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ON THE HASSE LOCUS OF A CALABI-YAU FAMILY

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A classic theorem of Deuring and Igusa asserts that the Hasse invariant of a modular family of elliptic curves has at most simple zeroes: in other words, the supersingular locus is reduced [4]. More generally, one can define the Hasse invariant for any family of Calabi-Yau varieties, and its zeroes form a well-defined closed subscheme of the base, called the "Hasse locus." Since it is locally defined by a single equation, this subscheme has codimension at most one, and one can ask, by analogy with the result of Deuring and Igusa, how singular it can be for modular families. For example, in the moduli space of polarized K3 surfaces of degree prime to the characteristic, the Hasse locus is a divisor with isolated singularities. In fact, the singularities are ordinary double points and occur exactly at the so-called "superspecial" K3 surfaces [8]. In other words, the Hasse invariant has simple zeroes except for superspecial surfaces, where it has zeroes of order exactly two. In this note we show that it is reasonable to expect that for versal Calabi-Yau families of relative dimension n, the Hasse invariant can vanish to order at most n, and we prove that this is so for hypersurfaces in projective space of characteristic greater than n. Philosophically, the reason for this is that the order of vanishing is controlled by the weight of the underlying variation of Hodge structure, rather than the order of the differential equation satisfied by the Hasse invariant.

A smooth projective and geometrically connected scheme X/k over a field k of dimension n is usually said to be a *Calabi-Yau* variety if $H^i(X, \mathcal{O}_X)$ vanishes for $i \neq 0, n$ and $\dim H^n(X, \mathcal{O}_X) = 1$. Serre duality implies that this condition is equivalent to the assertion that $H^i(X, \Omega^n_{X/k})$ vanishes for $i \neq 0, n$ and $H^0(X, \Omega^n_{X/k})$ is one-dimensional. By an n-dimensional "Calabi-Yau family" we shall mean a smooth projective morphism $f\colon X \to S$ all of whose fibers are Calabi-Yau varieties of dimension n. It then follows that $R^n f_*(\mathcal{O}_X)$ and $f_*(\Omega^n_{X/S})$ are invertible sheaves on S and that the natural map

$$f^* f_*(\Omega^n_{X/S}) \to \Omega^n_{X/S}$$

is an isomorphism. Let $L := f_*(\Omega^n_{X/S})$; there is a canonical duality isomorphism $L^{-1} \cong R^n f_*(\mathcal{O}_X)$. If S has characteristic p (as we shall assume from now on), then the absolute Frobenius endomorphism of X defines a map

$$h\colon F_S^*(R^n f_*(\mathcal{O}_X)) \to R^n f_*(\mathcal{O}_X) \tag{1}$$

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called the *Hasse invariant* of X/S; it can be viewed as map $L^{-p} \cong F_S^*(L^{-1}) \to L$, or equivalently as a section of the invertible sheaf L^{p-1}. A choice of a local basis for L determines also local bases for L^{-1} and $F_S^*(L^{-1})$; the matrix of h with respect to these bases is the *Hasse-Witt matrix* of X/S. The ideal it generates is independent of the choices, and the corresponding closed subscheme S_h of S is the *Hasse locus* of X/S. If the Hasse-Witt matrix is a nonzero divisor, then S_h is an effective Cartier divisor in S; in any case S_h has codimension at most one in S. If s is a closed point of S_h and $n \in \mathbf{N}$ then S_h has multiplicity at least n if the ideal I_h of S_h in S is contained in m_s^n, in which case we say that the Hasse invariant vanishes to order n.

Let X/S be a Calabi-Yau family, where S is a smooth scheme over an algebraically closed field k. Consider the following conditions (the last of which depends on the choice of a natural number m):

1. The Hodge spectral sequence of each of the fibers of X/S degenerates at E_1, and all the sheaves $R^i f_*(\Omega^j_{X/S})$ are locally free.
2. The Kodaira-Spencer mapping $T_{S/k} \to R^1 f_*(T_{X/S})$ is surjective.
3. For a fixed integer m, the natural maps induced by cup product and interior multiplication

$$\mathrm{Sym}^j\, R^1 f_*(T_{X/S}) \otimes f_*(\Omega^n_{X/S}) \to R^j f_*(\Omega^{n-j}_{X/S}))$$

   are surjective for all j less than m.

