THE EQUIVARIANT COHOMOLOGY OF HAMILTONIAN G-SPACES FROM RESIDUAL S^1 ACTIONS

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ABSTRACT. We show that for a Hamiltonian action of a compact torus G on a compact, connected symplectic manifold M, the G-equivariant cohomology is determined by the residual S^1 action on the submanifolds of M fixed by codimension-1 tori. This theorem allows us to compute the equivariant cohomology of certain manifolds, which have pieces that are four-dimensional or smaller. We give several examples of the computations that this allows.

1. Introduction

It has long been a "folk theorem" that, for a Hamiltonian torus action on a symplectic manifold, the associated equivariant cohomology is determined by S^1 actions on certain submanifolds. Recently, Tolman and Weitsman [TW] used equivariant Morse theory to prove that the cohomology is determined by that of the *one-skeleton*, the subspace given by the closure of all points whose orbit under the torus action is one-dimensional. Here we use a powerful result of Chang and Skjelbred [CS] to give a short proof of a slightly stronger statement.

Let M be a compact, connected symplectic manifold, and let G be a compact torus acting on M in a Hamiltonian fashion. Let M^G denote the fixed point set of the action. The inclusion $r : M^G \hookrightarrow M$ induces a map

$$r^* : H_G^*(M) \longrightarrow H_G^*(M^G).$$

F. Kirwan [K] proved that r^* is an injection. We find a simple description of the image of r^* in terms of the S^1-equivariant cohomology of submanifolds of M fixed by any codimension-1 subtorus of G.

We first fix some notation. Let $H \subset G$ be a codimension-1 torus and let M^H denote its fixed point set. Let X_H denote a connected component of M^H. Let

$$r_{X_H}^* : H_G^*(X_H) \to H_G^*(X_H^G)$$

be the map induced by the inclusion of the fixed points X_H^G into X_H and

$$i_{X_H}^* : H_G^*(M^G) \to H_G^*(X_H^G)$$

be the map induced by the inclusion of $X_H^G = X_H \cap M^G$ into M^G. We can reduce the computation of G-equivariant cohomology of M to the computation of S^1-equivariant cohomology of submanifolds of M as follows.

Theorem 1. *Let M be a compact, connected symplectic manifold with a Hamiltonian action of a torus G. Let $r^* : H_G^*(M) \longrightarrow H_G^*(M^G)$ be the map induced by the inclusion of the fixed point set. A class $f \in H_G^*(M^G)$ is in the image of r^* if and only if*

$$i_{X_H}^*(f) \in r_{X_H}^*(H_G^*(X_H)).$$

for all codimension-1 subtori $H \subset G$ and connected components X_H of M^H.

In particular, consider the case in which G acts with isolated fixed points, and $\dim X_H \leq 2$ for all X_H. Theorem 1 gives an explicit description of r^*, which was proven in significant generality in [GKM]. See, for example, [GS], Chapter 11. First we find the cohomology of each component X_H. If $\dim X_H = 2$, then X_H is diffeomorphic to S^2 with a Hamiltonian $S^1 \cong G/H$ action with fixed points denoted $\{N, S\}$. Suppose first that $G \cong S^1$ and $H = \{0\}$. In that case,

$$r_{X_H}^* : H_{S^1}^*(S^2) \hookrightarrow H_{S^1}^*(\{N, S\})$$

is the inclusion induced by $\{N, S\} \subset S^2$. By the Atiyah-Bott Berline-Vergne localization theorem [AB], [BV], any element $f \in H_{S^1}^*(\{N, S\})$ in the image of $r_{X_H}^*$ must satisfy the property that

(1)
$$f_N - f_S \in x \cdot \mathbb{C}[x]$$

where f_N and f_S are the restrictions of f to the points N and S, respectively. We have identified the equivariant cohomology of a point $H_{S^1}^*(pt)$ with $\mathbb{C}[x]$.

Let R be the graded ring $H_{S^1}^*(\{N, S\})$ subject to the above restriction. A quick dimension check shows that as modules over $H_{S^1}^*(pt)$, $H_{S^1}^*(S^2) = R$. However, the module structure forces the rings to be equal, so that condition (1) is the only condition for $f \in im(r_{X_H}^*)$.

