This expression is unique if we require that $\overline{J}_k$ do not vanish along any H_i. Let $X_k \subset W_k$ be the strict transform of X. Note that $\overline{J}_k = (\mathcal{I}(X)\mathcal{O}_{W_k} : \overline{L}_k)$, whereas $\mathcal{I}(X_k) = (\mathcal{I}(X)\mathcal{O}_{W_k} : \overline{L}_k^N)$ for

N large enough, so clearly $\overline{J}_k \subset \mathcal{I}(X_k)$. On the other hand $\overline{J}_k = \mathcal{I}(X_k) = \mathcal{I}(X)$ when restricted to the open subset $V_k(\simeq V_0)$ of W_k.

Assume, for simplicity, that $\mathrm{Reg}(X) = X \setminus \mathrm{Sing}(X) \subset V_0$ is of pure codimension e. Via the isomorphism of open subsets $V_k \simeq V_0$ we see that at any point $y \in V_k$, either $(J_k)_y = \mathcal{O}_{W_k,y}$, or there is a regular system of parameters $\{x_1, \dots, x_d\} \subset \mathcal{O}_{W_k,y}$ so that $(J_k)_y = \mathcal{I}((X_k))_y =< x_1, \dots, x_e >$. This gives an idea of what J_k looks like locally at points of $V_k \subset W_k$. To deal with all points of W_k we need an additional definition:

Definition 5.2. Let (W, E) be a pair, and let $I \subset \mathcal{O}_W$ be a sheaf of ideals.
1) We shall say that I has *relative local codimension* $\geq a$ *at a point* $y \in W$, if either $I_y = \mathcal{O}_{W,y}$ or there is a regular system of parameters $\{x_1, x_2, \dots, x_n\}$ at $\mathcal{O}_{W,y}$, such that:

i. $< x_1, x_2, \dots, x_a >\subset I_y \subset \mathcal{O}_{W,y}$, and
ii. any hypersurface $H_i \in E$ containing the point y has a local equation $\mathcal{I}(H_i) =< x_{i_j} >\subset \mathcal{O}_{W,y}$, with $i_j > a$.

2) We shall say that I has *relative codimension* $\geq a$ in (W, E), if both conditions in 1) hold at every point $y \in W$.

Remark 5.3. Note that if I is of relative codimension $\geq a$, then the closed subscheme $V(I)$ is in fact of codimension $\geq a$ in W. Note also that any ideal I is of relative codimension ≥ 0 since such condition is empty.

Let X be under the assumptions of Theorem 2.2. Let us first consider the case when X is of pure codimension e. Then the local codimension of $J_0 = I(X)$ at any point of $V_0 = W_0 - \mathrm{Sing}(X)$ is e. Therefore one can check that J_0 will be of relative codimension $\geq a$ in (W_0, J_0) for some $a \leq e$. Now Theorem 2.2 follows from Lemma 5.4 and 5.5.

Lemma 5.4. *Assume that sequence 5.1.1 is defined so that* $\mathcal{I}(X)\mathcal{O}_{W_k} = \overline{L}_k\overline{J}_k$ *and the relative codimension of* $\overline{J}_k$ *in* (W_k, E_k) *is* $\geq a$. *Let* $U_k \subset W_k$ *be a non empty open set such that for any* $y \in U_k$*, the local codimension of* $(\overline{J}_k)_y$ *is* $\geq a + 1$. *Then we can define an enlargement of the sequence* 5.1.1*,*

$$(W_k, E_k) \longleftarrow \dots \longleftarrow (W_N, E_N) \tag{5.4.1}$$

so that:

(i) $\mathcal{I}(X)\mathcal{O}_{W_N} = \overline{L}_N\overline{J}_N$ *and* $\overline{J}_N$ *has relative codimension* $\geq a + 1$.
(ii) *The birational morphism* $W_k \longleftarrow W_N$ *defines an isomorphism over the open set* $U_k \subset W_k$.

Sketch of the proof. Fix a point $y \in V(\overline{J}_k)$, and set $< x_1, \dots, x_a >\subset (\overline{J}_k)_y$ as in Definition 5.2. We replace W_k by an open neighborhood of y so that $V_k^a = V(< x_1, \dots, x_a >) \subset W_k$ is smooth of codimension a. We attach to $(W_k, (J_k, 1), E_k)$ a new basic object

$$(V_k^a, (C(J_k), 1), \overline{E}_k), \tag{5.4.2}$$

where $C(J_k) = J_k\mathcal{O}_{V_k^a}$, and $\overline{E}_k$ are the induced hypersurfaces in V_k^a, as in Definition 5.2 (ii).

We next apply [EV1, Proposition 4.15] with $b = 1$, to show that every sequence of transformations

$$(V_k^a, (C(J_k), 1), \overline{E}_k) \longleftarrow \cdots \longleftarrow (V_N^a, (C(J_N), 1), \overline{E}_N), \tag{5.4.3}$$

induces a sequence

$$(W_k, (J_k, 1), E_k) \longleftarrow \cdots \longleftarrow (W_N, (J_N, 1), E_N), \tag{5.4.4}$$

and that $\mathrm{Sing}(C(J_i), 1) = \mathrm{Sing}(J_i, 1)$ (see [EV2, Definition 9.3] for the notion of coefficient ideal). In particular, a resolution of $(C(J_k), 1)$ induces a resolution of $(J_k, 1)$.