Condition (1) is technical in nature; it seems to hold often, but not always [3]. It implies in particular that the Hodge filtration F of the relative De Rham cohomology $H^n_{DR}(X/S) := R^n f_*(\Omega^{\cdot}_{X/S})$ is a filtration by local direct factors, and that formation of the Hodge sheaves and the Hodge filtration is compatible with any base change $S' \to S$. Condition (2) is what we mean by saying that X/S is "versal"; in the case of polarized K3 surfaces one needs to work with a primitive version [8]. The last condition is more subtle. It is a nondegeneracy condition about the "Yukawa coupling." Since X/S is a Calabi-Yau family, there are natural isomorphisms

$$R^1 f_*(T_{X/S}) \cong R^1 f_* \,\mathrm{Hom}\,(\Omega^n_{X/S}, \Omega^{n-1}_{X/S}) \cong \mathrm{Hom}\,(L, R^1 f_*(\Omega^{n-1}_{X/S})).$$

Thus condition (3) is always verified with $m = 2$. Furthermore, the above isomorphisms and condition (1) imply that formation of $R^1 f_*(T_{X/S})$ commutes with base change and that condition (3) is compatible with passage to the fibers. In particular, (3) will hold in some neighborhood of a point s if and only if it holds for the fiber over s. Let us also note that if (2) holds, (3) is equivalent to the injectivity of the dual map

$$R^{n-j} f_*(\Omega^j_{X/S}) \to \Gamma^j(R^1 f_*(T_{X/S})) \otimes R^n f_*(\mathcal{O}_X).$$

Here we have identified the dual of $\mathrm{Sym}^j(T_{X/S})$ with the divided power construction $\Gamma^j(\Omega^1_{X/S})$, using [1, A10].

Recall that there is a second spectral sequence of hypercohomology: $E_2^{i,j} = H^i(X, \underline{H}^j)$, where $\underline{H}^j$ is the Zariski sheaf associated to the presheaf $U \mapsto H_{DR}(U/S)$. In characteristic p, this spectral sequence is often called the *conjugate spectral sequence*, and the associated filtration the *conjugate filtration*. Thanks to the Cartier isomorphism, (2) above implies that the conjugate spectral sequence degenerates at E_2, and that the conjugate filtration F_{con} is a filtration by local direct factors whose formation commutes with base change (see [5, 2.3.2] and the ensuing discussion).

Theorem 1. *Let X/S be a Calabi-Yau family, let $i < p$ be a natural number, and let s be a closed point of S. Suppose that conditions* (1) *and* (2) *hold at s, and let X_s be the fiber of X/S over s. If $F^n_{con}H_{DR}(X_s/k) \subseteq F^i H_{DR}(X_s/k)$ then the Hasse invariant of X/S vanishes to order i at s. The converse is true if in addition condition* (3) *holds with $m = i$.*

Since (2) trivially holds with $m = 1$, the following result of Katz [5, 2.3.4.1.7] is a special case of theorem (1). However, we hasten to point out that Katz's result is in fact the fundamental starting point of the proof of (1).

Corollary 2. *Let X/S be a Calabi-Yau family satisfying conditions* (1)–(3). *Then a closed point s of S lies in the Hasse locus S_h if and only if*

$$F^n_{con}H_{DR}(X_s/k) \subseteq F^1 H_{DR}(X_s/k).$$

Condition (1) implies that $F^n_{con}H_{DR}(X_s/k) \neq 0$, and since $F^{n+1}H_{DR}(X_s/k)$ vanishes, the theorem also provides a (conditional) bound on the order of vanishing of the Hasse invariant.

Corollary 3. *Let X/S be a Calabi-Yau family of relative dimension n less than p satisfying conditions* (1)–(3) *for all m. Then the order of vanishing of the Hasse invariant at any closed point of S is less than or equal to n.*

Proof of Theorem 1. The theorem can be conveniently explained in terms of the F-T crystal on S to which X/S gives rise [7]. However, for the sake of concreteness, we shall give a more direct treatment which does not make explicit use of this notion or of the crystalline topos.

Let $H_{DR}(X/S)$ or just E denote the relative De Rham cohomology of X/S, together with its Gauss-Manin connection ∇ and its Hodge and conjugate filtrations F and F_{con}. The Hasse invariant of X/S fits into a commutative diagram

$$\begin{array}{ccc} F_S^* H^n_{DR}(X/S) & \xrightarrow{\Phi} & H^n_{DR}(X/S) \\ \Big\downarrow F_S^*(\pi) & & \pi \Big\downarrow \\ F_S^* H^n(X, \mathcal{O}_X) & \xrightarrow{h} & H^n(X, \mathcal{O}_X) \end{array}$$