Suppose now that $G \cong T^n$.

Proposition 1.1. *Suppose that S^2 is a Hamitonian G-space for $G \cong T^n$. Let H be a codimension 1 subtorus which acts trivially. Then a function $f = (f_N, f_S) \in S(\mathfrak{g}^*) \oplus S(\mathfrak{g}^*)$ is in the image of $r^* : H_G^*(S^2) \to H_G^*(\{N, S\})$ if and only if*

$$f_N - f_S \in \ker(\pi_H),$$

where $\pi_H : S(\mathfrak{g}^*) \to S(\mathfrak{h}^*)$ is induced by the projection $\mathfrak{g}^* \to \mathfrak{h}^*$.

Proof. Choose a complement L to H in G and write $S(\mathfrak{l}^*) \cong \mathbb{C}[x]$. We note that $H_G^*(S^2) = H_L^*(S^2) \otimes H_H^*(pt)$ and $H_G^*(\{N, S\}) = H_L^*(\{N, S\}) \otimes H_H^*(pt)$ because H acts trivially on S^2. Furthermore, the map

$$r^* : H_L^*(S^2) \otimes S(\mathfrak{h}^*) \to H_L^*(\{N, S\}) \otimes S(\mathfrak{h}^*),$$

is the identity of the second component, where $S(\mathfrak{h}^*)$ is the H-equivariant cohomology of a point. Thus, by (1), $f \in H_G^*(\{N, S\})$ is in the image of $r^* H_G^*(S^2) \to H_G^*(\{N, S\})$ if and only if $f_N - f_S \in (x) \cdot \mathbb{C}[x] \otimes S(\mathfrak{h}^*)$. But this is precisely the kernel of π_H. $\square$

Using this description of $H_G^*(X_H)$ in the case that $\dim(X_H) \leq 2$, we have the following corollary, a theorem of Goresky, Kottwitz and MacPherson [GKM].

Corollary 1.2. [GKM] *Let M be a compact, symplectic manifold with a Hamiltonian action of a compact torus G. Assume that M^G consists of isolated fixed points $\{p_1, \ldots, p_d\}$ and that each component X_H of M^H has dimension 0 or 2 for $H \subset G$ a codimension-1 torus. Let f_i be the restriction of a class $f \in H_G^*(M)$ to the fixed point p_i. Let $\pi_H : \mathfrak{g}^* \to \mathfrak{h}^*$ be the projection induced by the inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$. Then the map*

$$r^* : H_G^*(M) \longrightarrow H_G^*(M^G) = \bigoplus_{p \in M^G} H_G^*(pt)$$

has image $(f_1, \ldots, f_d)$ such that

$$\pi_H(f_i) = \pi_H(f_j)$$

whenever $\{p_i, p_j\} = X_H \cap M^G$.

This theorem can be stated in terms of graphs (cf. [GZ]). The natural generalization of graphs are hypergraphs. These are explored by the authors in [GH].

We now discuss the more general case, which extends the above corollary. We allow the fixed point sets of codimension 1 subtori of G to have dimension 0, 2, or 4. The equivariant cohomology of M in this case can be computed as follows.

Theorem 2. *Suppose that M is a compact, connected symplectic manifold with an effective Hamiltonian G action. Suppose further that the G action has only isolated fixed points $M^G = \{p_1, \ldots, p_d\}$ and that $\dim X_H \leq 4$ for all $H \subset G$ of codimension 1 and X_H a connected component of M^H. As before, let $f_i \in H_G^*(pt)$ denote the restriction of $f \in H_G^*(M)$ to the fixed point p_i. The image of the injection $r^* : H_G^*(M) \to H_G^*(M^G)$ is the subalgebra of functions $(f_1, \ldots, f_d) \in \bigoplus_{i=1}^d S(\mathfrak{g}^*)$ which satisfy*

$$\begin{cases} \pi_H(f_{i_j}) = \pi_H(f_{i_k}) & \text{if } \{p_{i_1}, \ldots, p_{i_l}\} = X_H^G \\ \sum_{j=1}^l \frac{f_{i_j}}{\alpha_1^{i_j} \alpha_2^{i_j}} \in S(\mathfrak{g}^*) & \text{if } \{p_{i_1}, \ldots, p_{i_l}\} = X_H^G \text{ and } \dim X_H = 4 \end{cases}$$

for all $H \subset G$ codimension-1 tori, where $\alpha_1^{i_j}$ and $\alpha_2^{i_j}$ are the (linearly dependent) weights of the G action on $T_{p_{i_j}} X_H$.