Functions $w-ord_e$ ([EV1, 5.19]) and n_e ([EV1, 6.17]) are defined at the closed set $\mathrm{Sing}(C(J_i), 1) = \mathrm{Sing}(J_i, 1)$, and these functions globalize because they are independent of all choice (of the neighborhood, of the point, etc). At a point $y \in U_k \cap V(\overline{J}_k)$, we observe that $(w - ord_e^k(y), n_e^k(y)) = (1, 0)$. We assume that sequence (5.4.3) is defined by the first N steps of the constructive resolution, where N is the least index for which the maximum value of $(w - ord_e^N, n_e^N)$ is $(1, 0)$. We set $C(J)_N = \overline{L'}_N \cdot \overline{C(J)}_N$ as in 5.1. We argue now as in [EV1, 6.16.1] by blowing-up smooth centers of pure codimension $a + 1$ in W_N (codimension 1 in V_N^a). We may assume that $C(J)_N = \overline{C(J)}_N$. This last step amounts to blowing-up $I(L)$ in the case of our example (see Section 3 **A)**). We finally check in this case that the induced sequence (5.4.4) fulfills the requirements of the lemma. □

5.5. How does Theorem 2.2 follow from Lemma 5.4 ? Assume that the sequence of transformations in (5.4.1) can be defined so that we obtain a factorization of the total transform $\mathcal{I}(X)\mathcal{O}_{W_k} = \overline{L}_k\overline{J}_k$, where $\overline{J}_k$ is of relative local codimension $\geq e$. We will first make some elementary geometric remarks on the closed set $V(\overline{J}_k) \subset W_k$:

(a) $V(\overline{J}_k)$ has local codimension $\geq e$ at any point. Let

$$V(\overline{J}_k) = F_1 \cup \ldots \cup F_l \cup C_{l+1} \cup \ldots \cup C_m$$

be the union of irreducible components (each of codimension at least e in W_k), where the F_i are those components of codimension e. Set $F = F_1 \cup \ldots \cup F_l$.

(b) Each component F_i, is a smooth and connected component of $V(\overline{J}_k)$.

(c) $V(\overline{J}_k) \cap V_k = F \cap V_k \simeq X \setminus \mathrm{Sing}(X)$ (V_k and V_0 as in 5.1).

Conditions (a) and (b) can be checked from Definition 5.2. Condition (c) will follow because we will define the sequences (5.1.1) and (5.4.1) so that $V_k \simeq V_0$.

If we assume that X is of *pure* codimension e, then (b) and Definition 5.2 assert that:

(d) If $X_k \subset W_k$ denotes the strict transform of X, then $X_k = F_1 \cup \ldots \cup F_{l'}$ ($l' \leq l$) is a disjoint union of closed regular sets.

Condition (2) in Definition 5.2 asserts that conditions (i) and (ii) of Theorem 2.2 hold for $W_k \longrightarrow W_0$. Now by (d) above, we conclude:

(e) For any $y \in X_k = F_1 \cup \ldots \cup F_{l'}$,

$$(\overline{J}_k)_y = (\mathcal{I}(X_k))_y \subset \mathcal{O}_{W_k,y} \ .$$

In fact, we know that $I(X_k)_y$ is a primary component of $(\overline{J}_k)_y$, thus $(\overline{J}_k)_y \subset (\mathcal{I}(X_k))_y$; if this inclusion were proper, then $V(\overline{J}_k)$ would have codimension $> e$ locally at y (but $y \in F \subset V(J_k)$). Note that (e) is saying that condition (iii) of Theorem 2.2 holds locally at each point in $F_1 \cup \ldots \cup F_{l'}$.

Since $V(\overline{J}_k)$ is a *disjoint union* of $F_1 \cup \ldots \cup F_{l'}$, and $F_{l'+1} \cup \ldots \cup F_l \cup C_1 \cup \ldots \cup C_m$, we can express $\overline{J}_k$ as a product of two ideals, $\overline{J}_k = [A_k]_1 \cdot [A_k]_2$, so that

$$V([A_k]_1) = F_1 \cup \ldots \cup F_{l'} \text{ and } V([A_k]_2) = F_{l'+1} \cup \ldots \cup F_l \cup C_1 \cup \ldots \cup C_m.$$

Note finally that $V([A_k]_2) \subset W_k \setminus V_k$. In case X is of pure codimension e, then (iii) of Theorem 2.2 is finally achieved by setting

$$(W_k, E_k) \longleftarrow \ldots \longleftarrow (W_r, E_r)$$

so as to define a principalization of $[A_k]_2 \subset \mathcal{O}_{W_k}$. The non-pure-dimensional case follows in the same fashion.

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Department of Mathematics, Facultad de Ciencias, Universidad Autónoma de Madrid, Canto Blanco 28049 Madrid, Spain.

E-mail address: ana.bravo@uam.es

E-mail address: villamayor@uam.es

Current address of Ana Bravo: Department of Mathematics, University of Michigan, Ann Arbor, MI 48109-1109.

E-mail address: anabz@math.lsa.umich.edu