In fact, Φ annihilates $F_S^* F^1 H^n_{DR}(X/S)$; since this is the kernel of the surjective map $F_S^*(\pi)$, Φ factors through a map

$$\Phi_0 \colon F_S^* H^n(X, \mathcal{O}_X) \to H^n_{DR}(X/S).$$

As Katz proves in [5], the degeneration of the Hodge and conjugate spectral sequences imply that this map is injective, and its image is precisely the invertible $\mathcal{O}_S$-module $F^n_{con} H^n_{DR}(X/S)$. Let us recall the argument: if $F \colon X \to X'$ is the relative Frobenius map, then $F_* \underline{H}^0 \cong \mathcal{O}_{X'}$, so in the conjugate spectral sequence,

$$E_2^{n,0} = H^n(X, \underline{H}^0) \cong H^n(X', F_* \underline{H}^0) \cong H^n(X', \mathcal{O}_{X'}) \cong F_S^* H^n(X, \mathcal{O}_X).$$

Let η be a local generator for $F^n_{con} H^n_{DR}(X/S)$, let ζ be a local generator for $H^n(X, \mathcal{O}_X)$, and write $\pi(\eta) = a\zeta$ with $a \in \mathcal{O}_S$. Then a is a local equation for the Hasse locus. Note that the same formulation works after any base change $T \to S$.

It is now apparent how to prove corollary 2. By definition, s belongs to the Hasse-Witt locus if and only $h(s)$ is zero, *i.e.*, if and only if the image $F^n_{con} H_{DR}(X_s/k)$ is contained in the kernel $F^1 H_{DR}(X_s/k)$ of π.

The theorem is a consequence of Griffiths transversality and the invariance of $F^n_{con} H_{DR}(X/S)$ is invariant under the Gauss-Manin connection. Since $i < p$, the connection can be used to trivialize E up to order i. More precisely, let S_i denote the spectrum of $\mathcal{O}_{S,s}/m^{i+1}$. Since $i+1 \le p$, the maximal ideal of S_i has a divided power structure in which $\gamma_p(a) = 0$ for every $a \in m$ [1, 3.2.4]. In general, if f and g are two morphisms from a k-scheme T to S which agree modulo a divided power ideal (I, γ), the connection furnishes a canonical isomorphism

$$\epsilon(g, f) \colon g^* E \to f^* E.$$

To make this explicit, let us suppose that there exists a coordinate system $(t_1, \dots t_m)$ for S/k, defining an étale map from S to affine m-space. Note that $g^*(t_i) - f^*(t_i) \in I$, so that we can take its divided powers. Then, in standard multi-index notation, $\epsilon(g, f)$ is given by:

$$\epsilon(g, f)(g^*(e)) = \sum_I \gamma_I(g^*(t_i) - f^*(t_i)) f^*(\nabla_{\partial_I}(e)) \tag{2}$$

for any $e \in E$. Griffiths transversality asserts that the connection moves the Hodge filtration by at most one step, and so it follows from the formula above that if $e \in F^i E(X/S)$, then

$$\epsilon(g, f) g^*(e) \in f^* F^i E + I f^* F^{i-1} E + I^{[2]} f^* F^{i-2} E + \cdots + I^{[i]} f^* E.$$

Here $I^{[i]}$ means the i^{th} divided power of the ideal I [1, 3.24].

To use these ideas to prove the first statement of the theorem, choose a local basis η of $H^n(X, \mathcal{O}_X)$. Then $F^*(\eta) = (1 \otimes \eta)$ is a horizontal section of $F^* H^n(X, \mathcal{O}_X)$, and $\Phi(1 \otimes \eta)$ is a horizontal basis of $F^n_{con} H_{DR}(X/S)$. Let f be the natural inclusion $S_i \to S$ and let $g \colon S_i \to S$ be the projection $S_i \to \operatorname{Spec} k$ followed by the inclusion $\operatorname{Spec} k \to S$. Assume that $F^n_{con} E(s) \subseteq F^i E(s)$. Then

$$g^* F^n_{con} E \subseteq g^* F^i E,$$

and taking $I = m/m^{i+1}$ with its divided power structure γ, we find by (2)

$$\epsilon(g,f)g^*F^n_{con}E \subseteq \epsilon(g,f)g^*F^iE \subseteq f^*F^iE + If^*F^{i-1}E + \cdots I^{[i]}f^*E$$

Since η is horizontal $\eta = \epsilon(g,f)g^*\eta$, and so it follows that $F^n_{con}E \subseteq F^1E + I^{[i]}E$. Writing $\pi(\eta) = a\zeta$ as above, we find that $a \in I^{[i]}$, and since $i < p$, $I^{[i]} = I^i$ and $a \in m^i$.

To investigate the converse, we need to express the highest order term in the equation (2) in terms of the Kodaira-Spencer mapping. Again we give a self-contained and what we hope is a down-to earth treatment of a result which already appears in the literature [6], and is widely known in characteristic zero.