The results we present here are not generally true outside the symplectic setting. We rely heavily on the fact that the restriction map to the equivariant cohomology of the fixed point set is an injection. We also make use of the Morse theory associated to a Morse function obtained from the moment map for the Hamiltonian action.

In Section 2 we prove Theorem 1. In Section 3 we prove Theorem 2. Lastly, in Section 4 we give several examples of the computations allowed by this theorem.

2. Reduction to the study of circle actions

The proof of Theorem 1 is based on the following result of Chang and Skjelbred [CS]. The description given here is due to Brion and Vergne [BrV].

Lemma 2.1. [CS] *The image of $r^* : H_G^*(M) \to H_G^*(M^G)$ is the set*

$$\bigcap_H r_{M^H}^*(H_G^*(M^H)),$$

where the intersection in $H_G^*(M^G)$ is taken over all codimension-one subtori H of G and r_{M^H} is the inclusion of M^G into M^H.

Remark 2.2. In fact, the only nontrivial contributions to this intersection are those codimension-one subtori H which appear as isotropy groups of elements of M. Since M is compact, there are only finitely many such isotropy groups.

We will now restate and prove the theorem reducing the computation of $H_G^*(M)$ to computations of $H_{S^1}^*(X)$ for various submanifolds $X \subseteq M$.

Theorem 1. *Let $r^* : H_G^*(M) \longrightarrow H_G^*(M^G)$ be the map induced by the inclusion of the fixed point set. Let $i_{X_H}^* : H_G^*(M^G) \to H_G^*(X_H^G)$ and $r_{X_H}^* : H_G^*(X_H) \to H_G^*(X_H^G)$ be the maps defined in Section 1. A class $f \in H_G^*(M^G)$ is in the image of r^* if and only if*

$$i_{X_H}^*(f) \in r_{X_H}^*(H_G^*(X_H)).$$

for all codimension-1 subtori $H \subset G$ and connected components X_H of M^H.

Proof. By Lemma 2.1, $f \in im(r^*)$ if and only if f is in the intersection of $r_{M^H}^*(H_G^*(M^H))$ over all codimension-one subtori H, where $r_{M^H}^* : H_G^*(M^H) \to H_G^*(M^G)$. Equivalently,

$$f \in \bigcap_H r_{M^H}^*\left(\bigoplus_{X_H} H_G^*(X_H)\right),$$

where the direct sum is taken over all connected components X_H of M^H. Let $k_{X_H} : H_G^*(X_H) \to H_G^*(M^H)$ be the map which extends any class on X_H to 0 on other components of M^H. Let $k_{X_H^G} : H_G^*(X_H^G) \to H_G^*(M^G)$ be the same map on the fixed point sets. Then

$$r_{M^H}^*\left(\bigoplus_{X_H} H_G^*(X_H)\right) = \bigoplus_{X_H} r_{M^H}^* \circ k_{X_H}(H_G^*(X_H)).$$

As $k_{X_H^G} \circ r_{X_H}^* = r_{M^H}^* \circ k_{X_H}$, we have that f is in $im(r^*)$ if and only if

$$(2) \qquad\qquad f \in \bigoplus_{X_H} k_{X_H^G} \circ r_{X_H}^*(H_G^*(X_H)),$$

for all H. Now note that $i_{X_H}^* \circ k_{X_H^G} = id$. Because the X_H are disjoint, we can now apply $i_{X_H}^*$ to (2) to get

$$(3) \qquad\qquad i_{X_H}^*(f) \in r_{X_H}^*(H_G^*(X_H)),$$

for every H and X_H. However, since $\bigoplus_{X_H} i_{X_H}^*$ is an injection, we can apply $\bigoplus_{X_H} k_{X_H^G}$ to (3) to get (2). Thus, (3) and (2) are equivalent. This completes the proof. $\qquad\square$

This provides another proof for a result of Tolman and Weitsman [TW].

Definition 2.3. *Let $N \subset M$ be the set of points whose orbits under the G action are 1-dimensional. The* one-skeleton *of M is the closure $\overline{N}$.*

Tolman and Weitsman show that the image of $r^* : H_G^*(M) \to H_G^*(M^G)$ is equal to the image of the cohomology of the one-skeleton.

Theorem 2.4. [TW] *Let M be a compact symplectic manifold with a Hamiltonian torus action by G. Let M^G be the fixed point set, and $\overline{N}$ be the one-skeleton. Let $r : M^G \hookrightarrow M$ be the inclusion of the fixed point set to M and $j : M^G \hookrightarrow \overline{N}$ be the inclusion to $\overline{N}$. The induced maps $r^* : H_G^*(M) \to H_G^*(M^G)$ and $j^* : H_G^*(\overline{N}) \to H_G^*(M^G)$ on equivariant cohomology have the same image.*

Proof. Because G acts effectively, N consists of points fixed by some codimension-1 torus $H \subset G$ but not by all of G, i.e.

$$N = \bigcup_H M^H \backslash M^G$$

where the union is taken over all codimension-1 tori $H \subset G$. As noted above, this is a finite union over all codimension-1 H which appear as isotropy subgroups of points in M. Then $\overline{N} = \bigcup_H M^H$, and the inclusion $\gamma_H : M^H \hookrightarrow M$ factors through the inclusion $\gamma : \overline{N} \hookrightarrow M$ for each codimension-1 torus H in G. It follows that the induced maps in cohomology also factor. Furthermore, there is an inclusion

$$H_G^*(\overline{N}) \hookrightarrow \bigoplus_{i=1}^k H_G^*(M^{H_i}),$$

where $H_i, i = 1, \ldots, k$ are the codimension-1 tori which appear as isotropy subgroups of G. Theorem 1 implies that the map $r^* : H_G^*(M) \to H_G^*(M^G)$ factors through the map

$$\bigoplus_{i=1}^k r^*_{M^{H_i}} : \bigoplus_{i=1}^k H_G^*(M^{H_i}) \longrightarrow H_G^*(M^G)$$

But then r^* must factor through $j^* : H_G^*(\overline{N}) \to H_G^*(M^G)$. $\qquad\square$

Now suppose that M^G consists of isolated fixed points. Then

$$H_G^*(M^G) = \bigoplus_{p \in M^G} S(\mathfrak{g}^*)$$

and any $f \in H_G^*(M^G)$ is a map $f : M^G \to S(\mathfrak{g}^*)$. Furthermore, as X_H and X_H^G have trivial H actions, we can rewrite Theorem 1 in the following way.

Theorem 2.5. *Under the above hypotheses, the image of r^* is the set of $f : M^G \to S(\mathfrak{g}^*)$ such that*

$$i^*_{X_H}(f) : X_H^G \to S(\mathfrak{g}^*)$$

is in the image of

$$r^*_{X_H} : H^*_G(X_H) \to H^*_G(X_H^G) = \bigoplus_{p \in X_H^G} S(\mathfrak{g}^*),$$

where $r^*_{X_H}$ is restriction for each fixed point.

3. An extension of a theorem of GKM

We use Theorem 2.5 to compute the equivariant cohomology of M in the case in which $\dim X_H \leq 4$ for all codimension-1 tori $H \subset G$ and connected components X_H of the fixed point set M^H. In Section 1 we considered the case that $\dim X_H \leq 2$.

First, let X be a compact, connected symplectic four-manifold with an effective Hamiltonian $G = S^1$ action with isolated fixed points. Then the equivariant cohomology can be computed as follows.