The Kodaira-Spencer mapping

$$\xi\colon T_{S/k} \to \operatorname{End}_{-1} \operatorname{Gr}_F E$$

is the map induced by the connection, using Griffiths transversality. For any sequence $(D_1, \dots D_i)$ of elements of $T_{S/k}$, denote by $\xi(D_1, \dots D_i)$ the composition $\xi(D_1) \circ \cdots \xi(D_i) \in \operatorname{End}_i(\operatorname{Gr}_F E)$. It follows from the integrability of ∇ that this composition is independent of the ordering of $(D_1, \dots D_i)$, and hence ξ defines a map

$$\xi_i\colon \operatorname{Sym}^i(T_{S/k}) \to \operatorname{End}_{-i}(\operatorname{Gr}_F E),$$

or, equivalently, an element $\xi^{[i]}$ of $\Gamma^i(\Omega^1_{S/k}) \otimes \operatorname{End}_{-i}(\operatorname{Gr}_F E)$. If $T_{S/k}$ is free with basis $(\partial_1, \dots \partial_n)$ and $(\omega_1, \dots \omega_n)$ is the dual basis for $\Omega^1_{S/k}$, then in multi-index notation

$$\xi^{[i]} = \sum_{|I|=i} \omega^{[I]} \otimes \xi_i(\partial^I). \tag{3}$$

In particular, $\xi^{[1]} = \sum_j \omega_j \otimes \xi(\partial_j)$, and $\xi^{[i]} = (\xi^1)^{[i]}$.

If (I, γ) is a PD-ideal of T defining a closed subscheme T' of T and if f and g are maps $T \to S$ with the same restriction h to T', then for any i and $j \geq 0$, the reduction modulo $I^{[i+1]}E + F^{j-i+1}E$ of $\epsilon(g,f)$ is a map

$$\xi_{i,j}(g,f)\colon h^* \operatorname{Gr}^j_F E \to I^{[i]}/I^{[i+1]} \otimes h^* \operatorname{Gr}^{j-i}_F E.$$

Putting these together for various j, we find

$$\xi_i(g,f) \in I^{[i]}/I^{[i+1]} \otimes \operatorname{End}_{-i} h^*(\operatorname{Gr}_F E).$$

In fact, $\xi_i(g,f)$ can be expressed in terms of first-order data as follows. The maps (g,f) give a map $\delta\colon h^*\Omega^1_{S/k} \to I/I^2$, and hence one gets a commutative diagram

$$\begin{array}{ccc} h^*\Gamma^i(\Omega^1_{S/k}) & \xrightarrow{\Gamma^i(\delta)} & \Gamma^i(I/I^2) \\ & \searrow^{\gamma_i(\delta)} & \downarrow \\ & & I^{[i]}/I^{[i+1]} \end{array}$$

Then the key formula is

Lemma 4. *With the above notation,* $\xi_i(g,f) = (\gamma_i(\delta)\otimes \mathrm{id})(h^*(\xi^{[i]}))$.

To check this formula, suppose that there exists a coordinate system $(t_1,\dots t_m)$ for S/k, defining an étale map from S to affine m-space. Let $z_i := g^*(t_i) - f^*(t_i) \in I$, and let $\partial_1,\dots\partial_m$ be the basis for $T_{S/k}$ dual to $(dt_1,\dots dt_m)$. Then δ sends $h^*(dt_i)$ to the class $\overline{z}_i$ of z_i in $I/I^{[2]}$. If $e \in F^jE$, let us consider a typical term $e_I := \gamma_I(z)f^*(\nabla_{\partial_I}(e))$ of $\epsilon(g,f)(f^*(e))$. If $|I| < i$, then it follows from Griffiths transversality that e_I belongs to $g^*F^{j-i+1}E$, and if $|I| > i$, e_I belongs to $I^{[i+1]}E$. If $|I| = i$, the image of $f^*(\nabla_{\partial_I})(e)$ modulo $F^{j-i+1}E$ is $f^*(\xi_i(\partial_I)(e))$. Using these facts and equation (3), we see that the reduction of $\epsilon(g,f)(f^*(e))$ modulo $F^{j-i+1}E + I^{[i+1]}E$ is

$$\begin{aligned}\sum_{|I|=i}\gamma_I(z)f^*(\xi_i(\partial_I))(e) &= \sum_{|I|=i}\overline{z}^{|I|}\otimes h^*\xi_i(\partial_I)(e)\\ &= \sum_{|I|=i}\gamma_i(\delta)(h^*(dt^{|I|})\otimes h^*\xi_i(\partial_I)(e)\\ &= (\gamma_i(\delta)\otimes \mathrm{id})(h^*(\xi^{[i]}))(e).\end{aligned}$$