Proposition 3.1. *Let X be a compact, connected symplectic 4-manifold with an effective Hamiltonian S^1 action with isolated fixed points $X^{S^1} = \{p_1, \ldots, p_d\}$. The map $r^* : H^*_{S^1}(X) \to H^*_{S^1}(X^{S^1})$ induced by inclusion is an injection with image*

$$(4) \qquad \{(f_1, \ldots, f_d) \in \bigoplus_{i=1}^d S(\mathfrak{s}^*) \mid f_i - f_j \in x \cdot \mathbb{C}[x], \; \sum_{i=1}^d \frac{f_i}{\alpha_1^i \alpha_2^i} \in S(\mathfrak{s}^*)\},$$

where α_1^i and α_2^i are the (linearly dependent) weights of the $S = S^1$ isotropy action on $T_{p_i}X$.

Proof. The map r^* is injective because X is equivariantly formal. We know that the f_i must satisfy the first condition because the functions constant on all the vertices are the only equivariant classes in degree 0, as $\dim H^0_{S^1}(X) = 1$. The second condition is necessary as a direct result of the Atiyah-Bott Berline-Vergne (ABBV) localization theorem ([AB],[BV]). Notice that this condition gives us one relation in degree 2 cohomology. A dimension count shows us that these conditions are sufficient. As an $S(\mathfrak{s}^*)$-module, $H^*_S(X) \cong H^*(X) \otimes H^*_S(pt)$. Thus, the equivariant Poincaré polynomial is

$$\begin{aligned} P^S_t(X) &= (1 + (d-2)t^2 + t^4) \cdot (1 + t^2 + t^4 + \ldots) \\ &= 1 + (d-1)t^2 + dt^4 + \cdots + dt^{2n} + \cdots. \end{aligned}$$

As $H^*_S(X)$ is generated in degree 2, the $d - 1$ degree 2 classes given by the $(f_1, \ldots, f_d)$ subject to the ABBV condition generate the entire cohomology ring. Thus we have found all the conditions. $\qquad \square$

We now prove a slightly more general proposition.

Proposition 3.2. *Let X be a compact, connected symplectic 4-manifold with a Hamiltonian G action with isolated fixed points $X^G = \{p_1, \ldots, p_d\}$. Suppose*

further that there is a codimension-1 subtorus H which acts trivially. The map $r^* : H_G^*(X) \to H_G^*(X^G)$ induced by inclusion is an injection with image

$$(5) \qquad \{(f_1, \ldots, f_d) \in \bigoplus_{i=1}^d S(\mathfrak{g}^*) \mid f_i - f_j \in \ker(\pi_H), \; \sum_{i=1}^d \frac{f_i}{\alpha_1^i \alpha_2^i} \in S(\mathfrak{g}^*)\},$$

where π_H is the map $S(\mathfrak{g}^*) \to S(\mathfrak{h}^*)$, and α_1^i and α_2^i are the (linearly dependent) weights of the G isotropy action on $T_{p_i} X$.

Proof. As in the case where $X \cong S^2$,

$$H_G^*(X) = H_{G/H}^*(X) \otimes S(\mathfrak{h}^*).$$

Furthermore,

$$H_G^*(X^G) = H_{G/H}^*(X^G) \otimes S(\mathfrak{h}^*).$$

Again, choose a complement L to H, and write $S(\mathfrak{l}^*) \cong \mathbb{C}[x]$. Then $H_{G/H}^*(X^G)$ can be identified with $\bigoplus_{p \in X^G} \mathbb{C}[x]$. By Proposition 3.1, we have $f \in H_G^*(X^G)$ is in the image of $r^* : H_G^*(X) \to H_G^*(X^G)$ if and only if the component of f in $H_{G/H}^*(X^G) \cong \bigoplus_{p \in X^G} \mathbb{C}[x]$ satisfies the conditions (4) of Proposition 3.1. But then f must satisfy the conditions (5). $\square$

We now discuss the more general case, which extends the result due to [GKM] (Corollary 1.2). Suppose that M is a compact, connected symplectic manifold with an effective Hamiltonian G action. Suppose further that this G action has only isolated fixed points $M^G = \{p_1, \ldots, p_d\}$ and that $\dim X_H \leq 4$ for all $H \subset G$ and X_H a connected component of M^H, as above. As before, let $f_i \in H_G^*(pt)$ denote the restriction of $f \in H_G^*(M)$ to the fixed point p_i. The equivariant cohomology of M can be computed as follows.