□

Now suppose that the Hasse invariant of X/S vanishes to order i at s, where $i < p$, and assume that (3) holds with $m \geq i$. Let η be a local horizontal basis for $F^n_{con}H_{DR}(X/S)$; we prove that $\eta(s) \in F^iH_{DR}(X_s/k)$. Using induction on i, we may assume that $\eta(s) \in F^{i-1}H_{DR}(X_s/k)$. Since the Hasse invariant vanishes to order at least i, $\pi(\eta) \in m^i \operatorname{Gr}^0_F H_{DR}(X/S)$. Let f and g be as above. Since $\eta(s) \in F^{i-1}H_{DR}(X_s/k)$, $g^*(\eta) \in g^*F^{i-1}H_{DR}(X/S)$. Since $f^*\eta = \epsilon(g,f)g^*(\eta))$,

$$\pi(f^*(\eta))) = \pi\epsilon(g,f)g^*(\eta) = \xi_{i-1}(g,f)h^*(\eta)$$

As we have seen, the dual of (3) says that the map

$$\operatorname{Gr}^{i-1}_F H_{DR}(X_s/k) \to \Gamma^{[i-1]}(m/m^2) \cong m^{[i-1]}/m^{[i]} \cong m^{i-1}/m^i$$

is injective. Thus the image of $h^*(\eta)$ in $\operatorname{Gr}^{i-1}_F H_{DR}(X_s/k)$ vanishes, and $\eta(s) \in F^iH_{DR}(X_s/k)$. □

Remark 5. It is easy to prove that conditions (1)–(3) above are verified for the universal family of smooth hypersurfaces of degree $n+2$ in $\mathbf{P}^{n+1}$. For example, the degeneration of the Hodge spectral sequence follows from [2]. Condition (3) follows from the fact that the graded Hodge cohomology can be identified with the Jacobian algebra in the universal family of hypersurfaces, and cup product with multiplication [6]. To make this explicit, let S^i denote the set of homogenous polynomials of degree i in the variables $T_1,\cdots T_m$ with coefficients in k, and let $f \in S^m$ define a smooth X_0. Then if p does not divide m, the primitive Hodge cohomology $H^{n-i}_{prim}(X,\Omega^i_{X_0/k})$ can be identified with the quotient of the set $S^{(n-i)m}$ by the set of polynomials which are multiples of the partial derivatives of f. Moreover, there is an obvious map from S^m to the normal bundle to X_0 in its ambient projective space and hence to $H^1(X_0, T_{X_0/k})$, and the cup product pairing of condition (3) just identifies with multiplication of polynomials. Thus the map of (3) is certainly surjective.

Remark 6. For varieties of dimension greater than three, condition (3) doesn't look very reasonable, and should probably be replaced by a condition involving some sort of "very primitive" cohomology. For example, if X/k has dimension 4 the image of the map

$$H^0(\Omega^4_{X/k}) \otimes \mathrm{Sym}^2 H^1(X, T_{X/k}) \to H^2(\Omega^2_{X/k})$$

lands in the the annihilator of the image of

$$\mathrm{Sym}^2 H^1(X, \Omega^1_{X/k}) \to H^2(\Omega^2_{X/k}),$$

since $H^2(X, \mathcal{O}_X) = 0$.

Remark 7. For versal families of polarized K3 surfaces of odd degree in characteristic two, it is again true that the Hasse invariant can have zeroes of order at most two [8]. A key ingredient in the proof of this is the fact that the symmetric bilinear form on the crystalline cohomology of such a surface is *even.* This allows us to replace the Yukawa coupling $\mathrm{Sym}^2(T_{S/k}) \to \mathrm{End}_{-2}\,\mathrm{Gr}_F\, H^2(X/S)$ with a map $\Gamma^2(T_{S/k}) \to \mathrm{End}_{-2}\,\mathrm{Gr}_F\, H^2(X/S)$. Its dual is then an element of $\mathrm{Sym}^2(\Omega^2_{S/k}) \otimes \mathrm{End}_{-2}\,\mathrm{Gr}_F\, H^2(X/S)$, eliminating the use of divided powers powers on the maximal ideal of a point of s. It would be very interesting to know if the Yukawa coupling in dimension n is also induced by a map $\Gamma^n(T_{S/k}) \to \mathrm{End}_{-n}\,\mathrm{Gr}_F\, H^n(X/S)$, *i.e.*, a polynomial law of degree n in the sense of Roby [1, A].

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