Theorem 2. *Under the above hypotheses, the image of the injection $r^* : H_G^*(M) \to H_G^*(M^G)$ is the subalgebra of functions $(f_1, \ldots, f_d) \in \bigoplus_{i=1}^d S(\mathfrak{g}^*)$ which satisfy*

$$\begin{cases} \pi_H(f_{i_j}) = \pi_H(f_{i_k}) & \text{if } \{p_{i_1}, \ldots, p_{i_l}\} = X_H^G \\ \sum_{j=1}^l \frac{f_{i_j}}{\alpha_1^{i_j} \alpha_2^{i_j}} \in S(\mathfrak{g}^*) & \text{if } \{p_{i_1}, \ldots, p_{i_l}\} = X_H^G \text{ and } \dim X_H = 4 \end{cases}$$

for all $H \subset G$ codimension-1 tori, where $\alpha_1^{i_j}$ and $\alpha_2^{i_j}$ are the (linearly dependent) weights of the G action on $T_{p_{i_j}} X_H$.

Proof. By Theorem 2.5, $im(r^*)$ consists of $(f_1, \ldots, f_d)$ which have certain properties restricted to each X_H. Proposition 3.2 lists these restrictions for each X_H of dimension 4. The conditions for X_H of dimension 2 are discussed in Section 1. A quick check shows that these are exactly the conditions listed above. $\square$

4. Examples

Here we demonstrate the use of Theorem 2 in computing equivariant cohomology. In the first example, we compute the S^1-equivariant cohomology of $\mathbb{C}P^2$ with a Hamiltonian circle action. In the second, we calculate the T^2-equivariant cohomology of $\mathbb{C}P^3$. Finally, we find the S^1-equivariant cohomology of a 4-dimensional manifold obtained by symplectic reduction.

Example 1. Consider $\mathbb{C}P^2$ with homogeneous coordinates $[z_0 : z_1 : z_2]$. Let $T = S^1$ act on $\mathbb{C}P^2$ by

$$e^{i\theta} \cdot [z_0 : z_1 : z_2] = [e^{-i\theta}z_0 : z_1 : e^{i\theta}z_2].$$

This action has three fixed points: $[1 : 0 : 0]$, $[0 : 1 : 0]$ and $[0 : 0 : 1]$.

The weights at these fixed points are

$$
\begin{array}{cc}
\text{Fixed point} & \text{Weights} \\
p_1 = [1 : 0 : 0] & x, 2x, \\
p_2 = [0 : 1 : 0] & -x, x, \\
p_3 = [0 : 0 : 1] & -2x, -x,
\end{array}
$$

where we have identified $\mathfrak{t}^*$ with degree one polynomials in $\mathbb{C}[x]$. As cohomology elements, these are assigned degree two. The image of the equivariant cohomology $H^*_{S^1}(\mathbb{C}P^2)$ in $H^*_{S^1}(\{p_1, p_2, p_3\}) \cong \bigoplus_{i=1}^{3} \mathbb{C}[x]$ is the subalgebra generated by the triples of functions (f_1, f_2, f_3) such that

$$f_i - f_j \in x \cdot \mathbb{C}[x] \text{ for every } i \text{ and } j, \text{ and}$$

$$\frac{f_1}{2x^2} - \frac{f_2}{x^2} + \frac{f_3}{2x^2} \in \mathbb{C}[x].$$

We use the cohomology computed above to compute the T^2-equivarant cohomology of $\mathbb{C}P^3$.

Example 2. The second example we consider is a T^2 action on $\mathbb{C}P^3$. Consider $\mathbb{C}P^3$ with homogeneous coordinates $[z_0 : z_1 : z_2 : z_3]$. Let T^2 act on $\mathbb{C}P^3$ by

$$(e^{i\theta_1}, e^{i\theta_2}) \cdot [z_0 : z_1 : z_2 : z_3] = [e^{-i\theta_1}z_0 : z_1 : e^{i\theta_1}z_2 : e^{i\theta_2}z_3].$$

This action has four fixed points, $[1 : 0 : 0 : 0]$, $[0 : 1 : 0 : 0]$, $[0 : 0 : 1 : 0]$ and $[0 : 0 : 0 : 1]$. The image of the moment map for this action is show in the figure below.

The weights at these fixed points are

$$
\begin{array}{cc}
\text{Fixed point} & \text{Weights} \\
p_1 = [1 : 0 : 0 : 0] & x, 2x, x + y, \\
p_2 = [0 : 1 : 0 : 0] & -x, x, y, \\
p_3 = [0 : 0 : 1 : 0] & -2x, -x, y - x, \\
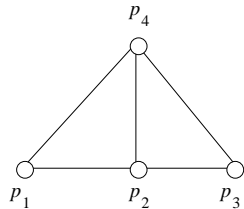p_4 = [0 : 0 : 0 : 1] & -x - y, -y, x - y.
\end{array}
$$

FIGURE 1. This shows the image of the moment map for T^2 acting on $M = \mathbb{C}P^3$, as described above.

Theorem 2 tells us that the image of the equivariant cohomology $H^*_{T^2}(\mathbb{C}P^3)$ in $H^*_{T^2}(\{p_1, p_2, p_3, p_4\}) \cong \bigoplus_{i=1}^4 \mathbb{C}[x, y]$ is the ring of functions (f_1, f_2, f_3, f_4) such that

$$f_i - f_j \in (x) \cdot \mathbb{C}[x, y] \quad \text{for every } i, j \in \{1, 2, 3\},$$

$$\frac{f_1}{2x^2} - \frac{f_2}{x^2} + \frac{f_3}{2x^2} \quad \in \quad \mathbb{C}[x, y],$$

$$f_1 - f_4 \quad \in \quad (y + x) \cdot \mathbb{C}[x, y],$$

$$f_2 - f_4 \quad \in \quad (y) \cdot \mathbb{C}[x, y],$$

$$f_3 - f_4 \quad \in \quad (y - x) \cdot \mathbb{C}[x, y].$$

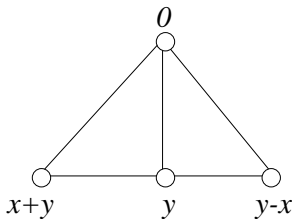

FIGURE 2. This shows an equivariant class of $H^*_{T^2}(\mathbb{C}P^3)$, shown as an element of the equivariant cohomology of the fixed points.

Example 3. Let $\mathcal{O}_\lambda$ be the coadjoint orbit of $SU(3)$ through the generic point $\lambda \in \mathfrak{t}^*$, the dual of the Lie algebra $\mathfrak{t}$ of the maximal 2-torus T in $SU(3)$. Recall that T acts on $\mathcal{O}_\lambda$ in a Hamiltonian fashion, and (one choice of) the moment map

$$\Phi_T : \mathcal{O}_\lambda \longrightarrow \mathfrak{t}^*$$

takes each matrix to its diagonal entries. Equivalently, Φ_T is the composition of the inclusion of $\mathcal{O}_\lambda$ into $\mathfrak{su}(3)^*$ and projection of $\mathfrak{su}(3)^*$ onto $\mathfrak{t}^*$.

We compute the equivariant cohomology of $M = \mathcal{O}_\lambda // H$, the *symplectic reduction* of $\mathcal{O}_\lambda$ by a circle H chosen such that the reduced space is a manifold. Let $H \subset T$ be any copy of S^1 which fixes a two-sphere in $\mathcal{O}_\lambda$. Then the moment map $\Phi_H : \mathcal{O}_\lambda \to \mathfrak{h}^*$ for the H action is the map Φ_T followed by the projection

$\pi_H : \mathfrak{t}^* \to \mathfrak{h}^*$ induced by the inclusion $\mathfrak{h} \hookrightarrow \mathfrak{t}$. The symplectic reduction at μ by H is by definition

$$M = \mathcal{O}_\lambda /\!/ H := \Phi_H^{-1}(\mu)/H,$$

where μ is a regular value for Φ_H. Note that there is a residual $T/H \cong S^1$ action on M. We use Theorem 1 to calculate the the corresponding equivariant cohomology of M.
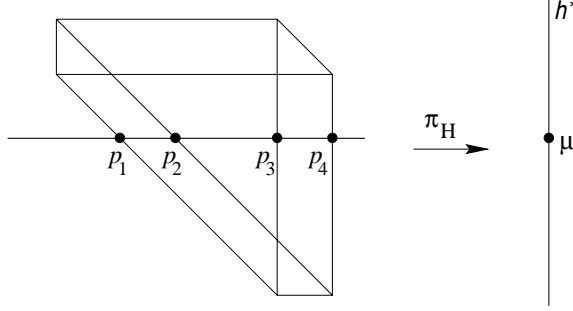

FIGURE 3. On the left is the image of the moment map for T acting on $\mathcal{O}_\lambda$. The cut through the moment polytope for $\mathcal{O}_\lambda$ corresponds to the symplectic reduction of $\mathcal{O}_\lambda$ by H at μ, for some choice of $\mathfrak{h}^\perp$.

One can easily see that there are four fixed points of this action, which we denote by p_i for $i = 1, \ldots, 4$. For each p_i, $\Phi_T^{-1}(p_i)$ lies on a two-sphere in $\mathcal{O}_\lambda$, denoted S_i^2, which is fixed by a a subgroup $H_i \cong S^1$ of T. Note that H_i is complementary to H in T.

The weights of the T/H action on the tangent space $T_{p_i} M$ are determined by the T action on S_i^2. Let n_i and s_i be fixed points of the T action on S_i^2. Note that the condition that μ be a regular value of Φ_H ensures that $\Phi_T^{-1}(p_i) \neq n_i, s_i$. Furthermore, by assumption the set $\Phi_T^{-1}(p_i)$ is point-wise fixed by H_i. Thus in the reduction, the T/H action on $T_{p_i} M$ is isomorphic to the H_i action on this space.

Denote the weights of the T action on $T_{n_i} \mathcal{O}_\lambda$ by $\pm\alpha_1, \pm\alpha_2$, and $\pm\alpha_3 = \pm(\alpha_1 + \alpha_2)$, where the signs depend on i. The weights of the H action on the reduction M are determined by projecting the α_i to $\mathfrak{h}_i{}^*$.

At p_1, the weight α_3 projects to 0 and the other two weights both project to the generator x of $S(\mathfrak{h}_1{}^*) \cong \mathbb{C}[x]$. Similarly, at p_2 the weights are x and $-x$, at p_3 they are x and $-x$ and at p_4 they are both $-x$. The image of the moment map $\Phi_H : M \to \mathfrak{h}^*$, with weights, is shown in Figure 4.

Finally, this tells us that the equivariant cohomology of M is

$$
\begin{aligned}
H_{S^1}^*(M) \quad \cong \quad & \{f : V \to \mathbb{C}[x] \mid f_i - f_j \in x \cdot \mathbb{C}[x], \\
& \frac{f_1}{x^2} - \frac{f_2}{x^2} - \frac{f_3}{x^2} + \frac{f_4}{x^2} \in \mathbb{C}[x]\}.
\end{aligned}
$$

Points $\quad p_1 \quad p_2 \qquad\quad p_3 \quad p_4$

Weights $\quad \begin{matrix} x \\ x \end{matrix} \quad \begin{matrix} -x \\ x \end{matrix} \qquad \begin{matrix} -x \\ x \end{matrix} \quad \begin{matrix} -x \\ -x \end{matrix}$
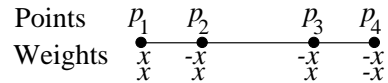
FIGURE 4. The image of the moment map for the T/H action on $M = \mathcal{O}_\lambda // H$, with the weights for the isotropy action on the tangent space of the fixed points.

Notice that this computation leads us to the T/S^1-equivariant cohomology of $M \cong \mathcal{O}_\lambda // S^1$ for a coadjoint orbit of $SU(n)$, as the submanifolds that appear are identical to those shown the above $SU(3)$ case.

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